RESCEU APCosPA Summer School 2015

Teleparallel Gravity in Five Dimensional Theories

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Ling-Wei Luo

Institute of Physics, National Chiao Tung University

Collaborators: **Chao-Qiang Geng** (*NTHU/NCTS*), **Chang Lai** (*CQUPT, China*), **Huan Hsin Tseng** (*NTHU/Tufts Univ.*),

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Outline

Outline

Absolute Parallelism

- The orthonormal frame in Weitzenböck geometry W_4

$$g_{\mu\nu} = \eta_{ij}\, e^i_\mu\, e^j_\nu \qquad \text{with} \qquad \eta_{ij} = \text{diag}(+1, -1, -1, -1)\,.$$

- Metric compatible condition $\nabla\, g_{\mu\nu} = 0$:

$$d\, e^i_\mu - \Gamma^\rho_\mu\, e^i_\rho + \omega^i{}_j\, e^j_\mu = 0 \qquad \text{and} \qquad \omega_{ij} = -\,\omega_{ji}\,.$$

- Absolute parallelism for parallel vectors (*Cartan 1922/Eisenhart 1925*)

$$\nabla_\nu\, e^i_\mu = \partial_\nu e^i_\mu - e^i_\rho\, \Gamma^\rho_{\mu\nu} = 0\,.$$

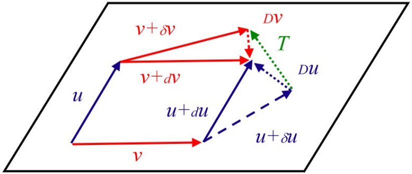
$$\Longrightarrow \quad \text{Weitzenböck connection} \quad \overset{w}{\Gamma}{}^\rho_{\mu\nu} = e^\rho_i \partial_\nu e^i_\mu \quad \longleftarrow \quad \omega_{ij\mu} = 0.$$

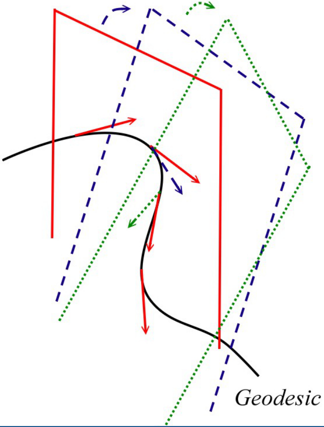
- Curvature-free $\quad R^\sigma{}_{\rho\mu\nu}(\Gamma) = e^\sigma_i\, e^j_\rho R^i{}_{j\mu\nu}(\omega) = 0.$
- Torsion tensor $T^i{}_{\mu\nu} \equiv \overset{w}{\Gamma}{}^i_{\nu\mu} - \overset{w}{\Gamma}{}^i_{\mu\nu} = \partial_\mu e^i_\nu - \partial_\nu e^i_\mu.$
- Contorsion tensor

$$K^\rho{}_{\mu\nu} = -\frac{1}{2}(T^\rho{}_{\mu\nu} - T_\mu{}^\rho{}_\nu - T_\nu{}^\rho{}_\mu) = -K_\mu{}^\rho{}_\nu\,.$$

Geometrical Meaning of Torsion

- Torsion free: a tangent vector does not rotate when we parallel transport it. (*P.371, John Baez and Javier P. Muniain, "Gauge Fields, Knots and Gravity," 1994*)
- $T(u,v) = \nabla_u v - \nabla_v u - \underbrace{[u,v]}_{\text{vanished in coordinate space}}$





Teleparallel Equivalent to GR in W_4

- Decomposition of the Weitzenböck connection

$$\overset{w}{\Gamma}{}^{\rho}_{\mu\nu} = \left\{ {}^{\rho}_{\mu\nu} \right\} + K^{\rho}{}_{\mu\nu} \,,$$

- Teleparallel Equivalent to GR ($GR_{\|}$ or TEGR) in W_4 based on the the relation ($T_\mu := T^{\nu}{}_{\nu\mu}$)

$$R(\Gamma) = \tilde{R}(e) + T - 2\,\tilde{\nabla}_\mu T^\mu = 0 \quad \Longrightarrow \quad -\tilde{R}(e) = T - 2\,\tilde{\nabla}_\mu T^\mu .$$

Torsion Scalar (*Einstein 1929*)

$$T \equiv \frac{1}{4}\,T^{\rho}{}_{\mu\nu}\,T_{\rho}{}^{\mu\nu} + \frac{1}{2}\,T^{\rho}{}_{\mu\nu}\,T^{\nu\mu}{}_{\rho} - T^{\nu}{}_{\mu\nu}\,T^{\sigma\mu}{}_{\sigma} = \frac{1}{2}\,T^{i}{}_{\mu\nu}\,S_{i}{}^{\mu\nu}$$

$S_\rho{}^{\mu\nu} \equiv K^{\mu\nu}{}_{\rho} + \delta^{\mu}_{\rho}\,T^{\sigma\nu}{}_{\sigma} - \delta^{\nu}_{\rho}\,T^{\sigma\mu}{}_{\sigma} = -S_{\rho}{}^{\nu\mu}$ is superpotential.

