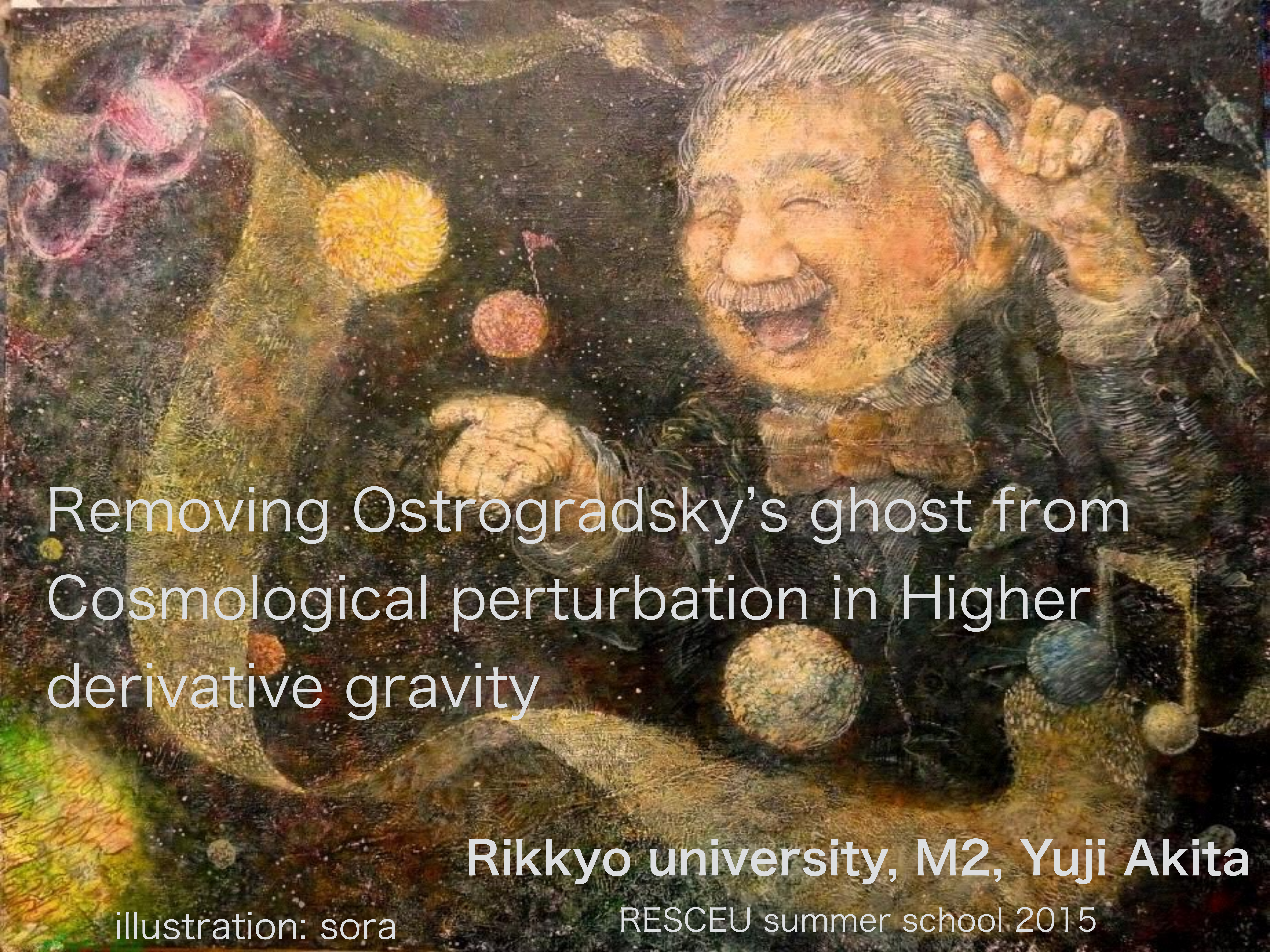


Removing Ostrogradsky's ghost from Cosmological perturbation in Higher derivative gravity

