The boundary effect of anomaly-induced action

arXiv:1505.00959
PRD accepted recently

Che-min Shen

Collaborate with Dr. Keisuke Izumi and Prof. Pisin Chen

National Taiwan University[1],
Leung Center for Cosmology and Particle Astrophysics[2],

Summary

- Scenario: Quantum field in curved spacetime
- Goal: Compute $\langle T_{\mu\nu} \rangle$ of some system.
- Method: Anomaly-induced action
- (Our work) Improvement: Including the Boundary term in above action

Summary

- Scenario: Quantum field in curved spacetime
- Goal: Compute $\langle T_{\mu\nu} \rangle$ of some system.
- Method: Anomaly-induced action
- (Our work) Improvement: Including the Boundary term in above action

Summary

- Scenario: Quantum field in curved spacetime
- Goal: Compute $\langle T_{\mu\nu} \rangle$ of some system. (Why?)
- Method: Anomaly-induced action
- (Our work) Improvement: Including the Boundary term in above action

Ans: Because it is important!

$$\underset{(?)}{\langle 0|} T_{\mu\nu} \underset{(?)}{|0\rangle}$$

However, the concept of the vacuum states is always ambiguous.
Different observers (frame) will realize different quantum states as "their vacuum state".
(Ex: Minkowski & Rindler vacuum? **Boulware & Hartle-Hawking & Unruh?**)

Summary

- Scenario: Quantum field in curved spacetime
- Goal: Compute $\langle T_{\mu\nu} \rangle$ of some system.
- **Method: Anomaly-induced action**
- (Our work) Improvement: Including the Boundary term in above action

$$S_{\rm anom} = \frac{1}{96\pi}\int \mathrm{d}^2x\sqrt{-g}\left[g^{\alpha\beta}\nabla_\alpha\varphi\nabla_\beta\varphi - 2R\varphi\right] \quad \text{and} \quad \Box\varphi = -R$$

$$\Rightarrow T^{\rm anom}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm anom}}{\delta g^{\mu\nu}} = \frac{1}{24\pi}\left[-\varphi_{;\mu\nu} + g_{\mu\nu}\Box\varphi - \frac{1}{2}\varphi_{;\mu}\varphi_{;\nu} + \frac{1}{4}g_{\mu\nu}\varphi_{;\alpha}\varphi^{;\alpha}\right]$$

People expected that this action can be variated to **get renormalized stress tensor of conformal theory** directly . However**, the correspondence between the solutions the and vacuum states are unclear**.

Summary

- Scenario: Quantum field in curved spacetime
- Goal: Compute $\langle \mathrm{T}_{\mu\nu} \rangle$ of some system.
- Method: Anomaly-induced action
- (Our work) Improvement: Including the Boundary term in above action

Key idea: **boundary -> information of specific frames** -> **vacuum.**

$$S_{\text{anom}} = \frac{1}{96\pi} \int \mathrm{d}^2x \sqrt{-g} \left[g^{\alpha\beta} \nabla_\alpha \varphi \nabla_\beta \varphi - 2R\varphi \right] \quad \text{and} \quad \Box \varphi = -R \quad \text{and} \quad \boxed{n^\nu \nabla_\nu \varphi = 2K}$$

(i) $\langle 0 | \mathrm{T}_{\mu\nu} | 0 \rangle$ VEV for what vacuum state is clear now. (Including excitation states.)

(?) (?)

(ii) The extended version of this approach can be used in bounded (finite) region in any given manifold.

Quantum fields in curved spacetime – Standard approach

Consider the simplest case, a scalar field in curved spacetime with the Lagrangian :

$$\mathcal{L} = \int \sqrt{-g}\,(-\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} m^2 \phi^2 - \xi \mathrm{R}\, \phi^2)$$

- The equation of motion: $(\Box + m^2 + \xi R)\phi(\mathrm{x}) = 0$
- Solving a complete solution set: $\{\mathrm{u}_k(\mathrm{x}) \mid \mathrm{all\ k}\}$
- Quantize the scalar field: $\phi(\mathrm{x}) = \int dk[\hat{\mathrm{a}}_k u_k(\mathrm{x}) + \hat{\mathrm{a}}_k^{\dagger} u_k^{*}(\mathrm{x})]$
- The vacuum state is thus defined by: $\hat{\mathrm{a}}_i \left|0\right\rangle = 0$

Conformally coupled scalar field in n-dim:

$\xi(\mathrm{n}) = \dfrac{n-2}{4(\mathrm{n}-1)}$ and m = 0.

