

Removing Ostrogradsky's ghost from Cosmological perturbation in Higher derivative gravity

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illustration: sora

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Overview

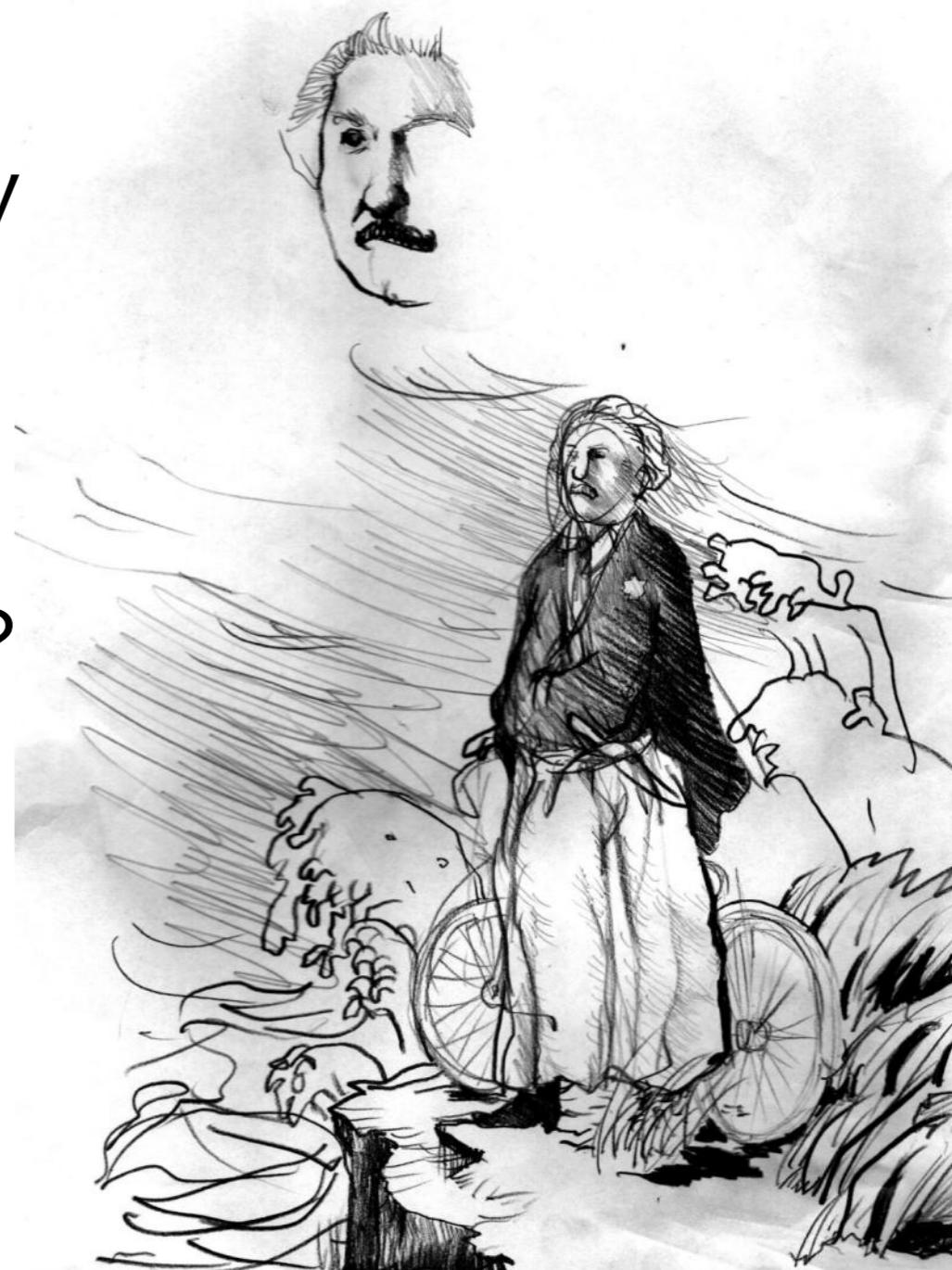
🍁 Introduction

- Higher derivative theory
- Ostrogradsky's instability

🍁 Instability in gravity

🍁 Stabilization: How to remove?

🍁 Cosmological implication



Introduction

 Equation of motions are 2nd order (usual case)

- Classical mechanics … Newton's e.o.m
- General Relativity … Einstein eqs
- Horndeski theory (and its beyond)

 What will happen
when we consider “Higher” derivatives?

