Non-linear effects associated with primordial black hole formation

Gravitational wave induced by peaked primordial fluctuations

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Introduction

- The primordial fluctuation generated in the inflationary epoch gives the origin of CMB anisotropy and large-scale structure (LSS).

CMB anisotropy & LSS observation

→ primordial curvature fluctuation

- nearly scale-invariant $1 - n_s = 0.014 \pm 0.022$
- small amplitude $\mathcal{P}_\zeta = (2.23 \pm 0.16) \times 10^{-9}$ (WMAP 5yr)

at $k = 0.002/\mathrm{Mpc}$

- However, this is not necessarily the case for small scale fluctuations.

Many possible behaviors of small scale fluctuations

- If fluctuations with an amplitude $O(1)$ exist, overdense regions collapse to black holes. (Hawking 71)

Primordial Black Hole (PBH)

Mass = Horizon mass at the time of formation $k/aH \simeq 1$

Fluctuations of scale k^{-1} $\longrightarrow$ PBHs with mass $\frac{4\pi}{3}\rho(H^{-1})^3|_{k/aH=1}$

● PBHs with wide range of mass can be produced.

an origin of intermediate mass black holes, dark matter,… ?

Large amplitude

→ Non-linear effect

Gravitational wave production through tensor-scalar mode couplings

- Such a large amplitude can be generated, for example, in an inflation model where slow-roll conditions are broken temporarily. (No effect to the tensor modes.)

(Lyth & Liddle 01, RS, Yokoyama & Nagata 08…)

Second-order gravitational waves

(Ananda, Clarkson, and Wands 07, Baumann, Steinhardt, Takahashi, and Ichiki 07, …)

- Perturbed metric

$$ds^2 = a^2 \left[-e^{2\Phi} d\eta^2 + e^{-2\Psi} (\delta_{ij} + h_{ij}/2)(dx^i + V^i d\eta)(dx^j + V^j d\eta) \right],$$

where

$$\partial_i V^i = 0, \quad h^i_i = 0, \quad \partial_i h^i_j = 0 \qquad (h^i_j \equiv \delta^{ik} h_{kj}).$$

Φ, Ψ : scalar mode V^i : vector mode h_{ij} : tensor mode

- At **linear order** of the perturbations, these modes decouple from each other. ⇒ Each mode evolves independently.

- However, this statement is not true at the **second** order of the perturbations.
⇒ **A mode can be a source of the other two modes.** (Tomita 67, …)

- Consider primordial scalar perturbations with a large amplitude, and gravitational waves (tensor modes) induced by them.

$$\Phi, \Psi \sim O(\epsilon), \qquad V^i, h_{ij} \sim O(\epsilon^2)$$

- Transverse, traceless part of (i,j) components of Einstein equation gives evolution equation of GW.

Up to the second order w.r.t. ϵ , $(8\pi G = 1)$

$$h^{i\,\prime\prime}_{j} + 2\mathcal{H}h^{i\,\prime}_{j} - \partial^2 h^i_j = 4\mathcal{P}^{is}_{rj} S^r_s,$$

$\left(\mathcal{P}^{is}_{rj}\right.$: projection operator to transverce, traceless part$\left.\right)$

where

$$S^r_s \equiv -2\Psi\partial^r\partial_s\Psi - \partial^r\Psi\partial_s\Psi + \partial^r\Phi\partial_s\Phi + 2\Psi\partial^r\partial_s\Phi + \partial^r\Phi\partial_s\Psi$$
$$+ \partial^r\Psi\partial_s\Phi - 3\mathcal{H}^2(1+w)u^r u_s.$$

(u^i : i component of 4-velocity)

At linear order w.r.t. ϵ

$$u^i = -\frac{2}{3\mathcal{H}(1+w)}\partial^i(\Phi + \mathcal{H}^{-1}\Psi'),$$
$$\Phi = \Psi.$$

(we neglect the anisotropic stress for simplicity. cf. Baumann et.al. 07)

- Evolution equation of GW

$$h^{i\,\prime\prime}_{j} + 2\mathcal{H}h^{i\,\prime}_{j} - \partial^2 h^i_j = 4\mathcal{P}^{is}_{rj}S^r_s,$$

where

$$S^r_s = 2\partial^r\Phi\partial_s\Phi - \frac{4}{3(1+w)}\partial^r(\Phi+\mathcal{H}^{-1}\Phi')\partial_s(\Phi+\mathcal{H}^{-1}\Phi').$$

Here, Φ evolves according to the linear-order equation,

$$\Phi'' + \frac{6(1+w)}{1+3w}\frac{1}{\eta}\Phi' - \partial^2\Phi = 0.$$

-We consider the modes which cross the horizon in the radiation-dominated era $(w=1/3)$.