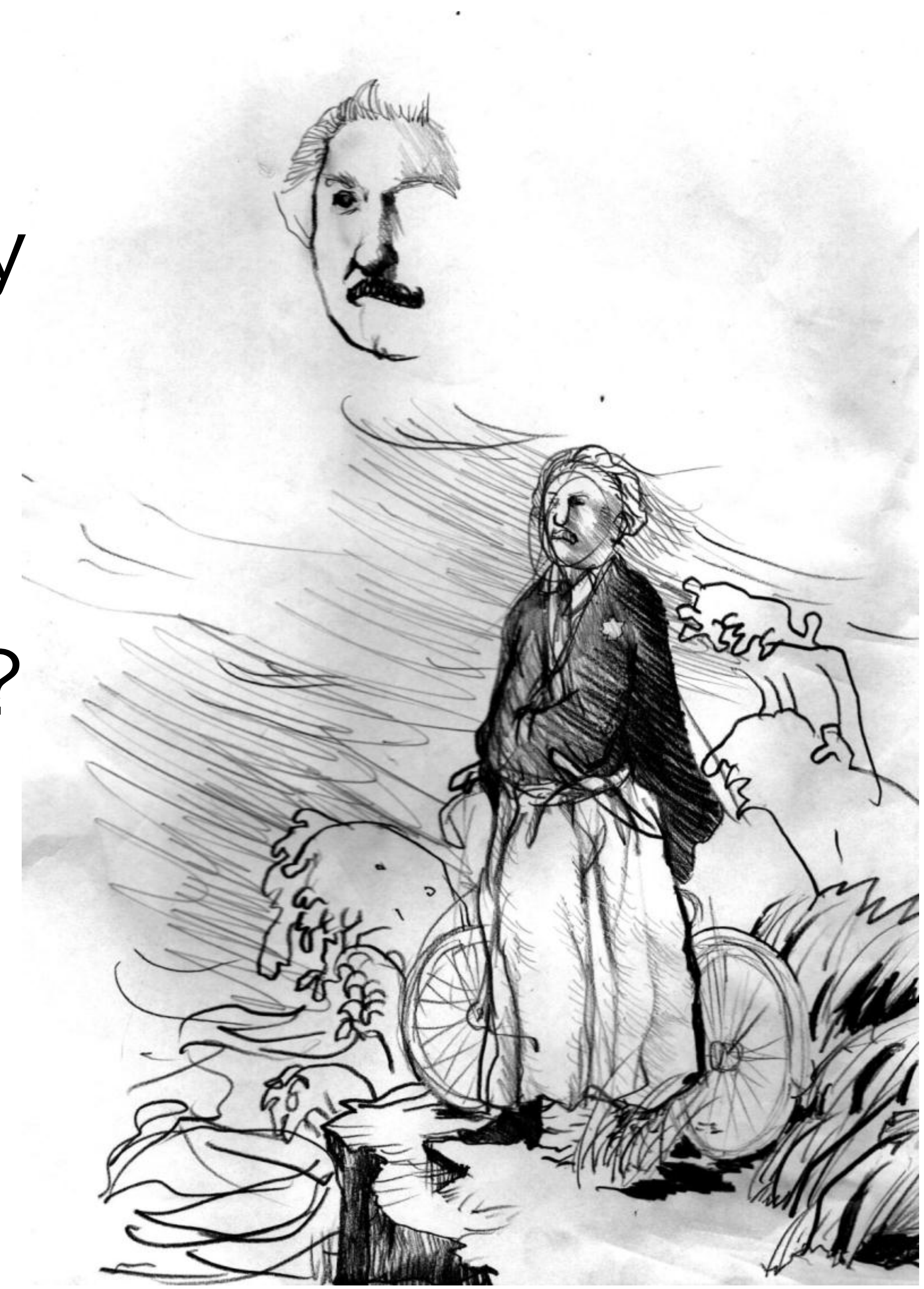
Rikkyo university, M2, Yuji Akita

RESCEU summer school 2015

illustration: sora

Overview

- Introduction
  - Higher derivative theory
  - Ostrogradsky's instability
- Instability in gravity
- Stabilization: How to remove?
- Cosmological implication



Introduction

🍁Equation of motions are 2nd order (usual case)

- Classical mechanics … Newton's e.o.m
- General Relativity … Einstein eqs
- Horndeski theory (and its beyond)

🍁**What will happen**
when we consider "Higher" derivatives?

