Hirotaka Yoshino, JGRG 22(2012)111412

"Axion bosenova"

RESCEU SYMPOSIUM ON

GENERAL RELATIVITY AND GRAVITATION

JGRG 22

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Axion Bosenova

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JGRG22 @ RESCEU University of Tokyo
(November 14, 2012)

Contents

Introduction

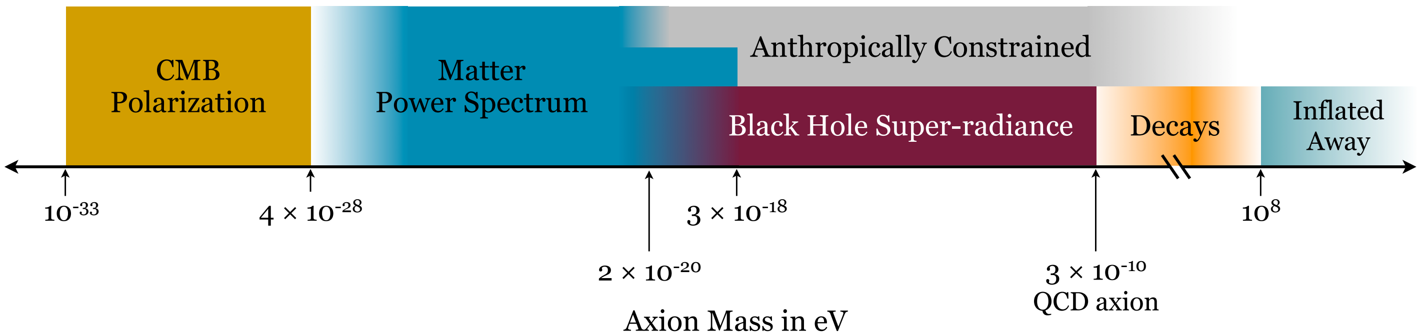
Axiverse

- QCD axion
  - QCD axion was introduced to solve the Strong CP problem.
  - It is one of the candidates of dark matter.

- String axions

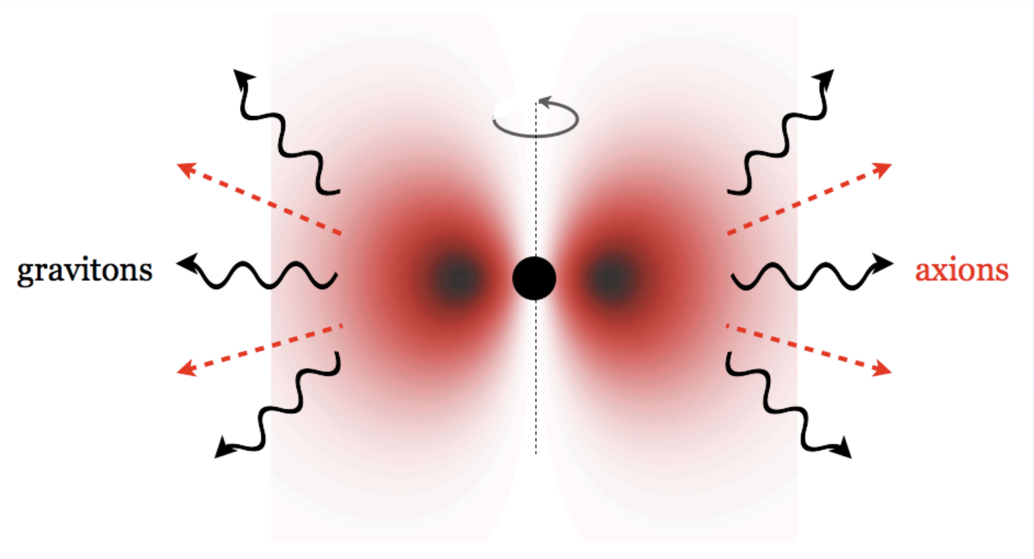
Arvanitaki, Dimopoulos, Dubvosky, Kaloper, March-Russel, PRD81 (2010), 123530.

  - String theory predicts the existence of 10-100 axion-like massive scalar fields.
  - There are various expected phenomena of string axions.



Axion field around a rotating black hole

- Axion field forms a cloud around a rotating BH and extract energy of the BH by "superradiant instability".