Introduction – Higher derivative theory



We will face to “Ostrogradsky’s instability”

The theory becomes unstable ! (in some case)

Simple model:

$$S = \int dt \left[\frac{1}{2} \ddot{q}^2 - V(q) \right] \rightarrow H = P_1 Q_2 + \frac{P_1^2}{2} + V(Q_1)$$

Canonical variables: $Q_1 := q$
 $Q_2 := \dot{q}$

Hamiltonian can take
arbitrary negative value.

Introduction – Higher derivative theory



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Hamiltonian can take
arbitrary negative value.

Quantum gravity suggest Higher derivative gravity

Gravity with instability

GR : effective theory at **low** energy

Quantum Gravity : effective at **high** energy

→ Suggestion: Curvature invariants

famous model: $\mathcal{L} = R + \alpha R^2$ (Starobinsky model)
[match to Planck observation]

Other curvature terms? $R_{\mu\nu}^2, R_{\mu\nu\rho\sigma}^2, C_{\mu\nu\rho\sigma}^2$

→ Ostrogradsky's instability

Gravity with instability

Appearance of instability (around Minkowski)

$$S = \frac{M_{\text{Pl}}^2}{2} \int d^4x \sqrt{-g} (R - 2\Lambda + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu})$$



$$\text{metric: } ds^2 = -dt^2 + (\delta_{ij} + h_{ij} + \frac{1}{2}h_{ik}h_{kj})dx^i dx^j$$

$$S = \frac{M_{\text{Pl}}^2}{8} \int d\eta d^3x \left\{ (h'_{ij})^2 + h^{ij} \partial^2 h_{ij} + \beta \left[(h''_{ij})^2 + 2h'_{ij} \partial^2 h'_{ij} + (\partial^2 h_{ij})^2 \right] \right\}$$

Hamiltonian:

$$H = \frac{M_{\text{Pl}}^2}{2} \int d^3x \left[\frac{1}{4\beta} p^{ij} p_{ij} + \pi^{ij} q_{ij} - 2\beta q_{ij} \partial^2 q_{ij} - q_{ij} q^{ij} - \beta \partial^2 h_{ij} \partial^2 h^{ij} - h_{ij} \partial^2 h_{ij} \right].$$

【Linear term → unstable】

Stabilization- add constraint

it can be stabilized! [Chen et. al. 2013]



Addition of suitable constraint

Lagrangian:

$$\mathcal{L} \supset (h''_{ij})^2 \rightarrow \mathcal{L} \supset (h''_{ij} - \lambda_{ij})^2, \lambda_{ij} \partial^2 h_{ij}$$

Time derivative \longleftrightarrow Spatial derivative

[extra d.o.f (ghost)]

[Good for renormalizability]

Hamiltonian:

$$H \supset \pi_{ij} q_{ij}$$

Positive! due to $\pi_{ij} \propto q_{ij}$

Stabilization- add constraint

it can be stabilized! [Chen et. al. 2013]

Reduced Hamiltonian

$$\begin{aligned}\mathcal{H}_R = & \frac{1}{4} \pi^{ij} (1 - 2\beta \partial^2) \pi_{ij} \\ & + h^{ij} (-\partial^2 + 3\beta \partial^2 \partial^2) h_{ij}\end{aligned}$$

【all terms are Quadratic】

Hamiltonian:

$$H \supset \pi_{ij} q_{ij}$$



Positive! due to $\pi_{ij} \propto q_{ij}$

A brief summary

Higher derivative term  Instability

ex) Gravitational theory: $R_{\mu\nu}^2, R_{\mu\nu\rho\sigma}^2, C_{\mu\nu\rho\sigma}^2$

It can be stabilized by Constraints

setting: $S = \frac{M_{\text{Pl}}^2}{2} \int d^4x \sqrt{-g} (R - 2\Lambda + \alpha R^2 + \beta \textcolor{red}{R}_{\mu\nu} R^{\mu\nu})$

Background is Minkowski.

Problems:

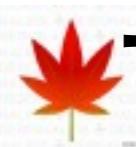


- Our universe is expanding
- Particular model is used

Improve at
these points!

Generalization:

Let's change ! Y.Akita and T. Kobayashi, [arXiv:1507.00812[gr-qc]]



The Background is Friedman sp.

$$ds^2 = a^2(\eta) \left[-d\eta^2 + (\delta_{ij} + h_{ij} + \frac{1}{2}h_{ik}h_{kj}) \right] dx^i dx^j$$



Model is Generalized.

$$S = \int d^4x \sqrt{-g} f(R, R_{\mu\nu}R^{\mu\nu}, C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma})$$

f: Arbitrary function of these curvatures.



Consider the stabilization.