- The TEGR action

$$S_{\text{TEGR}} = \frac{1}{2\kappa}\int d^4x\, e\, T \qquad (e = \sqrt{-g})\,.$$

Motivation

- Fundamental fields in GR:
  - Metric tensor $g_{\mu\nu}$

    $\Longrightarrow$ Levi-Civita connection $\{^{\,\rho}_{\mu\nu}\} = \frac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right)$
- Fundamental field in Teleparallel Gravity:
  - Veierbein fields e^i_μ

    $\Longrightarrow$ Weitzenböck connection $\Gamma^\rho_{\mu\nu} = e_i^\rho \partial_\nu e^i_\mu$.

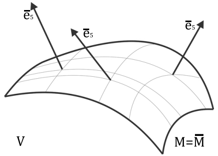
Question

Does there exist any different effect coming from the extra dimension?

Outline

Hypersurface in GR

- Gauss normal coordinate with the signature $(+---\varepsilon)$.
- The tensor $B_{MN} := -\tilde{\nabla}_M n_N$ is defined by the unit normal vector of the hypersurface n



$$\text{expansion} \quad \theta = h^{MN} B_{MN} \,,$$

$$\text{shear} \quad \sigma_{MN} = B_{(MN)} - \frac{1}{3}\theta h_{MN} \,,$$

$$\text{twist} \quad \omega_{MN} = B_{[MN]} \quad \longrightarrow \quad 0 \quad \text{(Hypersurface orthogonal)}\,,$$

where $h_{MN} = \bar{g}_{MN} - \varepsilon\, n_M n_N$ the projection operator.

Gauss's equation

$$\bar{R}^{\mu}{}_{\nu\rho\sigma} = R^{\mu}{}_{\nu\rho\sigma} + \varepsilon(K^{\mu}{}_{\sigma} K_{\nu\rho} - K^{\mu}{}_{\rho} K_{\nu\sigma})$$

- Extrinsic curvature
$K_{\mu\nu} = B_{(\mu\nu)} = -\varepsilon\, \tilde{\nabla}_\mu n \cdot e_\nu = -\varepsilon\, \frac{1}{2}\mathcal{L}_n g_{\mu\nu} = \varepsilon \left\{ {5 \atop \mu\nu} \right\} n_5$

5-Dimension Teleparallelism

- An embedding: $W_4 \longrightarrow W_5$.
- In Gauss normal coordinate

$$\bar{g}_{MN} = \begin{pmatrix} g_{\mu\nu}(x^\mu, y) & 0 \\ 0 & \varepsilon\phi^2(x^\mu, y) \end{pmatrix} .$$

- The 5D torsion scalar in the orthonormal frame

$${}^{(5)}T = \underbrace{\bar{T}}_{\text{induced 4D torsion scalar}} + \frac{1}{2}\left(\bar{T}_{i\hat{5}j}\,\bar{T}^{i\hat{5}j} + \bar{T}_{i\hat{5}j}\,\bar{T}^{j\hat{5}i}\right) + 2\,\bar{T}^{j}{}_{j}{}^{i}\,\bar{T}^{\hat{5}}{}_{i\hat{5}} - \bar{T}^{j}{}_{\hat{5}j}\,\bar{T}^{k\hat{5}}{}_{k} .$$

- The non-vanishing components of vielbein are e_μ^i and $e_5^{\hat{5}}$

Projection of the torsion tensor

$$\bar{T}^{\rho}{}_{\mu\nu} = T^{\rho}{}_{\mu\nu} \quad \text{(purely 4-dimensional object)}$$

$$\begin{array}{cccc} i & \longrightarrow & \mu & \bigcirc\,, \\ i & \longrightarrow & 5 & \times\,, \\ \hat{5} & \longrightarrow & \mu & \times\,, \\ \hat{5} & \longrightarrow & 5 & \bigcirc\,. \end{array}$$

- The 5D torsion scalar in the coordinate frame

$$^{(5)}T = \bar{T} + \frac{1}{2}\left(\bar{T}_{\rho 5\nu}\,\bar{T}^{\rho 5\nu} + \bar{T}_{\rho 5\nu}\,\bar{T}^{\nu 5\rho}\right) + 2\,\bar{T}^{\sigma}{}_{\sigma}{}^{\mu}\,\bar{T}^{5}{}_{\mu 5} - \bar{T}^{\nu}{}_{5\nu}\,\bar{T}^{\sigma 5}{}_{\sigma}\,.$$

Note:

In general, *induced torsion* $\bar{T}^{\rho}{}_{\mu\nu} = T^{\rho}{}_{\mu\nu} + \bar{C}^{\rho}{}_{\mu\nu}$, where

$$\bar{C}^{\rho}{}_{\mu\nu} = \bar{e}^{\rho}_{\hat{5}}\overbrace{(\partial_\mu \bar{e}^{\hat{5}}_{\nu} - \partial_\nu \bar{e}^{\hat{5}}_{\mu})}^{\bar{C}^{\hat{5}}{}_{\mu\nu}}\,.$$

$\bar{C}^{\hat{5}}{}_{\mu\nu} = \Gamma^{\hat{5}}_{\nu\mu} - \Gamma^{\hat{5}}_{\mu\nu} = h^{M}_{\mu}\,h^{N}_{\nu}\,T^{\hat{5}}{}_{MN} \sim \omega_{\mu\nu}$ is related to the extrinsic torsion or twist $\omega_{\mu\nu}$.

Outline

Braneworld Theory

- The vielbein in the Gauss normal coordinate

$$\bar{e}^I_M = \begin{pmatrix} e^i_\mu(x^\mu, y) & 0 \\ 0 & \phi(x^\mu, y) \end{pmatrix} .$$

- The induced torsion scalar $\bar{T} = T$.

- The bulk action in the orthonormal frame

$$S_{\text{bulk}} = \frac{1}{2\kappa_5} \int dvol^5 \Big(T + \frac{1}{2}(\bar{T}_{i5j}\, \bar{T}^{i5j} + \bar{T}_{i5j}\, \bar{T}^{j5i}) + \frac{2}{\phi}\, e_i(\phi)\, \bar{T}^a - \bar{T}_5\, \bar{T}^5 \Big) \text{ with } \bar{T}_A := \bar{T}^b{}_{bA} .$$

Note:
According to [*Ponce de Leon 2001*], the induced-matter theory ($R_{AB} = 0$, *Wesson 1998*) can be regarded as a mathematically equivalent formulation of the braneworld theory.

- Assume the bulk metric $\bar{g}$ is maximally symmetric 3-space with spatially flat $(k = 0)$

$$\bar{g}_{MN} = \text{diag}\left(1, -a^2(t,y), -a^2(t,y), -a^2(t,y), \varepsilon\, \phi^2(t,y)\right) ,$$

$$\bar{\vartheta}^{\hat{0}} = dt\,, \quad \bar{\vartheta}^{a} = a(t,y)\, dx^\alpha\,, \quad \bar{\vartheta}^{\hat{5}} = \phi(t,y)\, dy\,.$$

- First Cartan structure equation

$$\bar{T}^{\hat{0}} = \bar{d}\,\bar{\vartheta}^{\hat{0}} = 0, \quad \bar{T}^{a} = \frac{\dot{a}}{a}\,\bar{\vartheta}^{\hat{0}} \wedge \bar{\vartheta}^{a} + \frac{a'}{a\phi}\,\bar{\vartheta}^{\hat{5}} \wedge \bar{\vartheta}^{a}, \quad \bar{T}^{\hat{5}} = \frac{\dot{\phi}}{\phi}\,\bar{\vartheta}^{\hat{0}} \wedge \bar{\vartheta}^{\hat{5}}\,,$$

- Torsion 5-form reads

$$\bar{\mathcal{T}} = \left[T + \left(\frac{3 + 9\,\varepsilon}{\phi^2}\,\frac{a'^2}{a^2} + 6\,\frac{\dot{a}}{a}\,\frac{\dot{\phi}}{\phi}\right)\right] dvol^5\,.$$

- The energy-momentum tensor is $\bar{\Sigma}_A = \bar{T}_A^B \,\bar{\star}\bar{\vartheta}_B$

$$\bar{T}_A^B(t,y) = \left(\bar{T}_A^B\right)_{\text{bulk}} + \left(\bar{T}_A^B\right)_{\text{brane}} \,,$$
$$\left(\bar{T}_A^B\right)_{\text{brane}} = \frac{\delta(y)}{\phi}\, diag(\rho(t), -P(t), -P(t), -P(t), 0)\,,$$
$$\left(\bar{T}_A^B\right)_{\text{bulk}} = \frac{\Lambda_5}{\kappa_5}\eta_A^B \,.$$

- We have the $\hat{0}\hat{0}$-component equation

$$\left(\frac{\dot{a}^2}{a^2} + \frac{\dot{a}}{a}\frac{\dot{\phi}}{\phi}\right) - \frac{1}{\phi^2}\left(\frac{a''}{a} - \frac{a'}{a}\frac{\phi'}{\phi}\right) - \frac{1}{\phi^2}\frac{a'^2}{a^2} = \frac{\kappa_5}{3}\bar{T}_{\hat{0}\hat{0}} \,.$$