-Solution of the evolution equation

$$h_{\mathbf{k}\,j}^{\,i}(\eta) = \frac{4}{a}\int^{\eta} \mathrm{d}\tilde{\eta}\; g_{\mathbf{k}}(\eta-\tilde{\eta})a(\tilde{\eta})\mathcal{P}^{is}_{rj}S_{\mathbf{k}\,s}^{\,r}(\tilde{\eta}),$$

where

$$g_{\mathbf{k}}(\eta-\tilde{\eta}) \equiv \frac{1}{k}\sin[k(\eta-\tilde{\eta})] \quad (k=|\mathbf{k}|).$$

In terms of polarization modes, $+, \times,$

$$h_{\mathbf{k}}^{+(\times)}(\eta) = \frac{1}{a}\int^{\eta} \mathrm{d}\tilde{\eta}\; g_{\mathbf{k}}(\eta-\tilde{\eta})a(\tilde{\eta})S_{\mathbf{k}}^{+(\times)}(\tilde{\eta}),$$

and

$$S_{\mathbf{k}}^{+(\times)} = -\int \frac{\mathrm{d}^3\tilde{k}}{(2\pi)^{3/2}} e^{+(\times)}(\tilde{\mathbf{k}})\left[2\Phi_{\tilde{\mathbf{k}}}\Phi_{\mathbf{k}-\tilde{\mathbf{k}}} - (\Phi_{\tilde{\mathbf{k}}}+\mathcal{H}^{-1}\Phi'_{\tilde{\mathbf{k}}})(\Phi_{\mathbf{k}-\tilde{\mathbf{k}}}+\mathcal{H}^{-1}\Phi'_{\mathbf{k}-\tilde{\mathbf{k}}})\right].$$

$(e^{+}(\tilde{\mathbf{k}}) = \tilde{k}^2(1-\mu^2)\cos 2\phi_{\tilde{k}},\; e^{\times}(\tilde{\mathbf{k}}) = \tilde{k}^2(1-\mu^2)\sin 2\phi_{\tilde{k}},\quad \mu = \mathbf{k}\cdot\tilde{\mathbf{k}}/k\tilde{k},\; \phi_{\tilde{\mathbf{k}}}$: azimuthal angle)

- Power spectrum of the induced GW

$$\mathcal{P}_h(k,\eta) = \frac{k^3}{2\pi^2}(|h_{\mathbf{k}}^+(\eta)|^2 + |h_{\mathbf{k}}^\times(\eta)|^2)$$
$$= \frac{8k^3}{\pi^2 a(\eta)^2}\int^\eta \mathrm{d}\eta_1 \int^\eta \mathrm{d}\eta_2\; a(\eta_1)a(\eta_2)g_{\mathbf{k}}(\eta-\eta_1)g_{\mathbf{k}}(\eta-\eta_2)\left[S_{\mathbf{k}}^+(\eta_1)S_{-\mathbf{k}}^+(\eta_2) + S_{\mathbf{k}}^\times(\eta_1)S_{-\mathbf{k}}^\times(\eta_2)\right].$$

Here,

$$\frac{k^3}{2\pi^2}S_{\mathbf{k}}^{+(\times)}(\eta_1)S_{-\mathbf{k}}^{+(\times)}(\eta_2) = \frac{1}{2}\int_0^\infty \mathrm{d}\tilde{k}\int_{-1}^{1}\mathrm{d}\mu\; \frac{k^3\tilde{k}^3}{|\mathbf{k}-\tilde{\mathbf{k}}|^3}(1-\mu^2)^2 f(\tilde{k},|\mathbf{k}-\tilde{\mathbf{k}}|,\eta_1)f(\tilde{k},|\mathbf{k}-\tilde{\mathbf{k}}|,\eta_2)\mathcal{P}_\Phi(\tilde{k})\mathcal{P}_\Phi(|\mathbf{k}-\tilde{\mathbf{k}}|),$$

where

$$f(k_1,k_2,\eta) = 2\Phi_{k_1}(\eta)\Phi_{k_2}(\eta) - [\Phi_{k_1}(\eta) + \mathcal{H}^{-1}\Phi'_{k_1}(\eta)][\Phi_{k_2}(\eta) + \mathcal{H}^{-1}\Phi'_{k_2}(\eta)],$$

$$\Phi_k(\eta) = \frac{9}{(k\eta)^2}\left[-\cos(k\eta/\sqrt{3}) + \sqrt{3}\sin(k\eta/\sqrt{3})/(k\eta)\right] \quad \text{(transfer function for } \Phi\text{)}.$$