Introduction – Higher derivative theory

We will face to "Ostrogradsky's instability"

The theory becomes unstable ! (in some case)

Simple model:

$$S = \int \mathrm{d}t \left[\frac{1}{2}\ddot{q}^2 - V(q) \right] \Rightarrow H = P_1 Q_2 + \frac{P_1^2}{2} + V(Q_1)$$

Canonical variables: $Q_1 := q$
$Q_2 := \dot{q}$

Hamiltonian can take arbitrary negative value.

Introduction – Higher derivative theory

We will face to "Ostrogradsky's instability"

The theory becomes unstable ! (in some case)

Simple model:

$$S = \int \mathrm{d}t \left[\frac{1}{2} \ddot{q}^2 - V(q) \right] \quad \Rightarrow \quad H = P_1 Q_2 + \frac{P_1^2}{2} + V(Q_1)$$

Canonical variables: $Q_1 := q$
$Q_2 := \dot{q}$

Hamiltonian can take arbitrary negative value.

Quantum gravity suggest Higher derivative gravity

Gravity with instability

GR : effective theory at **low** energy

Quantum Gravity : effective at **high** energy

⇒ Suggestion: Curvature invariants

famous model: $\mathcal{L} = R + \alpha R^2$ (Starobinsky model)

【match to Planck observation】

Other curvature terms? $R_{\mu\nu}^2,\ R_{\mu\nu\rho\sigma}^2,\ C_{\mu\nu\rho\sigma}^2$

⇒ Ostrogradsky's instability

Gravity with instability

Appearance of instability (around Minkowski)

$$S=\frac{M_{\rm Pl}^2}{2}\int \mathrm{d}^4x\sqrt{-g}\left(R-2\Lambda+\alpha R^2+\beta R_{\mu\nu}R^{\mu\nu}\right)$$

metric: $\mathrm{d}s^2=-\mathrm{d}t^2+(\delta_{ij}+h_{ij}+\frac{1}{2}h_{ik}h_{kj})\mathrm{d}x^i\mathrm{d}x^j$

$$S=\frac{M_{\rm Pl}^2}{8}\int \mathrm{d}\eta\mathrm{d}^3x\left\{(h'_{ij})^2+h^{ij}\partial^2h_{ij}+\beta\left[(h''_{ij})^2+2h'_{ij}\partial^2h'_{ij}+(\partial^2h_{ij})^2\right]\right\}$$

Hamiltonian:

$$H=\frac{M_{\rm Pl}^2}{2}\int \mathrm{d}^3x\left[\frac{1}{4\beta}p^{ij}p_{ij}+\pi^{ij}q_{ij}-2\beta q_{ij}\partial^2q_{ij}\right.$$
$$\left.-q_{ij}q^{ij}-\beta\partial^2h_{ij}\partial^2h^{ij}-h_{ij}\partial^2h_{ij}\right]\ .$$

【Linear term→unstable】

Stabilization- add constraint

it can be stabilized! [Chen et. al. 2013]

Addition of suitable constraint

Lagrangian:

$$\mathcal{L} \supset (h''_{ij})^2 \quad \Rightarrow \quad \mathcal{L} \supset (h''_{ij} - \lambda_{ij})^2, \ \lambda_{ij}\partial^2 h_{ij}$$

Time derivative ⟷ Spatial derivative

【extra d.o.f (ghost)】 【Good for renormalizability】

Hamiltonian:

$$H \supset \pi_{ij} q_{ij} \quad \Rightarrow$$ **Positive!** due to $\pi_{ij} \propto q_{ij}$

Stabilization- add constraint

it can be stabilized! [Chen et. al. 2013]

Reduced Hamiltonian

$$\mathcal{H}_R = \frac{1}{4}\pi^{ij}(1 - 2\beta\partial^2)\pi_{ij} + h^{ij}(-\partial^2 + 3\beta\partial^2\partial^2)h_{ij}$$

【all terms are Quadratic】

Hamiltonian:

$H \supset \pi_{ij} q_{ij}$ ➡ **Positive!** due to $\pi_{ij} \propto q_{ij}$

A brief summary

Higher derivative term → Instability

ex) Gravitational theory: $R_{\mu\nu}^2, \; R_{\mu\nu\rho\sigma}^2, \; C_{\mu\nu\rho\sigma}^2$

<u>It can be stabilized by Constraints</u>

setting: $S = \frac{M_{\rm Pl}^2}{2} \int \mathrm{d}^4 x \sqrt{-g} \left(R - 2\Lambda + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} \right)$

Background is Minkowski.