However, there is no unique way to choose a complete solution set, therefore the concept of **“vacuum” is ambiguous** in general.

Quantum fields in curved spacetime – Renormalization

Compute the (one-loop) effective action S_{eff} by path integral : $e^{-S_{eff}} = \int [D\phi] e^{-S[\phi]}$

Derive the expectation value of stress tensor from S_{eff}: $-\frac{2}{\sqrt{-g}} \frac{\delta S_{eff}[g_{\mu\nu}]}{\delta g^{\mu\nu}} = \frac{< out, 0|T_{\mu\nu}|0, in >}{< out, 0|0, in >} =< T_{\mu\nu} >$

By the DeWitt-Schwinger expansion, this effective Lagrangian in **n**-dim can be expanded as

$$L_{eff}(x) \approx \frac{1}{2(4\pi)^{n/2}} \sum_{j=0}^{\infty} a_j(x) m^{n-2j} \Gamma(j - \frac{n}{2})$$

The singularities of the **gamma function** are simple poles at zero and the negative integers

L_{div} 2-dim →, 4-dim →

$$\begin{cases} a_0(x) = 1, \\ a_1(x) = (\frac{1}{6} - \xi)R, \\ a_2(x) = \frac{1}{2}(\frac{1}{6} - \xi)^2 R^2 + \frac{1}{180}(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - R_{\mu\nu}R^{\mu\nu}) - \frac{1}{6}(\frac{1}{5} - \xi)\Box R \end{cases}$$

PS: The **UV divergent** behavior here is **independent of** choice of **quantum state**.

Method: Conformal anomaly & Anomaly-induced action…1

- In **conformal theory**, the stress tensor is **traceless** in general.

$$S[\bar{g}_{\mu\nu}(x)] = S[g_{\mu\nu}(x)] + \int d^n x \frac{\delta S[g_{\mu\nu}]}{\delta \bar{g}^{\alpha\beta}} \delta \bar{g}^{\alpha\beta}(x) = S[g_{\mu\nu}(x)] - \int d^n x \sqrt{-\bar{g}} T^{\lambda}_{\lambda}[\bar{g}_{\mu\nu}(x)] \delta\sigma$$

$$\Rightarrow \quad T^{\lambda}_{\lambda}[g_{\mu\nu}] = -\frac{1}{\sqrt{-g}} \frac{\delta S[\bar{g}_{\mu\nu}]}{\delta\sigma}|_{\sigma=0}$$

- However, after renormalization, due to the counter terms the conformal symmetry is broken and it results in an **nonzero trace** of stress tensor, which is called **conformal anomaly**.

- For example, in 2-dim: $< T^{\mu}_{\mu} >_{ren} = -\frac{R}{24\pi}$

Method: Conformal anomaly & Anomaly-induced action…2

- By using **conformal symmetry** and **the counter terms**, the anomaly-induced action can be built.

2-dim Anomaly-induced action (**Polyakov action** [3]):

$$S_{\rm anom}=\frac{1}{96\pi}\int {\rm d}^2x\sqrt{-g}\left[g^{\alpha\beta}\nabla_\alpha\varphi\nabla_\beta\varphi-2R\varphi\right]$$

where the auxiliary scalar field should satisfy the field equation: $\Box\varphi=-R$

$$\Rightarrow\ T^{\rm anom}_{\mu\nu}=-\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm anom}}{\delta g^{\mu\nu}}=\frac{1}{24\pi}\left[-\varphi_{;\mu\nu}+g_{\mu\nu}\Box\varphi-\frac{1}{2}\varphi_{;\mu}\varphi_{;\nu}+\frac{1}{4}g_{\mu\nu}\varphi_{;\alpha}\varphi^{;\alpha}\right]$$

$$\Rightarrow\ g^{\alpha\beta}T^{\rm anom}_{\alpha\beta}=\frac{1}{24\pi}\Box\varphi=-\frac{1}{24\pi}R$$

- People expected and checked that the stress tensor from above action can be used to solve renormalized stress tensor corresponding to various quantum states. [1][2]

[1] R. Balbinot, A. Fabbri and I. L. Shapiro, Nucl. Phys. B 559, 301 (1999).
[2] E. Mottola and R. Vaulin, Phys. Rev. D 74, 064004 (2006).
[3] A. M. Polyakov, Phys. Lett. B 103, 207 (1981).