Gravitational waves induced by peaked primordial scalar fluctuations

- For simplicity, we assumed the form of the power spectrum $\mathcal{P}_\Phi(k)$ to be the delta-function:

$$\mathcal{P}_\Phi(k) = A^2 \delta(\ln(k/k_p)).$$

- **Power spectrum of the GW** from peaked $\mathcal{P}_\Phi(k)$

$$\mathcal{P}_h(k,\eta) = A^4 \left[1 - \left(\frac{k}{2k_p}\right)^2\right]^2 \theta\left(1 - \frac{k}{2k_p}\right) I(k, k_p, \eta)^2,$$

where

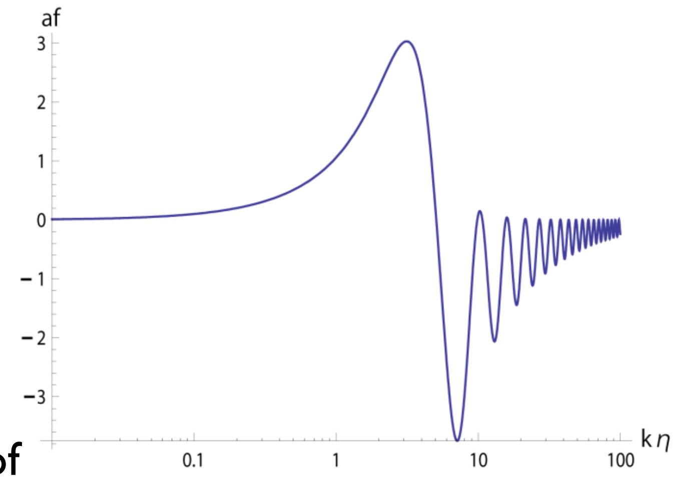
$$I(k, k_p, \eta) \equiv \frac{4}{a(\eta)} \int^{k_p \eta} \mathrm{d}(k_p \tilde{\eta}) \ \sin[k(\eta - \tilde{\eta})] a(\tilde{\eta}) f(k_p, k_p, \tilde{\eta}).$$

- The function $af(k,k,\eta)$ has a peak at $k\eta \sim O(1)$

$$\Phi_k \propto \begin{cases} \eta^0 & k\eta \ll 1 \\ \eta^{-2} & k\eta \gg 1 \end{cases}, \quad \eta\Phi'_k \propto \begin{cases} \eta^0 & k\eta \ll 1 \\ \eta^{-1} & k\eta \gg 1 \end{cases},$$

(In radiation-dominated era, $\eta = \mathcal{H}^{-1} \propto a$)

$$\Rightarrow \quad af(k,k,\eta) \propto \begin{cases} \eta & k\eta \ll 1 \\ \eta^{-1} & k\eta \gg 1 \end{cases},$$



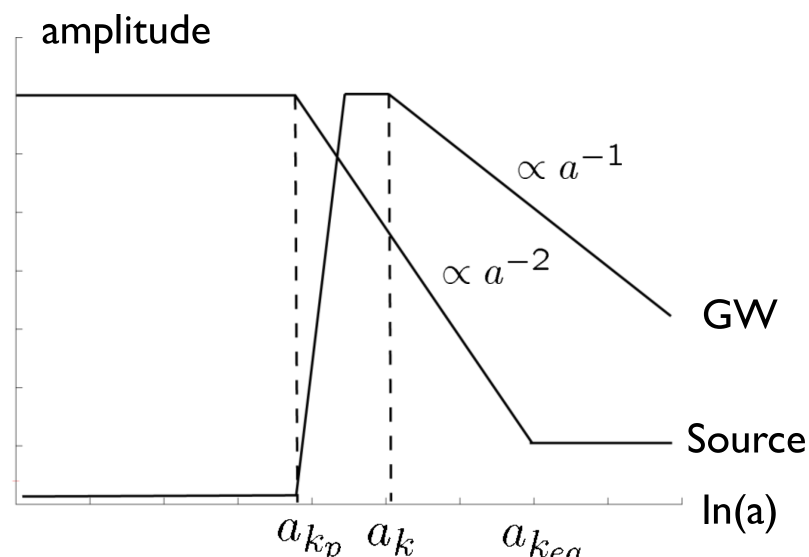
Time evolution of $af(k,k,\eta)$

- GWs are generated instantaneously at the time of horizon cross of the mode k_p .

- After the time, GWs evolves according to the linear-order evolution equation (evolution equation without sources)

(At the scale $k \sim 2k_p/\sqrt{3}$, resonant amplification occurs. The amplitude of GW decays as $\eta^{-1}\ln(\eta)$. In the matter-dominated era, the amplification doesn't occurs.)

- Although the sources don't decay in the matter-dominated era, these are insignificant at the scales which we are interested in.