Arvanitaki, Dimopoulos, Dubvosky, Kaloper, March-Russel, PRD81 (2010), 123530.

Arvanitaki and Dubovsky, PRD83 (2011), 044026.

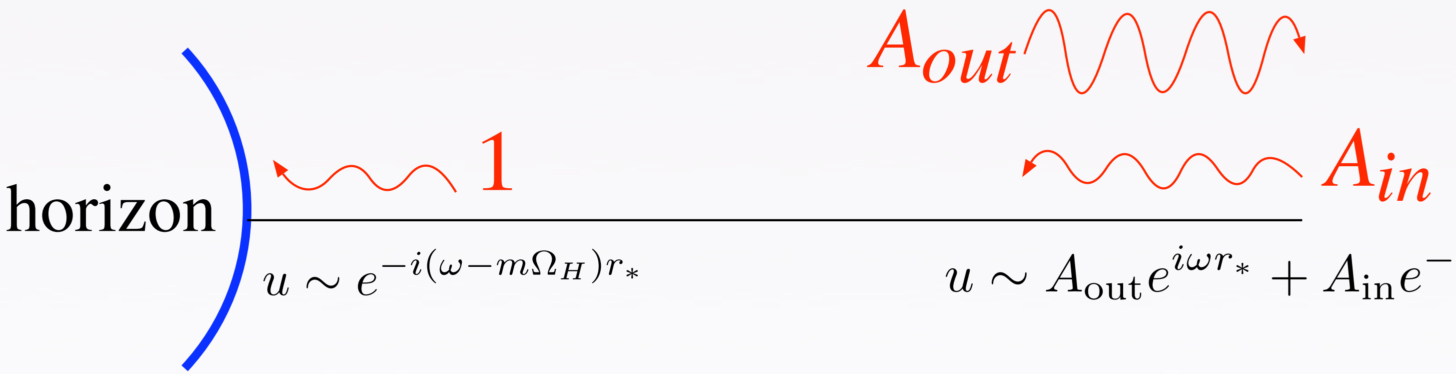
Superradiance

Zel'dovich (1971)

- Massless Klein-Gordon field $\nabla^2\Phi = 0$

$$\Phi = \mathrm{Re}[e^{-i\omega t}R(r)S(\theta)e^{im\phi}]$$

$$R = \frac{u}{\sqrt{r^2+a^2}} \Rightarrow \frac{d^2u}{dr_*^2} + \left[\omega^2 - V(\omega)\right]u = 0$$