Generalization: Check the instability

Start with: $S = \int d^4x \sqrt{-g} f(R, R_{\mu\nu}R^{\mu\nu}, C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma})$

$$r_1 := R, \quad r_2 := R_{\mu\nu}R^{\mu\nu}, \quad r_3 := C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$$

Quadratic action:

$$S = \int d^4x \left\{ \frac{a^2}{8} (Ah'_{ij}^2 + Ch_{ij}\partial^2 h_{ij}) + \frac{\beta}{8} [h''_{ij}^2 + 2h'_{ij}\partial^2 h'_{ij} + (\partial^2 h_{ij})^2] \right\}$$

Definitions $f_i := \partial f / \partial r_i$

$$\beta := 2f_2 + 4f_3$$

$$A := 2f_1 + \frac{8f_2}{a^2} (\mathcal{H}^2 + \mathcal{H}') - 4f'_2 \frac{\mathcal{H}}{a^2},$$

$$C := 2f_1 + \frac{8f_2}{a^2} (\mathcal{H}^2 + \mathcal{H}') + \frac{4f'_2}{a^2} \mathcal{H} - \frac{2}{a^2} (f''_2 - 2f''_3)$$

Time dependent
Coefficients

Generalization: Check the instability

Start with: $S = \int d^4x \sqrt{-g} f(R, R_{\mu\nu}R^{\mu\nu}, C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma})$

$r_1 := R, \quad r_2 := R_{\mu\nu}R^{\mu\nu}, \quad r_3 := C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$

Quadratic action:

$$S = \int d^4x \left\{ \frac{a^2}{8} (Ah'_{ij}^2 + Ch_{ij}\partial^2 h_{ij}) + \frac{\beta}{8} [h''_{ij}^2 + 2h'_{ij}\partial^2 h'_{ij} + (\partial^2 h_{ij})^2] \right\}$$

Hamiltonian:

$$H = \int d^3x \left\{ \pi^{ij}q_{ij} + \frac{2}{\beta}p_{ij}^2 - \frac{1}{8}h_{ij} (a^2C\partial^2 + \beta\partial^2\partial^2) h_{ij} \right.$$
$$\left. - \frac{1}{8}q_{ij} (a^2A + 2\beta\partial^2) q_{ij} \right\}$$

Generalization: Add constraints

$$S = \int d^4x \left\{ \frac{a^2}{8} (Ah_{ij}'^2 + Ch_{ij}\partial^2 h_{ij}) + \frac{\beta}{8} [h_{ij}''^2 + 2h_{ij}'\partial^2 h_{ij}' + (\partial^2 h_{ij})^2] \right\}$$

Constraints: same form

$$\mathcal{L} \supset (h_{ij}'')^2 \quad \rightarrow \quad \mathcal{L} \supset (h_{ij}'' - \lambda_{ij})^2, \quad \lambda_{ij}\partial^2 h_{ij}$$

Time derivative \longleftrightarrow Spatial derivative

【extra d.o.f (ghost)】 【Good for renormalizability】

Constructing $h_{ij} := h_{ij} \longleftrightarrow \pi^{ij} := \partial\mathcal{L}/\partial h'_{ij}$

Hamiltonian: $q_{ij} := h'_{ij} \longleftrightarrow p^{ij} := \partial\mathcal{L}/\partial h''_{ij}$

$\lambda_{ij} := \lambda_{ij} \longleftrightarrow \pi_\lambda^{ij} = 0$

Generalization: Add constraints

Hamiltonian with constraint

$$H = \int d^3x \left[\pi^{ij} q_{ij} + \frac{2}{\beta} p_{ij}^2 + \lambda_{ij} \left(p_{ij} - \frac{\beta}{2} \partial^2 h_{ij} \right) - \frac{1}{8} h_{ij} (a^2 C \partial^2 + \beta \partial^2 \partial^2) h_{ij} - \frac{1}{8} q_{ij} (a^2 A + 2\beta \partial^2) q_{ij} \right]$$

Primary constraint $\phi_1 : \pi_\lambda^{ij} = 0$



Secondary constraints \cdots **Consistency of primary constraint**

$\phi_2 : \dot{\phi}_1 = 0 \Leftrightarrow \{\phi_1, H\}_{\text{PB}} = 0$ and so on.



several constraints including $\pi_{ij} \propto q_{ij}$

Generalization: Reduced Hamiltonian

Results of Stabilization



Reduced Hamiltonian - tensor perturbation

$$H_R = \int d^3x \left\{ \frac{2}{a^4 A^2} \pi_{ij} (a^2 A - 2\beta \partial^2) \pi_{ij} + \frac{1}{8} h_{ij} (-a^2 C \partial^2 + 3\beta \partial^2 \partial^2) h_{ij} \right\}$$