Evolution of GWs

- Energy density of GW

$$\Omega_{\mathrm{GW}}(k,\eta) \equiv \frac{1}{3H^2}\frac{\mathrm{d}\rho_{gw}}{\mathrm{d}\ln k}$$

$$= \frac{1}{3}(k\eta)^2 \mathcal{P}_h(k,\eta). \qquad \text{(at subhorizon scales)}$$

For the GWs induced by peaked $\mathcal{P}_\Phi(k)$,

$$\Omega_{\mathrm{GW}}(k,\eta) = \frac{16}{3}A^4\left(\frac{k}{k_p}\right)^2\left[1-\left(\frac{k}{2k_p}\right)^2\right]^2\theta\left(1-\frac{k}{2k_p}\right)\tilde{I}(k/k_p,k_p\eta)^2,$$

where

$$\tilde{I}(k/k_p,k_p\eta) \equiv \int^{k_p\eta}\mathrm{d}(k_p\tilde{\eta})\ \sin[k(\eta-\tilde{\eta})](k_p\tilde{\eta})f(k_p,k_p,\tilde{\eta}).$$

Constraints on the amplitude of scalar perturbations

- At the scale $k \sim 2k_p/\sqrt{3}$ where resonant amplification occurs, $\tilde{I}^2 \sim 30[\log(k_p\eta)]^2$ for large $k_p\eta$.
- In the matter-dominated era, the energy density of GWs evolves at the same rate as that of radiation.

$$\Omega_{\rm GW}/\Omega_{\rm rad} = \begin{cases} 96A^4[\ln(k_p\eta)]^2 & \eta < \eta_{\rm eq} \\ 96A^4[\ln(k_p\eta_{\rm eq})]^2 & \eta > \eta_{\rm eq} \end{cases}.$$

- msec pulsar

$$h_0^2\Omega_{\rm GW} < 4.8 \times 10^{-8} \quad (f = 4.4 \times 10^{-9}\,{\rm Hz})$$

$$\Rightarrow \quad A < 1.3 \times 10^{-2} \quad (k_p = 2.8\ {\rm pc}^{-1})$$

- Binary pulsars

$$h_0^2\Omega_{\rm GW} < 2.7\times10^{-4} \quad (f = 1.0\times10^{-11}\text{–}4.4\times10^{-9}\,{\rm Hz})$$

$$\Rightarrow \quad A < (1.4\text{–}1.2)\times10^{-1} \quad (k_p = 6.5\times10^{-3}\text{–}2.8\ {\rm pc}^{-1})$$

Abundance of primordial black holes

- Abundance of PBHs

$$\beta(M) \equiv \frac{\rho_{\mathrm{PBH}(M)}}{3H^2} \sim \frac{1}{4}\exp\left(\frac{1}{8\mathcal{P}_\Phi(k_M)}\right).$$

(Radiation-dominated era)

- In the radiation-dominated era, the relation between the mass o PBHs and the scale of the scalar perturbations is given by

$$M = 7\times 10^{50}\mathrm{g}\left(\frac{k_M}{k_{\mathrm{eq}}}\right)^{-2}.$$

Constraint on the primordial black hole abundance

- msec pulsar

$$A < 1.3 \times 10^{-2} \quad (k_p = 2.8 \ \mathrm{pc}^{-1})$$

$$\Rightarrow \quad \beta < 10^{-1.7\times 10^3} \sim 0 \quad (M_{\mathrm{PBH}} = 8 \times 10^{33}\mathrm{g})$$

- Binary pulsars

$$A < (1.4\text{–}1.2)\times 10^{-1} \quad (k_p = 6.5\times 10^{-3}\text{–}2.8 \ \mathrm{pc}^{-1})$$

$$\Rightarrow \quad \beta < 4.2\times 10^{-5}\text{–}4.2\times 10^{-4} \quad (M_{\mathrm{PBH}} = 2\times 10^{39}\text{–}8\times 10^{33}\mathrm{g})$$

Direct detection of the indeced GWs

- For optimal frequency of ground-based interferometers $\sim$ 100Hz ,

$$\Omega_{\rm GW} h^2 = 2.0 A^4.$$

The frequency $\sim$ 100Hz corresponds to the mass $\sim 10^{13}$g .

For typical amplitude of the scalar perturbations which lead to large number of PBHs, $A^2 \sim O(10^{-3})$,

$$\Omega_{\rm GW} h^2 = 2.0 \times 10^{-7}.$$

$\Rightarrow$ The energy density of GWs induced by the scalar perturbations is large enough to be detected by ground-based interferometers (sensitivity $\sim 10^{-7} - 10^{-9}$) if a large number of PBHs with mass $\sim 10^{13}$g were generated .

Summary

- The primordial scalar perturbations with large amplitude generate gravitational waves through tensor-scalar mode couplings.

- The induced gravitational waves can be used to constrain the abundance of primordial black holes.

- The induced gravitational waves can be detected by ground-based interferometers if a large number of PBHs with mass $\sim 10^{13}\mathrm{g}$ were generated.