Problems:

→ ・Our universe is expanding
・Particular model is used

Improve at these points!

Generalization:

Let's change ! Y.Akita and T. Kobayashi, [arXiv:1507.00812[gr-qc]]

The Background is Friedman sp.

$$\mathrm{d}s^2 = a^2(\eta)\left[-\mathrm{d}\eta^2 + \left(\delta_{ij} + h_{ij} + \frac{1}{2}h_{ik}h_{kj}\right)\right]\mathrm{d}x^i\mathrm{d}x^j$$

Model is Generalized.

$$S = \int \mathrm{d}^4x\sqrt{-g}\; f(R, R_{\mu\nu}R^{\mu\nu}, C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma})$$

f: Arbitrary function of these curvatures.

⇒ **Consider the stabilization.**

Generalization: Check the instability

Start with: $S = \int d^4x\sqrt{-g}\; f(R, R_{\mu\nu}R^{\mu\nu}, C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma})$

$$r_1 := R, \quad r_2 := R_{\mu\nu}R^{\mu\nu}, \quad r_3 := C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$$

Quadratic action:

$$S = \int d^4x \left\{ \frac{a^2}{8}\left(A h_{ij}'^2 + C h_{ij}\partial^2 h_{ij}\right) + \frac{\beta}{8}\left[h_{ij}''^2 + 2h_{ij}'\partial^2 h_{ij}' + \left(\partial^2 h_{ij}\right)^2\right]\right\}$$

Definitions $\quad f_i := \partial f / \partial r_i$

$$\beta := 2f_2 + 4f_3,$$

$$A := 2f_1 + \frac{8f_2}{a^2}\left(\mathcal{H}^2 + \mathcal{H}'\right) - 4f_2'\frac{\mathcal{H}}{a^2},$$

$$C := 2f_1 + \frac{8f_2}{a^2}\left(\mathcal{H}^2 + \mathcal{H}'\right) + \frac{4f_2'}{a^2}\mathcal{H} - \frac{2}{a^2}\left(f_2'' - 2f_3''\right)$$

Time dependent Coefficients

Generalization: Check the instability

Start with: $S=\int \mathrm{d}^4x\sqrt{-g}\ f(R, R_{\mu\nu}R^{\mu\nu}, C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma})$

$$r_1 := R, \quad r_2 := R_{\mu\nu}R^{\mu\nu}, \quad r_3 := C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$$

Quadratic action:

$$S=\int \mathrm{d}^4x\left\{\frac{a^2}{8}\left(Ah_{ij}'^2+Ch_{ij}\partial^2h_{ij}\right)+\frac{\beta}{8}\left[h_{ij}''^2+2h_{ij}'\partial^2h_{ij}'+\left(\partial^2h_{ij}\right)^2\right]\right\}$$

Hamiltonian:

$$H=\int \mathrm{d}^3x\left\{\pi^{ij}q_{ij}+\frac{2}{\beta}p_{ij}^2-\frac{1}{8}h_{ij}\left(a^2C\partial^2+\beta\partial^2\partial^2\right)h_{ij}-\frac{1}{8}q_{ij}\left(a^2A+2\beta\partial^2\right)q_{ij}\right\}$$

Generalization: Add constraints

$$S = \int \mathrm{d}^4x \left\{ \frac{a^2}{8} \left(A h_{ij}'^2 + C h_{ij} \partial^2 h_{ij} \right) + \frac{\beta}{8} \left[h_{ij}''^2 + 2 h_{ij}' \partial^2 h_{ij}' + \left(\partial^2 h_{ij} \right)^2 \right] \right\}$$

Constraints: same form

$$\mathcal{L} \supset (h_{ij}'')^2 \quad \Rightarrow \quad \mathcal{L} \supset (h_{ij}'' - \lambda_{ij})^2, \ \lambda_{ij} \partial^2 h_{ij}$$