- The scale factor $a(t,y) = \theta(y)a^{(+)}(t,y) + \theta(-y)a^{(-)}(t,y)$

$$\Longrightarrow \quad a''(t,y) = \delta(y)\,[a'](t,0) + \tilde{a}''(t,y) \text{ with } [a'] = a^{(+)\prime} - a^{(-)\prime}.$$

- The junction condition:

$$[a'](t,0) = -\frac{\kappa_5}{3\varepsilon}\rho\, a_0(t)\,\phi_0(t) \quad \xRightarrow{\mathbb{Z}_2 \text{ symmetry}} \quad a'(t,0) = -\frac{\kappa_5}{6\varepsilon}\rho\, a_0(t)\,\phi_0(t)\,.$$

- Modified Friedmann equation on the brane

$$\frac{\dot{a}_0^2(t)}{a_0^2(t)} + \frac{\ddot{a}_0(t)}{a_0(t)} = -\frac{\kappa_5^2}{36}\,\rho(t)(\rho(t) + 3P(t)) - \frac{k_5}{3\phi_0^2(t)}\,\left(\bar{T}_{55}\right)_{\text{bulk}}\,.$$

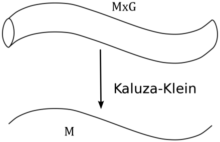
$\Longrightarrow$ Coincides with GR! (See *Binetruy, Deffayet and Langlois 2000*), but the *junction condition* comes from torsion itself!

Remark:
$\bar{T} = T$ a purely 4-dimensional object in Gauss normal coordinate.
$\Longrightarrow$ No extrinsic torsion contribution on the brane in TEGR.

Outline

Kaluza-Klein Theory

- KK ansatz:
  - Cylindrical condition (no y dependency)
  - Compactify to S^1 and only consider zero KK mode



- The manifold is $M_4 \times S^1$ $(y = r\,\theta)$
- The metric is reduced to

$$\bar{g}_{MN} = \begin{pmatrix} g_{\mu\nu}(x^\mu) & 0 \\ 0 & -\phi^2(x^\mu) \end{pmatrix} .$$

- The residual components are $T^\rho{}_{\mu\nu}$ and $\bar{T}^5{}_{\mu 5} = \partial_\mu \phi / \phi$.
- The 5D torsion scalar

$$^{(5)}T = T + 2\,T^\sigma{}_\sigma{}^\mu\,\bar{T}^5{}_{\mu 5}$$

The effective action of 5D TEGR

$$S_{\text{eff}} = \frac{1}{2\kappa_4} \int d^4x\, e\,(\phi\, T + 2\,T^\mu\,\partial_\mu \phi)$$

Minimal and Non-minimal coupling

Minimal coupled case

$$T \sim -R \quad \text{(TEGR)}.$$

- TEGR in 5D KK scenario with the metric given by

$$\bar{g}_{MN} = \begin{pmatrix} g_{\mu\nu} - k^2 A_\mu A_\nu & k\, A_\mu \\ k\, A_\nu & -1 \end{pmatrix} \text{ with } k^2 = \kappa_4,$$

The effective Lagrangian is

$$e^{-1}\mathcal{L}_{\text{eff}} = \frac{1}{2\kappa_4} T - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad \text{(the form coincides with GR).}$$

(*de Andrade et al. 2000*)

Non-minimal coupled case

$$\phi\, T \not\sim -\phi\, R$$

Remark:

The curvature-torsion relation in TEGR: $-\tilde{R}(e) = T - 2\,\tilde{\nabla}_\mu T^\mu$.

5D GR vs. 5D TEGR in KK Scenario

The Effective Lagrangian of 5D GR

- $\omega_{\text{BD}} = 0$ case in Brans-Dicke theory.

$$-\sqrt{-{}^{(5)}g}\,{}^{(5)}\tilde{R} \quad \longrightarrow \quad -\sqrt{-g}\left(\phi\tilde{R} - \Box\phi\right).$$

The Effective Lagrangian of 5D TEGR

$${}^{(5)}e\,{}^{(5)}T \quad \longrightarrow \quad e\Big(\phi T + \underbrace{2\,T^{\mu}\,\partial_{\mu}\phi}_{\text{no analogy}}\Big).$$

- Substituting the relation $-\tilde{R}(e) = T - 2\,\tilde{\nabla}_{\mu}T^{\mu}$ into the 4D effective Lagrangian

Equivalent to Scalar-Tensor Theory

$$\frac{-1}{2\kappa_4}\int d^4x\, e\Big(\phi\tilde{R}(e)\underbrace{-2\,\tilde{\nabla}_{\mu}(\phi\,T^{\mu})}_{\text{surface term}}\Big).$$

