$\mathrm{rot}\left(\mathrm{rot}\mathbf{A}\right)$の計算

ベクトル演算公式　$\mathrm{rot}\left(\mathrm{rot}\mathbf{A}\right)=\mathrm{grad}(\mathrm{div}\mathbf{A})-\nabla^2$　を導く。

定義から、$\mathrm{rot}\mathbf{A}$ はベクトルで次のような成分をもつ。

$$\mathrm{rot}\mathbf{A}=\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z},\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x},\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)$$

従って、これにもう一度 rot を演算すると次のようになる。

$$\left[\mathrm{rot}\left(\mathrm{rot}\mathbf{A}\right)\right]_x=\frac{\partial}{\partial y}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right) \qquad ①$$

$$\left[\mathrm{rot}\left(\mathrm{rot}\mathbf{A}\right)\right]_y=\frac{\partial}{\partial z}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)-\frac{\partial}{\partial x}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right) \qquad ②$$

$$\left[\mathrm{rot}\left(\mathrm{rot}\mathbf{A}\right)\right]_z=\frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right) \qquad ③$$

一方、右辺の第一項を成分で書くと次のようになる。

$$\left[\mathrm{grad}\left(\mathrm{div}\mathbf{A}\right)\right]_x=\frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right) \qquad ④$$

$$\left[\mathrm{grad}\left(\mathrm{div}\mathbf{A}\right)\right]_y=\frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right) \qquad ⑤$$

$$\left[\mathrm{grad}\left(\mathrm{div}\mathbf{A}\right)\right]_z=\frac{\partial}{\partial z}\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right) \qquad ⑥$$

また、右辺第二項は

$$-\nabla^2 A_x=-\left(\frac{\partial^2 A_x}{\partial x^2}+\frac{\partial^2 A_x}{\partial y^2}+\frac{\partial^2 A_x}{\partial z^2}\right) \qquad ⑦$$

$$-\nabla^2 A_y=-\left(\frac{\partial^2 A_y}{\partial x^2}+\frac{\partial^2 A_y}{\partial y^2}+\frac{\partial^2 A_y}{\partial z^2}\right) \qquad ⑧$$

$$-\nabla^2 A_z = -\left(\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2}\right) \qquad ⑨$$

これらから、それぞれ成分ごとに調べると

①=④＋⑦

②=⑤＋⑧

③=⑥＋⑨

が成り立つので、$\mathrm{rot}(\mathrm{rot}\mathbf{A}) = \mathrm{grad}(\mathrm{div}\mathbf{A}) - \nabla^2$ が成り立っていることがわかる。