Method: Conformal anomaly & Anomaly-induced action…3

Consider the Wess-Zumino action, which was first found by Polyakov:

$$\begin{aligned}
\Gamma_{WZ}[\bar{g},\sigma] &= S_{ren}[\bar{g}] - S_{ren}[g] \\
&= \left[S_{eff}[\bar{g}] - S_{eff}[g]\right] - \left[S_{ct}[\bar{g}] - S_{ct}[g]\right] \\
&= 0 - \left[S_{ct}[\bar{g}] - S_{ct}[g]\right] \\
&= \frac{1}{24\pi}\lim_{n\to 2}\left[\frac{\int d^2x\sqrt{-\bar{g}}\bar{R} - \int d^2x\sqrt{-g}R}{n-2}\right] \\
&= S_{anom}[\bar{g}] - S_{anom}[g]
\end{aligned}$$

Recall the divergent term in 2-dim come from a_1:

$$\boxed{S_{ct} = \frac{-1}{24\pi}\lim_{n\to 2}\int d^2x\sqrt{-g}\frac{R}{(n-2)}}$$

$$S_{\rm anom}[g] = \frac{1}{96\pi}\int \mathrm{d}^2x\sqrt{-g}\int \mathrm{d}^2x'\sqrt{-g'}RD_2(x,x')R'$$

where $\Box D_2(x,x') = -\dfrac{\delta(x-x')}{\sqrt{-g}}$

Method: Conformal anomaly & Anomaly-induced action…4

Take the Schwarzschild metric as an example: $ds^2 = -(1-\frac{2M}{r})dt^2 + (1-\frac{2M}{r})^{-1}dr^2$

$$\Box\varphi = -R \Rightarrow$$

(The most general solutions which result in stationary stress tensor)

$$\varphi = At - ln(1-\frac{2M}{r}) + B\frac{1}{2M}r^*$$

where $r^* \equiv r + 2M\ln(\frac{r}{2M}-1)$

(i) A = B = 0 $\Rightarrow$ **Boulware vacuum** solution: $T_{\mu\nu} = \frac{1}{24\pi}\begin{pmatrix} \frac{-4Mr+7M^2}{r^4} & 0 \\ 0 & -\frac{M^2}{r^2(r-2M)^2} \end{pmatrix} = \langle B|\hat{T}_{\mu\nu}|B\rangle$

(ii) A = 0, B = 1/(2M)

Hartle-Hawking vacuum solution:

$$\Rightarrow \begin{cases} T_{tt} = \frac{1}{24\pi}\left(\frac{7M^2}{r^4} - \frac{4M}{r^3} + \frac{1}{16M^2}\right) \\ T_{rr} = \frac{1}{24\pi}\left(1-\frac{2M}{r}\right)^{-2}\left(\frac{1}{16M^2} - \frac{M^2}{r^4}\right) \\ T_{tr} = T_{rt} = 0 \end{cases} = \langle H|\hat{T}_{\mu\nu}|H\rangle$$

(iii) A = -B = -1/(4M)

Unruh vacuum solution:

$$\Rightarrow \begin{cases} T_{tt} = \frac{1}{24\pi}\left(\frac{1}{32M^2} + \frac{7M^2}{r^4} - \frac{4M}{r^3}\right) \\ T_{rr} = \frac{1}{24\pi}\left(1-\frac{2M}{r}\right)^{-2}\left(\frac{-M^2}{r^4} + \frac{1}{32M^2}\right) \\ T_{tr} = T_{rt} = \frac{1}{24\pi}\left[\frac{-1}{32M^2}\left(1-\frac{2M}{r}\right)^{-1}\right] \end{cases} = \langle U|\hat{T}_{\mu\nu}|U\rangle$$