Conformal Transformation

- Doing the conformal transformation ($\tilde{g}_{\mu\nu} = \Omega^2(x)\, g_{\mu\nu}$):

$$\begin{aligned} T &= \Omega^2\, \tilde{T} - 4\,\Omega\, \tilde{g}^{\mu\nu}\, \tilde{T}_\mu \partial_\nu \Omega + 6\, \tilde{g}^{\mu\nu}\, \partial_\mu \Omega\, \partial_\nu \Omega\,, \\ T_\mu &= \tilde{T}_\mu + 3\,\Omega^{-1}\, \partial_\mu \Omega\,. \end{aligned}$$

- Choosing $\phi = \Omega^2$, the action reads

$$S_{\text{eff}} = \int d^4x\, \tilde{e}\, \left[\frac{1}{2\,\kappa_4}\, \tilde{T} + \frac{1}{2}\, \tilde{g}^{\mu\nu} \partial_\mu \psi\, \partial_\nu \psi \right]\,,$$

  where $\psi = (6/\kappa_4)^{1/2} \ln \Omega$ is the dilaton field.

- There exists an Einstein frame for such a non-minimal coupled effective Lagrangian in teleparallel gravity.

Looking for the Topological Effect (current work)

- Including spinor

$$\mathcal{L}_{\mathsf{D}} = e\,\bar{\psi}\,i\,\gamma^j\,e_j^\mu\left[\partial_\mu - \frac{i}{2}\Big(\omega_{jk\mu}(e) + K_{jk\mu}\Big)\sigma^{jk}\right]\psi \quad \text{with } \sigma^{jk} = \frac{i}{4}[\gamma^j, \gamma^k]\,.$$

- Gravitational chiral anomaly in GR (*Kimura 1969*)

$$d\star j_A = \frac{1}{384\,\pi^2}\,\mathcal{R}_{ij}\wedge\mathcal{R}^{ij} = d\Big(\Omega\wedge d\,\Omega + \frac{2}{3}\,\Omega\wedge\Omega\wedge\Omega\Big)\,.$$

- The modified Chern-Simons gravity (*Jackiw and Pi 2003*)

$$\mathcal{L}_{\mathsf{mCS}} = \frac{1}{16\,\pi\,G}\,\frac{\theta}{4}\,\mathcal{R}_{ij}\wedge\mathcal{R}^{ij} \quad \longrightarrow \quad \text{leptogenesis}$$

(*Alexander, Peskin and Sheikh-Jabbari 2004*)

Question

How about the anomaly induced gravity in W_4?

$$\text{Nieh-Tan term} \quad \mathcal{T}_i\wedge\mathcal{T}^i - \underbrace{\mathcal{R}_{ij}}_{=0}\wedge\vartheta^i\wedge\vartheta^j = d\Big(\mathcal{T}_i\wedge\vartheta^i\Big)$$

Outline

Summary

- In GR, the extrinsic curvature plays an important role to give the projected effect in the lower dimension.
- The effect on the lower dimensional manifold is totally determined by a higher dimensional object for TEGR in the braneworld scenario.
- Braneworld theory of teleparallel gravity in the FLRW cosmology still provides an equivalent viewpoint as Einstein's general relativity.
- In the FLRW universe, we found that the accelerated expansion of the universe can be achieved by the effective teleparallel gravity, which is the same as the effective KK scenario in general relativity.
- The KK reduction of telaparallel gravity generates an additional coupling in the effective Lagrangian, which leads to an Einstein frame by the conformal transformation for the non-minimal coupled teleparallel gravity.

End

Thank You!!!

Outline

Alternative Gravitational Theory

- Einstein's unified field theory: *Riemannian Geometry with Maintaining the Notion of Distant Parallelism*

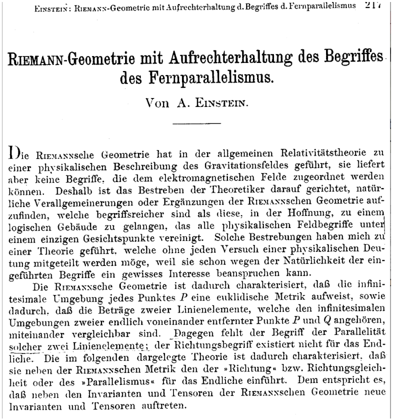
  (Teleparallelism, *Einstein 1928*)

- Torsion scalar (*Einstein 1929*)

- Teleparallel Lagrangian is equivalent to the Riemann scalar (*Lanczos 1929*)

- Generalization: *New General Relativity* (NGR)

  (*Hayashi & Shirafuji 1979*)



EINSTEIN: RIEMANN-Geometrie mit Aufrechterhaltung d. Begriffes d. Fernparallelismus 217

RIEMANN-Geometrie mit Aufrechterhaltung des Begriffes des Fernparallelismus.

Von A. EINSTEIN.