Time derivative ⟷ Spatial derivative

【extra d.o.f (ghost)】 【Good for renormalizability】

Constructing Hamiltonian:

$$h_{ij} := h_{ij} \longleftrightarrow \pi^{ij} := \partial \mathcal{L} / \partial h_{ij}'$$

$$q_{ij} := h_{ij}' \longleftrightarrow p^{ij} := \partial \mathcal{L} / \partial h_{ij}''$$

$$\lambda_{ij} := \lambda_{ij} \longleftrightarrow \pi_{\lambda}^{ij} = 0$$

Generalization: Add constraints

Hamiltonian with constraint

$$H = \int \mathrm{d}^3x \left[\pi^{ij} q_{ij} + \frac{2}{\beta} p_{ij}^2 + \lambda_{ij} \left(p_{ij} - \frac{\beta}{2} \partial^2 h_{ij} \right) \right.$$
$$\left. - \frac{1}{8} h_{ij} \left(a^2 C \partial^2 + \beta \partial^2 \partial^2 \right) h_{ij} - \frac{1}{8} q_{ij} \left(a^2 A + 2\beta \partial^2 \right) q_{ij} \right]$$

Primary constraint $\phi_1 : \ \pi_\lambda^{ij} = 0$

Secondary constraints $\cdots$ **Consistency of primary constraint**

$\phi_2 : \ \dot{\phi}_1 = 0 \Leftrightarrow \{\phi_1, H\}_{\mathrm{PB}} = 0$ and so on.

→ several constraints including $\pi_{ij} \propto q_{ij}$

Generalization: Reduced Hamiltonian

Results of Stabilization

Reduced Hamiltonian - tensor perturbation

$$H_R = \int d^3x \left\{ \frac{2}{a^4 A^2} \pi_{ij} \left(a^2 A - 2\beta\partial^2 \right) \pi_{ij} + \frac{1}{8} h_{ij} \left(-a^2 C \partial^2 + 3\beta\partial^2\partial^2 \right) h_{ij} \right\}$$

Stable for $A,\ C,\ \beta > 0$

Spectrum of G.W.

model independent form

Suppression factor

$$\mathcal{P}_h = \frac{1}{A + s\beta H^2} \frac{2H^2}{\pi^2} \quad , \quad s \simeq \text{const.} = \mathcal{O}(1)$$

Generalization: Reduced Hamiltonian

Results of Stabilization

Phase space dimensionality:

Original: 8 d.o.f $\cdots h_{ij}, q_{ij}$ and their conjugate

add λ_{ij} and conjugate $\rightarrow$ +4 d.o.f ▶12 d.o.f

+8 second class constraint

➡ 4 d.o.f : same with GR 【ghosts are removed】

Generalization: Vector sector

$$\mathrm{d}s^2 = a^2\left[-\left(1 - B_i B^i\right)\mathrm{d}\eta^2 + 2B_i \mathrm{d}x^i \mathrm{d}\eta + \delta_{ij}\mathrm{d}x^i \mathrm{d}x^j\right]$$

Vector mode is dynamical:

$$S = \int \mathrm{d}^4x \left[\frac{\beta}{4}\left(\hat{v}_{ij}'^2 + \hat{v}_{ij}\partial^2\hat{v}_{ij}\right) + \frac{a^2 A}{4}\hat{v}_{ij}^2\right]$$ with $\hat{v}_{ij} := \partial_i B_j$

Stable for $\beta > 0$ and $A < 0$

【Conflict with stable condition of tensor sector】

→ **Remove the vector mode from theory**

Generalization: Vector sector

Constraint

$$S = \int \mathrm{d}^4x \left\{ \frac{\beta}{4} \left[\left(\hat{v}'_{ij} - \hat{\lambda}_{ij} \right)^2 + \hat{v}_{ij} \partial^2 \hat{v}_{ij} \right] + \frac{a^2 A}{4} \hat{v}_{ij}^2 \right\}$$

$$\Rightarrow \quad H = \int \mathrm{d}^3x \left(\frac{\hat{\pi}_{ij}^2}{\beta} + \hat{\pi}_{ij} \hat{\lambda}_{ij} - \frac{\beta}{4} \hat{v}_{ij} \partial^2 \hat{v}_{ij} - \frac{a^2 A}{4} \hat{v}_{ij}^2 \right)$$

primary constraint, Secondary constraints,……

⇒ **reduced Hamiltonian Vanishes!**

Generalization: Scalar sector

Scalar perturbation can be stabilized.
(expression is complicated)