Improvement: Boundary effect of anomaly-induced action…1

- Introduce the boundary part of divergent terms.
- Recalculate the anomaly-induced action with boundary effect.

$$S_{anom}[g] = \frac{1}{96\pi}\{\int_{\mathcal{M}} d^2x\sqrt{-g}(-\varphi\Box\varphi - 2\varphi R) + \int_{\Sigma} d^1x\sqrt{\gamma}(\varphi n_\mu \nabla^\mu\varphi - 4\varphi K)\}$$

where the auxiliary scalar field should satisfy the original equation and an additional boundary constraint:

$$\begin{cases} \Box\varphi = -R \\ \boxed{n^\nu\nabla_\nu\varphi = 2K}, \qquad x \in \Sigma \end{cases}$$

The boundary effect of anomaly-induced action
Che-Min Shen, Keisuke Izumi, Pisin Chen, **arXiv:1505.00959**

Improvement: Boundary effect of anomaly-induced action…2

Introducing the boundary part of divergent term: [4][5]

$$S_{ct} = \frac{-1}{24\pi} \lim_{n\to 2} \int d^2x\sqrt{-g}\frac{R}{(n-2)} \longrightarrow \frac{-1}{24\pi}\frac{\int_{\mathcal{M}} d^2x\sqrt{-g}R + 2\int_{\Sigma} d^1x\sqrt{-\gamma}K}{n-2}$$

$$\begin{aligned}
\Gamma_{WZ}[\bar{g},\sigma] &= S_{ren}[\bar{g}] - S_{ren}[g] \\
&= \left[S_{eff}[\bar{g}] - S_{eff}[g]\right] - \left[S_{ct}[\bar{g}] - S_{ct}[g]\right] \\
&= 0 - \left[S_{ct}[\bar{g}] - S_{ct}[g]\right] \\
&= \frac{1}{24\pi}\lim_{n\to 2}\left[\frac{(\int d^2x\sqrt{-\bar{g}}\bar{R} + 2\int d^1x\sqrt{-\bar{\gamma}}\bar{K}) - (\int d^2x\sqrt{-g}R + 2\int d^1x\sqrt{-\gamma}K)}{n-2}\right] \\
&= S_{anom}[\bar{g}] - S_{anom}[g]
\end{aligned}$$

[4] T. P. Branson and P. B. Gilkey, Commun. Part. Di. Eq. 15, 245 (1990).
[5] I. G. Avramidi, Phys. Atom. Nucl. 56, 138 (1993) [Yad. Fiz. 56N1, 243 (1993)].

Schwarzschild metric and its various coordinates

$$
\begin{aligned}
ds^2 &= -(1-\frac{2M}{r})dt^2 + (1-\frac{2M}{r})^{-1}dr^2 = (1-\frac{2M}{r})(-dt^2 + dr^{*2}) = -(1-\frac{2M}{r})dudv \\
&= -\frac{32M^3}{r}e^{-\frac{r}{2M}}dU\,dV = -\frac{32M^3}{r}e^{-\frac{r}{2M}}(-dT_H{}^2 + dR_H{}^2) \\
&= -\frac{8M^2}{r}(\frac{r}{2M}-1)^{\frac{1}{2}}e^{\frac{t-r}{2M}}dU\,dv = \frac{8M}{r}(\frac{r}{2M}-1)^{\frac{1}{2}}e^{\frac{t-r}{2M}}(-dT_U{}^2 + dR_U{}^2),
\end{aligned}
$$

where $r^* := r + 2M\ln(\frac{r}{2M}-1), \qquad u := t - r^*, \qquad v := t + r^*,$