Die RIEMANNsche Geometrie hat in der allgemeinen Relativitätstheorie zu einer physikalischen Beschreibung des Gravitationsfeldes geführt, sie liefert aber keine Begriffe, die dem elektromagnetischen Felde zugeordnet werden können. Deshalb ist das Bestreben der Theoretiker darauf gerichtet, natürliche Verallgemeinerungen oder Ergänzungen der RIEMANNschen Geometrie aufzufinden, welche begriffsreicher sind als diese, in der Hoffnung, zu einem logischen Gebäude zu gelangen, das alle physikalischen Feldbegriffe unter einem einzigen Gesichtspunkte vereinigt. Solche Bestrebungen haben mich zu einer Theorie geführt, welche ohne jeden Versuch einer physikalischen Deutung mitgeteilt werden möge, weil sie schon wegen der Natürlichkeit der eingeführten Begriffe ein gewisses Interesse beanspruchen kann.

Die RIEMANNsche Geometrie ist dadurch charakterisiert, daß die infinitesimale Umgebung jedes Punktes P eine euklidische Metrik aufweist, sowie dadurch, daß die Beträge zweier Linienelemente, welche den infinitesimalen Umgebungen zweier endlich voneinander entfernter Punkte P und Q angehören, miteinander vergleichbar sind. Dagegen fehlt der Begriff der Parallelität solcher zwei Linienelemente; der Richtungsbegriff existiert nicht für das Endliche. Die im folgenden dargelegte Theorie ist dadurch charakterisiert, daß sie neben der RIEMANNschen Metrik den der »Richtung« bzw. Richtungsgleichheit oder des »Parallelismus« für das Endliche einführt. Dem entspricht es, daß neben den Invarianten und Tensoren der RIEMANNschen Geometrie neue Invarianten und Tensoren auftreten.

Riemann-Cartan Geometry U_4

- Einstein's general relativity was established in 1915 and described on Riemannian geometry V_4 with metric $g_{\mu\nu}$ and the metric-compatible Levi-Civita connection $\{^{\rho}_{\mu\nu}\} = \frac{1}{2} g^{\rho\lambda}(g_{\nu\lambda,\mu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda})$.
- The metric-compatible affine connection in U_4 is

$$\Gamma^{\rho}_{\mu\nu} = \{^{\rho}_{\mu\nu}\} + K^{\rho}{}_{\mu\nu} \,,$$

which can be decomposed into torsion and torsionless parts.

- Torsion: couple to spin angular momentum in gravity (*Élie Cartan 1922*.)
- Introducing the orthonormal frame $g_{\mu\nu} = \eta_{ij} e^i_\mu e^j_\nu$, and the relation of affine and *spin* connections in U_4 is

$$\Gamma^{\nu}_{\rho\mu} = e^{\nu}_{i} \partial_\mu e^i_\rho + e^{\nu}_{i} \, \omega^{i}{}_{j\mu} \, e^j_\rho .$$

Poincaré Gauge Theory (PGT)

- $\delta_0\phi = (\frac{1}{2}\,\omega(x)\cdot M + \xi(x)\cdot P)\phi$
- Gauge fields are $\omega^i{}_j = \omega^i{}_{j\mu}dx^\mu$ and $\theta^i = e^i_\mu dx^\mu$
- Fields strength is

$$D\circ D = \mathcal{R}^{ij}M_{ij} + \mathcal{T}^i P_i \quad \text{or} \quad [D_\rho, D_\sigma] = R^{ij}{}_{\rho\sigma}M_{ij} - T^i{}_{\rho\sigma}P_i$$

  where M_{ij} and P_i are the *rotational* and the *translational* generators, respectively.
- Cartan equations: $\mathcal{R}^i{}_j = d\omega^i{}_j + \omega^i{}_k\wedge\omega^k{}_j$ and $\mathcal{T}^i = D\theta^i$.

Einstein-Cartan-Sciama-Kibble (ECSK) Theory

The simplest Poincaré gauge theory:

$$S_{\text{ECSK}} = \int d^4x\; e\left[-\frac{1}{2\kappa}R(e,\omega)\right]$$

- ECSK extension: include *supersymmetry* and massless *Rarita-Schwinger field* (*Rarita & Schwinger 1941*)
  $\longrightarrow$ $N=1$ $D=4$ Supergravity.

Notation in 5D

- In orthonormal frame, 5D metric is $\bar{g}_{MN} = \bar{\eta}_{IJ}\, e^I_M\, e^J_N$,
  $\bar{\eta}_{IJ} = \mathrm{diag}(+1, -1, -1, -1, \varepsilon)$ with $\varepsilon = -1$.
- Coordinate frame
  $M, N = 0, 1, 2, 3, 5, \quad \mu, \nu = 0, 1, 2, 3, \quad \alpha, \beta = 1, 2, 3.$
- Orthonormal frame (anholonomic frame)
  $A, B, I, J, K = \hat{0}, \hat{1}, \hat{2}, \hat{3}, \hat{5}, \quad i, j, k = \hat{0}, \hat{1}, \hat{2}, \hat{3}, \quad a, b = \hat{1}, \hat{2}, \hat{3}.$