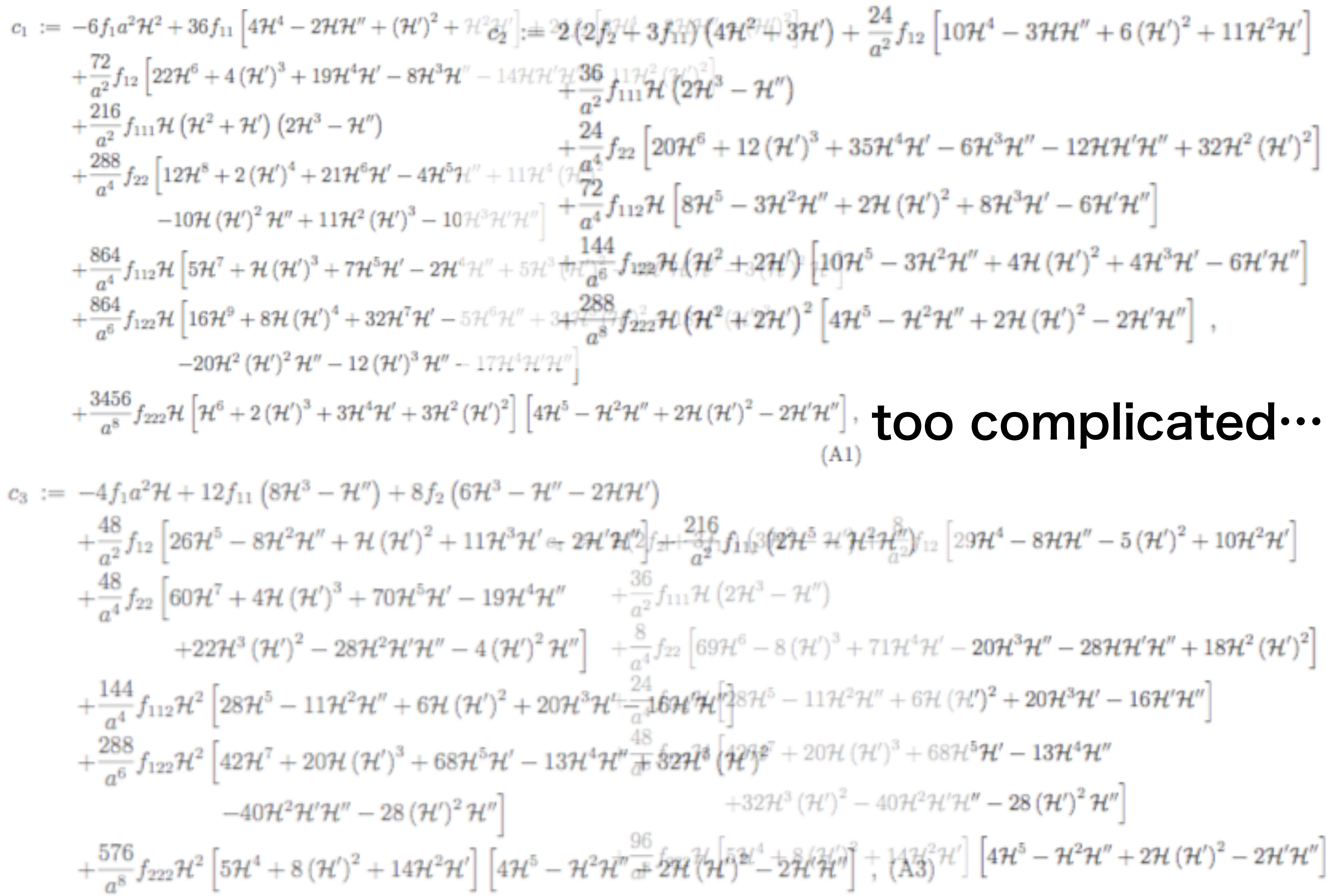
Flat gauge metric:

$$\mathrm{d}s^2 = a^2\left[-(1+2\Phi)\mathrm{d}\eta^2 + 2\partial_i B\mathrm{d}\eta\mathrm{d}x^i + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right]$$

Quadratic action:

$$S = \int \mathrm{d}^4x\Big[b_0\left(\Phi'\right)^2 + b_1\left(\partial^2\Phi + \mathcal{B}'\right)^2 + b_2\Phi'\mathcal{B}' + b_3\Phi\mathcal{B}' + c_1\Phi^2 + c_2\Phi\partial^2\Phi + c_3\Phi\mathcal{B} + c_4\mathcal{B}^2 + c_5\mathcal{B}\partial^2\Phi\Big]$$

$$\mathcal{B} := \partial_i^2 B$$


too complicated…

Coefficients contain…

- Hubble param.
- model dependent param.