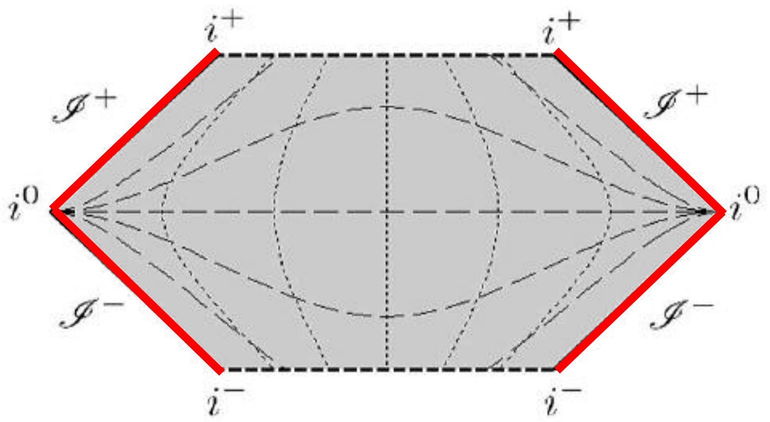
$$U := -e^{\frac{-u}{4M}}, \qquad V := e^{\frac{v}{4M}}, \qquad T_H := \frac{1}{2}(V+U), \qquad R_H := \frac{1}{2}(V-U),$$

$$T_U := \frac{1}{2}(v+U), \qquad R_U := \frac{1}{2}(v-U).$$

Improvement: Boundary effect of anomaly-induced action…3

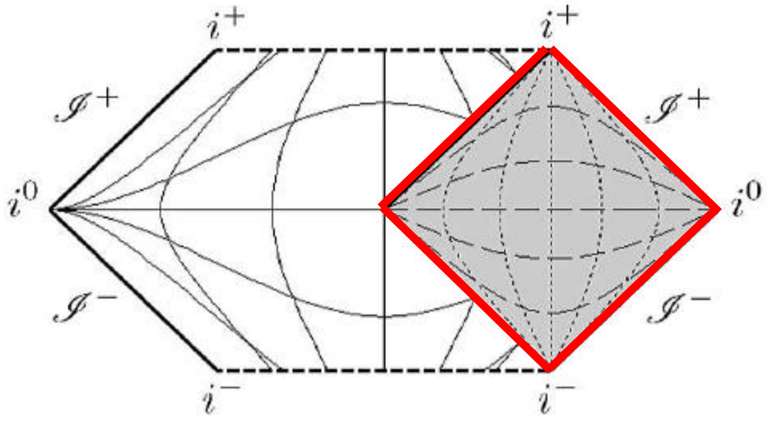
- The vacuum states depend on observer (frame) choosing.
- Key idea: introduce **boundaries corresponding to specific frames** can naturally give us the corresponding vacuum solutions.
- For example, we consider Schwarzschild black hole metric again with the appropriate choices of boundaries corresponding to different vacuum states.

Hartle-Hawking vacuum state: $|H\rangle$



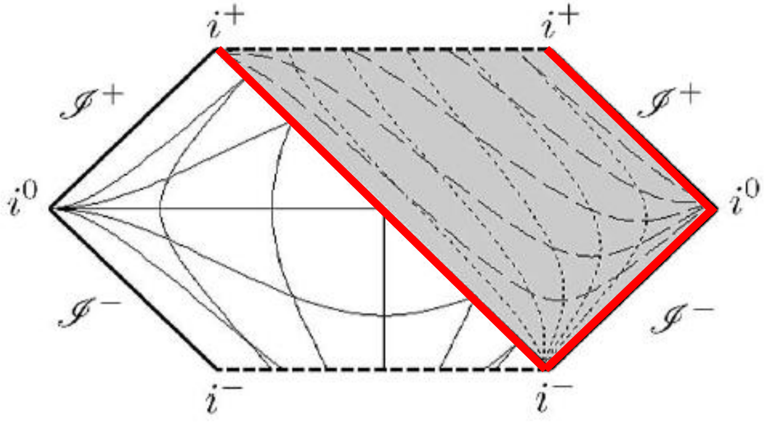
The dashed lines are (T_H,R_H) coordinate.

Boulware vacuum state: $|B\rangle$



The dashed lines are (t,r) coordinate.

Unruh vacuum state: $|U\rangle$



The dashed lines are (T_U,R_U) coordinate.