Affine Connection and Spin Connection

- Consider noncoordinate basis (orthonormal frame)

$$\begin{aligned}
e_i^{\nu} D_\mu V^i &= e_i^{\nu}(\partial_\mu V^i + \omega^i{}_{j\mu} V^j) \\
&= e_i^{\nu}[\partial_\mu(e_\rho^i V^\rho) + \omega^i{}_{j\mu} V^j] \\
&= e_i^{\nu}[(\partial_\mu e_\rho^i) V^\rho + e_\rho^i(\partial_\mu V^\rho) + \omega^i{}_{j\mu} e_\rho^j V^\rho] \\
&= (e_i^{\nu}\partial_\mu e_\rho^i) V^\rho + \underbrace{\delta_\rho^\nu \partial_\mu V^\rho}_{\partial_\mu V^\nu} + e_i^{\nu} \omega^i{}_{j\mu} e_\rho^j V^\rho] \\
&= \partial_\mu V^\nu + (e_i^{\nu}\partial_\mu e_\rho^i + e_i^{\nu} \omega^i{}_{j\mu} e_\rho^j) V^\rho \\
&\equiv \partial_\mu V^\nu + \Gamma^\nu_{\rho\mu} V^\rho = \nabla_\mu V^\nu .
\end{aligned}$$

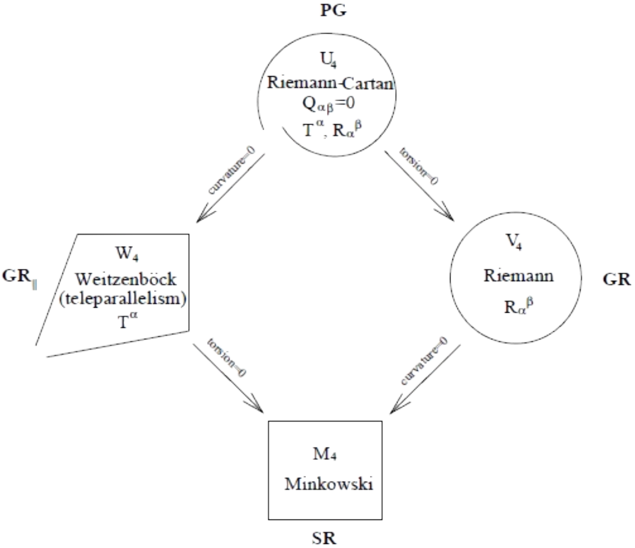
The relation between affine connection and spin connection

$$\Gamma^\nu_{\rho\mu} \equiv e_i^{\nu}\partial_\mu e_\rho^i + e_i^{\nu} \omega^i{}_{j\mu} e_\rho^j$$

- We have the definition of the total covariant derivative $\boldsymbol{\nabla}_\mu$

$$\Longrightarrow \quad \partial_\mu e_\rho^i - \Gamma^\nu_{\rho\mu} e_\nu^i + \omega^i{}_{j\mu} e_\rho^j = 0$$

$$\Longrightarrow \quad \boldsymbol{\nabla}_\mu e_\rho^i = 0 \text{ (vielbein postulate).}$$



Different gravitational theories with geometry (arXiv:9602013[gr-qc]).

Brief History of 5-Dimensional Theories

- Kaluza-Klein (KK) theory: to unify electromagnetism and gravity by gauge theory
  - Cylindrical condition (*Kaluza 1921*)
  - Compactification to a small scale (*Klein 1926*)
- Generalization of KK: induced-matter theory
  $\Longrightarrow$ matter from the 5th-dimension (*Wesson 1998*)
- Large Extra dimension (*Arkani-Hamed, Dimopoulos and Dvali (ADD) 1998*)
  - Solving hierarchy problem
  - SM particles confined on the 3-brane

- Randall-Sundrum model in AdS_5 spacetime (*Randall and Sundrum 1999*)
  - RS-I (UV-brane and SM particles confined on IR-brane)
    $\Longrightarrow$ solving hierarchy problem
  - RS-II (only one UV brane)
    $\Longrightarrow$ compactification to generate 4-dimensional gravity

- DGP brane model (*Dvali, Gabadadze and Porrati 2000*)
  $\Longrightarrow$ accelerating universe

- Universal Extra Dimension (*Appelquist, Cheng and Dobrescu 2001*)
  - Not only graviton but SM particles can propagate to the extra dimension $\Longrightarrow$ low compactification scale: reach to the electroweak scale