Generalization: Scalar sector

Quadratic action:

$$S = \int d^4x \Big[b_0 (\Phi')^2 + b_1 \left(\partial^2\Phi + \mathcal{B}'\right)^2 + b_2 \Phi' \mathcal{B}' + b_3 \Phi \mathcal{B}' + c_1 \Phi^2 + c_2 \Phi \partial^2 \Phi + c_3 \Phi \mathcal{B} + c_4 \mathcal{B}^2 + c_5 \mathcal{B} \partial^2 \Phi \Big]$$

Hamiltonian:

$$H = \int d^3x \left[\frac{3}{4\beta} \left(\pi_{\mathcal{B}} - \frac{1}{3}\frac{\pi_\Phi}{\mathcal{H}} - b_3\Phi - 2b_1 \partial^2\Phi \right)^2 + \frac{\pi_\Phi^2}{24\mathcal{H}^2\mathcal{I}} - \frac{2\mathcal{I}+\beta}{3}(\partial^2\Phi)^2 - c_1\Phi^2 - c_2\Phi\partial^2\Phi - c_3\Phi\mathcal{B} - c_4\mathcal{B}^2 - c_5\mathcal{B}\partial^2\Phi \right].$$

To see instability:

Need to do canonical transformation

Generalization: Scalar sector

$$\widetilde{H} = \int d^3x \left\{ \underline{\frac{3}{4\beta}\left(\widetilde{\pi}_{\mathcal{B}} - \widetilde{\Phi} b_3 - 2b_1\partial^2\widetilde{\Phi}\right)^2 + \frac{\widetilde{\pi}_\Phi^2}{24\mathcal{H}^2\mathcal{I}}} \right.$$
$$\left. - \frac{2\mathcal{I}+\beta}{3}\left[\partial^2\left(\widetilde{\Phi} - \frac{\widetilde{\mathcal{B}}}{3\mathcal{H}}\right)\right]^2 + \cdots \right\}$$

Stability conditions:

- Kinematic terms: $\beta > 0 \text{ and } \mathcal{I} > 0$

➡ **The last term becomes negative**

(unstable at high momentum scale.)

Generalization: Scalar sector

Constraint:

$$S=\int \mathrm{d}^4x\Big[b_0(\Phi')^2+b_1\left(\partial^2\Phi+\mathcal{B}'-\lambda\right)^2+b_2\Phi'(\mathcal{B}'-\lambda)+b_3\Phi(\mathcal{B}'-\lambda)$$
$$+c_1\Phi^2+c_2\Phi\partial^2\Phi+c_3\Phi\mathcal{B}+c_4\mathcal{B}^2+c_5\mathcal{B}\partial^2\Phi\Big].$$

Just modify $\mathcal{B}'\longrightarrow\mathcal{B}'-\lambda$

Reduced Hamiltonian:

$$\widetilde{H}_R=\int \mathrm{d}^3x\left[\frac{2\mathcal{I}+\beta}{24\mathcal{H}^2\mathcal{I}\beta}\widetilde{\pi}_\Phi^2+\frac{c_5^2}{4c_4}\left(\partial^2\widetilde{\Phi}\right)^2+d_1\widetilde{\Phi}^2-d_2\widetilde{\Phi}\partial^2\widetilde{\Phi}\right]$$

Summary

- Higher derivative term → **Ostrogradsky's ghost**

Gravitational theory: $R_{\mu\nu}^2,\ R_{\mu\nu\rho\sigma}^2,\ C_{\mu\nu\rho\sigma}^2$ cf. Q.G.

⇒ Instability can be **removed** by the addition of **suitable constraint**

What we did:

Stabilize

the general model $f(R, R_{\mu\nu}^2, C_{\mu\nu\rho\sigma}^2)$

on cosmological background