Improvement: Boundary effect of anomaly-induced action…4

At first, consider "Boulware" boundary:

$$\begin{cases} \Box\varphi = -R \\ \boxed{n^\nu \nabla_\nu \varphi = 2K} \end{cases} \Rightarrow \quad \varphi = \ln(1-\frac{2M}{r}) + A_2 t + \int d\omega\, c(\omega)\cos[\omega r^*]e^{i\omega t}$$

The most general solutions which result in "stationary" stress tensor is

$$\Rightarrow \quad \varphi = \ln(1-\frac{2M}{r}) + A_2 t$$

$$\Rightarrow \quad T_{\mu\nu} = \frac{1}{24\pi}\begin{pmatrix} \frac{-4Mr+7M^2}{r^4} + \frac{A_2^2}{4} & 0 \\ 0 & -\frac{M^2}{r^2(r-2M)^2} + \frac{A_2^2 r^2}{4(r-2M)^2} \end{pmatrix}$$

When A_2=0, the result is indeed the Boulware vacuum solution.
A_2 characterizes the thermal excitation based on the Boulware vacuum.

Improvement: Boundary effect of anomaly-induced action…5

Similarly, by considering the "Hartle-Hawking" and "Unruh" boundaries, we will get

The Hartle-Hawking vacuum solution and the corresponding thermal excitation states (which is parametrized by A_H):

$$\begin{cases} T_{tt} = \frac{1}{24\pi}\left[\left(\frac{7M^2}{r^4}-\frac{4M}{r^3}+\frac{1}{16M^2}\right)+\frac{{A_H}^2}{64M^2}\left({R_H}^2+{T_H}^2\right)\right] \\ T_{rr} = \frac{1}{24\pi}\left[\left(1-\frac{2M}{r}\right)^{-2}\left(\frac{1}{16M^2}-\frac{M^2}{r^4}\right)+\frac{{A_H}^2}{64M^2}\left(1-\frac{2M}{r}\right)^{-2}\left({R_H}^2+{T_H}^2\right)\right] \\ T_{tr} = T_{rt} = \frac{1}{24\pi}\left[\frac{{A_H}^2}{32M^2}\left(1-\frac{2M}{r}\right)^{-1}\left(T_H R_H\right)\right] \end{cases}$$

The Unruh vacuum solution and the corresponding thermal excitation states (which is parametrized by A_U):

$$\begin{cases} T_{tt} = \frac{1}{24\pi}\left[\left(\frac{1}{32M^2}+\frac{7M^2}{r^4}-\frac{4M}{r^3}\right)+\frac{{A_U}^2}{8}\left(1+\frac{r-2M}{32M^3}\mathrm{e}^{\frac{r-t}{2M}}\right)\right] \\ T_{rr} = \frac{1}{24\pi}\left[\left(1-\frac{2M}{r}\right)^{-2}\left(\frac{-M^2}{r^4}+\frac{1}{32M^2}\right)+\frac{{A_U}^2}{8}\left(\left(1-\frac{2M}{r}\right)^{-2}+\left(1-\frac{2M}{r}\right)^{-1}\frac{r}{32M^3}\mathrm{e}^{\frac{r-t}{2M}}\right)\right] \\ T_{tr} = T_{rt} = \frac{1}{24\pi}\left[\frac{-1}{32M^2}\left(1-\frac{2M}{r}\right)^{-1}+\frac{{A_U}^2}{8}\left(\left(1-\frac{2M}{r}\right)^{-1}-\frac{r}{32M^3}\mathrm{e}^{\frac{r-t}{2M}}\right)\right] \end{cases}$$

Note: When A2 = 1/(2M), the result in previous page is the same as that of the Hartle-Hawking vacuum state. This fact is consistent with the usual understanding about Hawking temperature.

Summary

- Anomaly-induced action is a useful approach to solve $\left\langle \mathrm{T}_{\mu\nu} \right\rangle$
- However, the **correspondence** between its solutions and the vacuum states which people are interested in **is missing**.
- Take **boundary** effect into consideration **introduce** some **information related to frame** (and thus the corresponding vaccum).
- The extended version of anomaly-induced action now really become an independent tool to solve $\left\langle 0 \middle| \mathrm{T}_{\mu\nu} \middle| 0 \right\rangle$
  (The original one is unable to do this.)

Thank you