The 4-Form Equations of Motion

- The 4-form equations of motion $\bar{D}\bar{H}_A - \bar{E}_A = \kappa_5\, \bar{\Sigma}_A$, where

$$\bar{H}_A := \frac{\delta \bar{\mathcal{T}}}{\delta \bar{T}^A} = (-2)\bar{\star}\left({}^{(1)}\bar{T}_A - 2\, {}^{(2)}\bar{T}_A - \frac{1}{2}\, {}^{(3)}\bar{T}_A \right) ,$$

$$\bar{E}_A := i_{\bar{e}_A}(\bar{\mathcal{T}}) + i_{\bar{e}_A}(\bar{T}^B) \wedge \bar{H}_B ,$$

$$\bar{\Sigma}_A := -\frac{\delta \bar{L}_{mat}}{\delta \bar{\vartheta}^A} .$$

- $A = \hat{0}, \hat{5}$ components:

$$\begin{aligned}\bar{D}\bar{H}_{\hat{0}} - \bar{E}_{\hat{0}} =& 3\left[\left(\frac{\dot{a}^2}{a^2} + \frac{\dot{a}}{a}\frac{\dot{\phi}}{\phi}\right) + \frac{\varepsilon}{\phi^2}\left(\frac{a''}{a} - \frac{a'}{a}\frac{\phi'}{\phi}\right) - \left(\frac{1-\varepsilon}{2\phi^2}\right)\frac{a'^2}{a^2}\right]\bar{\star}\bar{\vartheta}_{\hat{0}} \\ &+ \frac{3\varepsilon}{\phi}\left(\frac{\dot{a}'}{a} - \frac{a'}{a}\frac{\dot{\phi}}{\phi}\right)\bar{\star}\bar{\vartheta}_{\hat{5}} = \kappa_5\, \bar{\Sigma}_{\hat{0}} ,\end{aligned}$$

$$\begin{aligned}\bar{D}\bar{H}_{\hat{5}} - \bar{E}_{\hat{5}} =& \frac{3}{\phi}\left(\frac{a'}{a}\frac{\dot{\phi}}{\phi} - \frac{\dot{a}'}{a}\right)\bar{\star}\bar{\vartheta}_{\hat{0}} + 3\left[\left(\frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2}\right) - \left(\frac{1-\varepsilon}{2\phi^2}\right)\frac{a'^2}{a^2}\right]\bar{\star}\bar{\vartheta}_{\hat{5}} \\ =& \kappa_5\, \bar{\Sigma}_{\hat{5}} .\end{aligned}$$

Equation of Motion of The Effective Action

- The gravitational equation of motion

$$\begin{aligned}\frac{1}{2}\, e_i^{\mu}\Big(\phi\, T + 2\, T^{\sigma}\,\partial_\sigma\phi\Big) - e_i^{\rho}\Big(\phi\, T^{j}{}_{\rho\nu}\, S_j{}^{\mu\nu}\Big) \\ - e_i^{\nu}\Big(\partial_\sigma\phi\, T^{\mu}{}_{\nu}{}^{\sigma} + \partial_\nu\phi\, T^{\mu} + \partial^{\mu}\phi\, T_\nu\Big) \\ + \frac{1}{e}\,\partial_\nu\Big(e\,(\phi\, S_i{}^{\mu\nu} + e_i^{\mu}\,\partial^{\nu}\phi - e_i^{\nu}\,\partial^{\mu}\phi)\Big) &= \kappa_4\,\Theta_i^{\mu}\end{aligned}$$

with $\Theta^{\mu}_{\nu} = \mathrm{diag}(\rho, -P, -P, -P)$

- The modified Friedmann equations in flat FLRW universe are

$$\begin{aligned} 3\,\phi\, H^2 + 3\, H\,\dot{\phi} &= \kappa_4\,\rho\,, \\ 3\,\phi\, H^2 + 2\,\dot{\phi}\, H + 2\,\phi\,\dot{H} + \ddot{\phi} &= -\,\kappa_4\, P\,, \end{aligned}$$

where $H = \dot{a}/a$ is the Hubble parameter (here $\rho = P = 0$ is assumed.)

- The equation of motion of scalar field ϕ in the

$$T - 2\,\partial_\mu T^\mu - 2T^\mu \Gamma^\nu_{\nu\mu} + e\,L_m = 0 \xRightarrow[\Gamma^\nu_{\nu\mu}=\Gamma^\alpha_{\alpha 0}=3\,(\dot{a}/a)]{\text{absence of matter}} a\,\ddot{a} + \dot{a}^2 = 0\,.$$

- Suppose the solution of $a(t)$ is proportional to t^m, the solution is

$$a(t) = a_s + b\,\sqrt{t}\,.$$

- The constraint of the coefficient: $a_s\,b = 0$
- $b = 0$ case:
  - $a(t) = a_s \Rightarrow$ the static universe.
- $a_s = 0$ case:
  - The Hubble parameter $H = 1/(2\,t) > 0$
  - The the acceleration of scale factor $\ddot{a} = -b/(4\,t^{2/3}) > 0$ for $b < 0$ $\Longrightarrow$ accelerated expanding universe.
- In general relativity, the equation of motion of ϕ is $\tilde{R}(e) = 0$ $\Longrightarrow$ the same solution for the scale factor.