Stable for $A, C, \beta > 0$



Spectrum of G.W.

model independent form

$$\mathcal{P}_h = \frac{1}{A + s\beta H^2} \frac{2H^2}{\pi^2}, \quad s \simeq \text{const.} = \mathcal{O}(1)$$

Suppression factor

Generalization: Reduced Hamiltonian

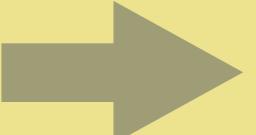
Results of Stabilization

Phase space dimensionality:

Original: 8 d.o.f $\cdots h_{ij}, q_{ij}$ and their conjugate

add λ_{ij} and conjugate $\rightarrow +4$ d.o.f $\blacktriangleright 12$ d.o.f

+8 second class constraint

 4 d.o.f : same with GR [ghosts are removed]

Generalization: Vector sector

$$ds^2 = a^2 \left[- (1 - B_i B^i) d\eta^2 + 2B_i dx^i d\eta + \delta_{ij} dx^i dx^j \right]$$

Vector mode is dynamical:

$$S = \int d^4x \left[\frac{\beta}{4} (\hat{v}_{ij}'^2 + \hat{v}_{ij} \partial^2 \hat{v}_{ij}) + \frac{a^2 A}{4} \hat{v}_{ij}^2 \right] \text{ with } \hat{v}_{ij} := \partial_i B_j$$

Stable for $\beta > 0$ and $A < 0$

【Conflict with stable condition of tensor sector】

→ Remove the vector mode from theory

Generalization: Vector sector

Constraint

$$S = \int d^4x \left\{ \frac{\beta}{4} \left[(\hat{v}'_{ij} - \hat{\lambda}_{ij})^2 + \hat{v}_{ij} \partial^2 \hat{v}_{ij} \right] + \frac{a^2 A}{4} \hat{v}_{ij}^2 \right\}$$


$$H = \int d^3x \left(\frac{\hat{\pi}_{ij}^2}{\beta} + \hat{\pi}_{ij} \hat{\lambda}_{ij} - \frac{\beta}{4} \hat{v}_{ij} \partial^2 \hat{v}_{ij} - \frac{a^2 A}{4} \hat{v}_{ij}^2 \right)$$

primary constraint, Secondary constraints,.....



reduced Hamiltonian Vanishes!

Generalization: Scalar sector

Scalar perturbation can be stabilized.
(expression is complicated)

Flat gauge metric:

$$ds^2 = a^2 [-(1 + 2\Phi)d\eta^2 + 2\partial_i B d\eta dx^i + \delta_{ij} dx^i dx^j]$$

Quadratic action:

$$\begin{aligned} S = & \int d^4x [b_0 (\Phi')^2 + b_1 (\partial^2 \Phi + \mathcal{B}')^2 + b_2 \Phi' \mathcal{B}' + b_3 \Phi \mathcal{B}' \\ & + c_1 \Phi^2 + c_2 \Phi \partial^2 \Phi + c_3 \Phi \mathcal{B} + c_4 \mathcal{B}^2 + c_5 \mathcal{B} \partial^2 \Phi] \end{aligned}$$

$$\mathcal{B} := \partial_i^2 B$$

$$c_1 := -6f_1a^2\mathcal{H}^2 + 36f_{11}\left[4\mathcal{H}^4 - 2\mathcal{H}\mathcal{H}'' + (\mathcal{H}')^2 + \mathcal{H}^2\mathcal{H}'\right] + 2(2f_2^2 + 3f_{11}^2)(4\mathcal{H}^2 + 3\mathcal{H}') + \frac{24}{a^2}f_{12}\left[10\mathcal{H}^4 - 3\mathcal{H}\mathcal{H}'' + 6(\mathcal{H}')^2 + 11\mathcal{H}^2\mathcal{H}'\right] + \frac{72}{a^2}f_{12}\left[22\mathcal{H}^6 + 4(\mathcal{H}')^3 + 19\mathcal{H}^4\mathcal{H}' - 8\mathcal{H}^3\mathcal{H}'' - 14\mathcal{H}\mathcal{H}'\mathcal{H} + \frac{36}{a^2}f_{111}\mathcal{H}(2\mathcal{H}^3 - \mathcal{H}'')\right] + \frac{216}{a^2}f_{111}\mathcal{H}(\mathcal{H}^2 + \mathcal{H}')(2\mathcal{H}^3 - \mathcal{H}'') + \frac{24}{a^4}f_{22}\left[20\mathcal{H}^6 + 12(\mathcal{H}')^3 + 35\mathcal{H}^4\mathcal{H}' - 6\mathcal{H}^3\mathcal{H}'' - 12\mathcal{H}\mathcal{H}'\mathcal{H}'' + 32\mathcal{H}^2(\mathcal{H}')^2\right] + \frac{288}{a^4}f_{22}\left[12\mathcal{H}^8 + 2(\mathcal{H}')^4 + 21\mathcal{H}^6\mathcal{H}' - 4\mathcal{H}^5\mathcal{H}'' + 11\mathcal{H}^4(\mathcal{H}'')^2 - 10\mathcal{H}(\mathcal{H}')^2\mathcal{H}'' + 11\mathcal{H}^2(\mathcal{H}')^3 - 10\mathcal{H}^3\mathcal{H}'\mathcal{H}''\right] + \frac{72}{a^4}f_{112}\mathcal{H}\left[8\mathcal{H}^5 - 3\mathcal{H}^2\mathcal{H}'' + 2\mathcal{H}(\mathcal{H}')^2 + 8\mathcal{H}^3\mathcal{H}' - 6\mathcal{H}'\mathcal{H}''\right] + \frac{864}{a^4}f_{112}\mathcal{H}\left[5\mathcal{H}^7 + \mathcal{H}(\mathcal{H}')^3 + 7\mathcal{H}^5\mathcal{H}' - 2\mathcal{H}^4\mathcal{H}'' + 5\mathcal{H}^3(\mathcal{H}'')^2 + \frac{144}{a^5}f_{122}\mathcal{H}(\mathcal{H}^2 + 2\mathcal{H}')\left[10\mathcal{H}^5 - 3\mathcal{H}^2\mathcal{H}'' + 4\mathcal{H}(\mathcal{H}')^2 + 4\mathcal{H}^3\mathcal{H}' - 6\mathcal{H}'\mathcal{H}''\right]\right] + \frac{864}{a^6}f_{122}\mathcal{H}\left[16\mathcal{H}^9 + 8\mathcal{H}(\mathcal{H}')^4 + 32\mathcal{H}^7\mathcal{H}' - 5\mathcal{H}^6\mathcal{H}'' + 34\frac{288}{a^8}f_{222}\mathcal{H}(\mathcal{H}^2 + 2\mathcal{H}')^2\left[4\mathcal{H}^5 - \mathcal{H}^2\mathcal{H}'' + 2\mathcal{H}(\mathcal{H}')^2 - 2\mathcal{H}'\mathcal{H}''\right] - 20\mathcal{H}^2(\mathcal{H}')^2\mathcal{H}'' - 12(\mathcal{H}')^3\mathcal{H}'' - 17\mathcal{H}^4\mathcal{H}'\mathcal{H}''\right] + \frac{3456}{a^8}f_{222}\mathcal{H}\left[\mathcal{H}^6 + 2(\mathcal{H}')^3 + 3\mathcal{H}^4\mathcal{H}' + 3\mathcal{H}^2(\mathcal{H}')^2\right]\left[4\mathcal{H}^5 - \mathcal{H}^2\mathcal{H}'' + 2\mathcal{H}(\mathcal{H}')^2 - 2\mathcal{H}'\mathcal{H}''\right],$$

$$\begin{aligned}
c_3 := & -4f_1a^2\mathcal{H} + 12f_{11}(8\mathcal{H}^3 - \mathcal{H}'') + 8f_2(6\mathcal{H}^3 - \mathcal{H}'' - 2\mathcal{H}\mathcal{H}') \\
& + \frac{48}{a^2}f_{12}[26\mathcal{H}^5 - 8\mathcal{H}^2\mathcal{H}'' + \mathcal{H}(\mathcal{H}')^2 + 11\mathcal{H}^3\mathcal{H}' + 2\mathcal{H}'\mathcal{H}''] + \frac{216}{a^2}f_{11}(2\mathcal{H}^5\mathcal{H}'\mathcal{H}^2) + \frac{8}{a^2}f_{12}[29\mathcal{H}^4 - 8\mathcal{H}\mathcal{H}'' - 5(\mathcal{H}')^2 + 10\mathcal{H}^2\mathcal{H}'] \\
& + \frac{48}{a^4}f_{22}[60\mathcal{H}^7 + 4\mathcal{H}(\mathcal{H}')^3 + 70\mathcal{H}^5\mathcal{H}' - 19\mathcal{H}^4\mathcal{H}'' + 36\mathcal{H}f_{111}\mathcal{H}(2\mathcal{H}^3 - \mathcal{H}'')] \\
& + 22\mathcal{H}^3(\mathcal{H}')^2 - 28\mathcal{H}^2\mathcal{H}'\mathcal{H}'' - 4(\mathcal{H}')^2\mathcal{H}''] + \frac{8}{a^4}f_{22}[69\mathcal{H}^6 - 8(\mathcal{H}')^3 + 71\mathcal{H}^4\mathcal{H}' - 20\mathcal{H}^3\mathcal{H}'' - 28\mathcal{H}\mathcal{H}'\mathcal{H}'' + 18\mathcal{H}^2(\mathcal{H}')^2] \\
& + \frac{144}{a^4}f_{112}\mathcal{H}^2[28\mathcal{H}^5 - 11\mathcal{H}^2\mathcal{H}'' + 6\mathcal{H}(\mathcal{H}')^2 + 20\mathcal{H}^3\mathcal{H}''] + \frac{24}{a^4}f_{16}\mathcal{H}'\mathcal{H}[28\mathcal{H}^5 - 11\mathcal{H}^2\mathcal{H}'' + 6\mathcal{H}(\mathcal{H}')^2 + 20\mathcal{H}^3\mathcal{H}' - 16\mathcal{H}'\mathcal{H}''] \\
& + \frac{288}{a^6}f_{122}\mathcal{H}^2[42\mathcal{H}^7 + 20\mathcal{H}(\mathcal{H}')^3 + 68\mathcal{H}^5\mathcal{H}' - 13\mathcal{H}^4\mathcal{H}''] + \frac{48}{a^6}f_{32}\mathcal{H}^6[\mathcal{H}^7 + 20\mathcal{H}(\mathcal{H}')^3 + 68\mathcal{H}^5\mathcal{H}' - 13\mathcal{H}^4\mathcal{H}'' \\
& - 40\mathcal{H}^2\mathcal{H}'\mathcal{H}'' - 28(\mathcal{H}')^2\mathcal{H}''] + \frac{96}{a^8}f_{222}\mathcal{H}^2[5\mathcal{H}^4 + 8(\mathcal{H}')^2 + 14\mathcal{H}^2\mathcal{H}'][4\mathcal{H}^5 - \mathcal{H}^2\mathcal{H}'' + 2\mathcal{H}^2\mathcal{H}(\mathcal{H}')^2 + 8(\mathcal{H}')^3 + 14\mathcal{H}^2\mathcal{H}'] [4\mathcal{H}^5 - \mathcal{H}^2\mathcal{H}'' + 2\mathcal{H}(\mathcal{H}')^2 - 2\mathcal{H}'\mathcal{H}''] \\
& + \frac{576}{a^8}f_{222}\mathcal{H}^2[5\mathcal{H}^4 + 8(\mathcal{H}')^2 + 14\mathcal{H}^2\mathcal{H}'][4\mathcal{H}^5 - \mathcal{H}^2\mathcal{H}'' + 2\mathcal{H}^2\mathcal{H}(\mathcal{H}')^2 + 8(\mathcal{H}')^3 + 14\mathcal{H}^2\mathcal{H}'] [4\mathcal{H}^5 - \mathcal{H}^2\mathcal{H}'' + 2\mathcal{H}(\mathcal{H}')^2 - 2\mathcal{H}'\mathcal{H}'']
\end{aligned}$$

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c_1 := & -6f_1a^2\mathcal{H}^2 + 36f_{11}\left[4\mathcal{H}^4 - 2\mathcal{H}\mathcal{H}'' + (\mathcal{H}')^2 + \mathcal{H}^2\mathcal{H}'\right] + 2\left(2f_2^2 + 3f_{11}^2\right)\left(4\mathcal{H}^2\mathcal{H}'\right)^2 + \frac{24}{a^2}f_{12}\left[10\mathcal{H}^4 - 3\mathcal{H}\mathcal{H}'' + 6(\mathcal{H}')^2 + 11\mathcal{H}^2\mathcal{H}'\right] \\
& + \frac{72}{a^2}f_{12}\left[22\mathcal{H}^6 + 4(\mathcal{H}')^3 + 19\mathcal{H}^4\mathcal{H}' - 8\mathcal{H}^3\mathcal{H}'' - 14\mathcal{H}\mathcal{H}'\mathcal{H}^2 + \frac{36}{a^2}f_{111}\mathcal{H}\left(2\mathcal{H}^3 - \mathcal{H}''\right)\right. \\
& + \frac{216}{a^2}f_{111}\mathcal{H}(\mathcal{H}^2 + \mathcal{H}')(2\mathcal{H}^3 - \mathcal{H}'') \\
& + \frac{288}{a^4}f_{22}\left[12\mathcal{H}^8 + 2(\mathcal{H}')^2 - 10\mathcal{H}(\mathcal{H}')^2\right] \\
& + \frac{864}{a^4}f_{112}\mathcal{H}\left[5\mathcal{H}^7 + \mathcal{H}'' + 4\mathcal{H}(\mathcal{H}')^2 + 4\mathcal{H}^3\mathcal{H}' - 6\mathcal{H}'\mathcal{H}''\right] \\
& + \frac{864}{a^6}f_{122}\mathcal{H}\left[16\mathcal{H}^9 + 8\mathcal{H}^7 + 20\mathcal{H}^2(\mathcal{H}')^2 + 2\mathcal{H}(\mathcal{H}')^2 - 2\mathcal{H}'\mathcal{H}''\right], \\
& + \frac{3456}{a^8}f_{222}\mathcal{H}\left[\mathcal{H}^6 + 2(\mathcal{H}')^3 + 12\mathcal{H}^5\mathcal{H}' + 16\mathcal{H}^4\mathcal{H}'' + 24\mathcal{H}^3\mathcal{H}'\mathcal{H}'' + 16\mathcal{H}^2\mathcal{H}'^2 + 8\mathcal{H}^2\mathcal{H}'\mathcal{H}'' + 8\mathcal{H}^3\mathcal{H}' - 6\mathcal{H}'\mathcal{H}''\right]
\end{aligned}$$

Coefficients contain...

- Hubble param.
- model dependent param.

$$\begin{aligned}
c_3 := & -4f_1a^2\mathcal{H} + 12f_{11}\left(8\mathcal{H}^5 - 8\mathcal{H}^2\mathcal{H}'' + \mathcal{H}(\mathcal{H}')^2 + 11\mathcal{H}^3\mathcal{H}' + 2\mathcal{H}'\mathcal{H}''\right) + \frac{216}{a^2}f_{11}\left(2\mathcal{H}^5\mathcal{H}'\mathcal{H}^2\right)f_{12}\left[29\mathcal{H}^4 - 8\mathcal{H}\mathcal{H}'' - 5(\mathcal{H}')^2 + 10\mathcal{H}^2\mathcal{H}'\right] \\
& + \frac{48}{a^2}f_{12}\left[26\mathcal{H}^5 - 8\mathcal{H}^2\mathcal{H}'' + \mathcal{H}(\mathcal{H}')^2 + 11\mathcal{H}^3\mathcal{H}' + 2\mathcal{H}'\mathcal{H}''\right] + \frac{36}{a^2}f_{111}\mathcal{H}(2\mathcal{H}^3 - \mathcal{H}'') \\
& + \frac{48}{a^4}f_{22}\left[60\mathcal{H}^7 + 4\mathcal{H}(\mathcal{H}')^3 + 70\mathcal{H}^5\mathcal{H}' - 19\mathcal{H}^4\mathcal{H}'' + 22\mathcal{H}^3(\mathcal{H}')^2 - 28\mathcal{H}^2\mathcal{H}'\mathcal{H}'' - 4(\mathcal{H}')^2\mathcal{H}''\right] + \frac{8}{a^4}f_{22}\left[69\mathcal{H}^6 - 8(\mathcal{H}')^3 + 71\mathcal{H}^4\mathcal{H}' - 20\mathcal{H}^3\mathcal{H}'' - 28\mathcal{H}\mathcal{H}'\mathcal{H}'' + 18\mathcal{H}^2(\mathcal{H}')^2\right] \\
& + \frac{144}{a^4}f_{112}\mathcal{H}^2\left[28\mathcal{H}^5 - 11\mathcal{H}^2\mathcal{H}'' + 6\mathcal{H}(\mathcal{H}')^2 + 20\mathcal{H}^3\mathcal{H}' + \frac{24}{a^4}16\mathcal{H}'\mathcal{H}^2\right] + \frac{48}{a^6}f_{122}\mathcal{H}^2\left[42\mathcal{H}^7 + 20\mathcal{H}(\mathcal{H}')^3 + 68\mathcal{H}^5\mathcal{H}' - 13\mathcal{H}^4\mathcal{H}'' + 32\mathcal{H}^3(\mathcal{H}')^2 + 20\mathcal{H}(\mathcal{H}')^3 + 68\mathcal{H}^5\mathcal{H}' - 13\mathcal{H}^4\mathcal{H}'' - 40\mathcal{H}^2\mathcal{H}'\mathcal{H}'' - 28(\mathcal{H}')^2\mathcal{H}''\right. \\
& \left. + 32\mathcal{H}^3(\mathcal{H}')^2 - 40\mathcal{H}^2\mathcal{H}'\mathcal{H}'' - 28(\mathcal{H}')^2\mathcal{H}''\right] + \frac{576}{a^8}f_{222}\mathcal{H}^2\left[5\mathcal{H}^4 + 8(\mathcal{H}')^2 + 14\mathcal{H}^2\mathcal{H}'\right]\left[4\mathcal{H}^5 - \mathcal{H}^2\mathcal{H}'' + \frac{96}{a^2}f_2\mathcal{H}^2\left(\mathcal{H}'\right)^2 - 2\mathcal{H}'\mathcal{H}''\right] + \frac{144}{a^2}\mathcal{H}^2\mathcal{H}'\left[4\mathcal{H}^5 - \mathcal{H}^2\mathcal{H}'' + 2\mathcal{H}(\mathcal{H}')^2 - 2\mathcal{H}'\mathcal{H}''\right]
\end{aligned}$$

Generalization: Scalar sector

Quadratic action:

$$S = \int d^4x [b_0 (\Phi')^2 + b_1 (\partial^2 \Phi + \mathcal{B}')^2 + b_2 \Phi' \mathcal{B}' + b_3 \Phi \mathcal{B}' + c_1 \Phi^2 + c_2 \Phi \partial^2 \Phi + c_3 \Phi \mathcal{B} + c_4 \mathcal{B}^2 + c_5 \mathcal{B} \partial^2 \Phi]$$

Hamiltonian:

$$H = \int d^3x \left[\frac{3}{4\beta} \left(\pi_{\mathcal{B}} - \frac{1}{3} \frac{\pi_{\Phi}}{\mathcal{H}} - b_3 \Phi - 2b_1 \partial^2 \Phi \right)^2 + \frac{\pi_{\Phi}^2}{24\mathcal{H}^2 \mathcal{I}} - \frac{2\mathcal{I} + \beta}{3} (\partial^2 \Phi)^2 - c_1 \Phi^2 - c_2 \Phi \partial^2 \Phi - c_3 \Phi \mathcal{B} - c_4 \mathcal{B}^2 - c_5 \mathcal{B} \partial^2 \Phi \right].$$

To see instability:

Need to do canonical transformation

Generalization: Scalar sector

$$\tilde{H} = \int d^3x \left\{ \frac{3}{4\beta} \left(\tilde{\pi}_{\mathcal{B}} - \tilde{\Phi} b_3 - 2b_1 \partial^2 \tilde{\Phi} \right)^2 + \frac{\tilde{\pi}_{\Phi}^2}{24\mathcal{H}^2 \mathcal{I}} \right.$$
$$\left. - \frac{2\mathcal{I} + \beta}{3} \left[\partial^2 \left(\tilde{\Phi} - \frac{\tilde{\mathcal{B}}}{3\mathcal{H}} \right) \right]^2 + \dots \right\}$$

Stability conditions:

- Kinematic terms: $\beta > 0$ and $\mathcal{I} > 0$

→ **The last term becomes negative**

(unstable at high momentum scale.)

Generalization: Scalar sector

Constraint:

$$S = \int d^4x [b_0(\Phi')^2 + b_1 (\partial^2\Phi + \mathcal{B}' - \lambda)^2 + b_2\Phi'(\mathcal{B}' - \lambda) + b_3\Phi(\mathcal{B}' - \lambda) + c_1\Phi^2 + c_2\Phi\partial^2\Phi + c_3\Phi\mathcal{B} + c_4\mathcal{B}^2 + c_5\mathcal{B}\partial^2\Phi].$$

Just modify $\mathcal{B}' \rightarrow \mathcal{B}' - \lambda$

Reduced Hamiltonian:

$$\tilde{H}_R = \int d^3x \left[\frac{2\mathcal{I} + \beta}{24\mathcal{H}^2\mathcal{I}\beta} \tilde{\pi}_\Phi^2 + \frac{c_5^2}{4c_4} \left(\partial^2\tilde{\Phi} \right)^2 + d_1\tilde{\Phi}^2 - d_2\tilde{\Phi}\partial^2\tilde{\Phi} \right]$$

Summary

- Higher derivative term → **Ostrogradsky's ghost**



Gravitational theory: $R_{\mu\nu}^2, R_{\mu\nu\rho\sigma}^2, C_{\mu\nu\rho\sigma}^2$ cf. Q.G.

→ Instability can be **removed** by
the addition of **suitable constraint**

What we did:

Stabilize

the general model

$$f(R, R_{\mu\nu}^2, C_{\mu\nu\rho\sigma}^2)$$

on cosmological background