$$u \sim e^{-i(\omega - m\Omega_H)r_*}$$

$$u \sim A_{\mathrm{out}}e^{i\omega r_*} + A_{\mathrm{in}}e^{-i\omega r_*}$$

$$\left(1 - \frac{m\Omega_H}{\omega}\right)|T|^2 = 1 - |R|^2$$

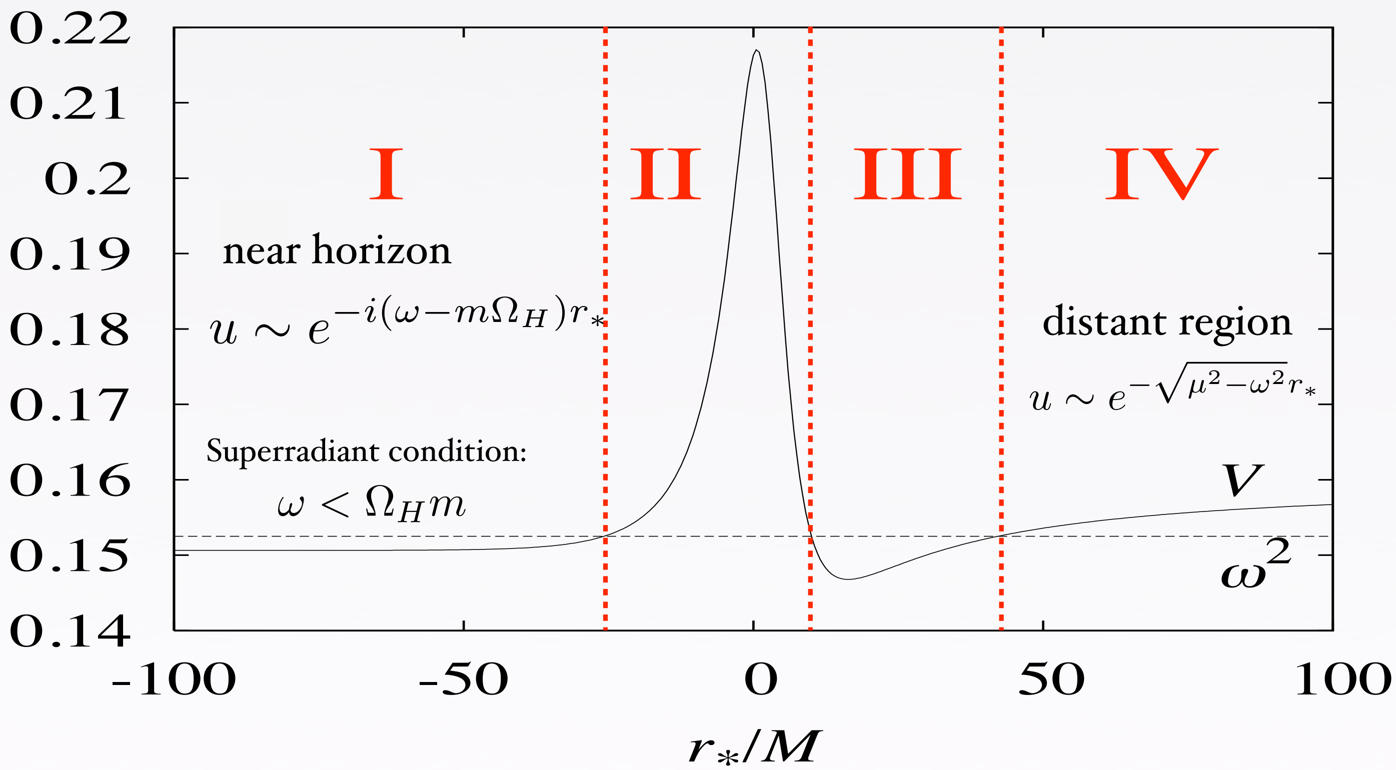
Superradiant condition:

$$\omega < \Omega_H m$$

Bound state

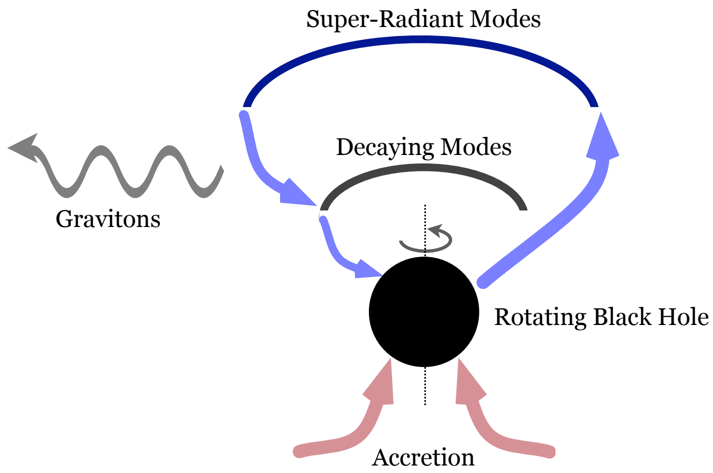
Zouros and Eardley, Ann. Phys. 118 (1979), 139.
Detweiler, PRD22 (1980), 2323.

- Massive Klein-Gordon field $\nabla^2\Phi - \mu^2\Phi = 0$



Typical time scale $\sim 10^7 M \sim \begin{cases} 50 \text{ s} & (M = 10 M_\odot) \\ 1.6 \text{ year} & (M = 10^6 M_\odot) \end{cases}$

BH-axion system



Arvanitaki and Dubovsky, PRD83 (2011), 044026.

- Superradiant instability
  - Emission of gravitational waves (Level transition, Pair annihilation of axions)
- Effects of nonlinear self-interaction
  - Bosenova
  - Mode mixing

Nonlinear effect

- Typically, the potential of axion field becomes periodic

$$V = f_a^2 \mu^2 [1 - \cos(\Phi / f_a)]$$

$$\Rightarrow \quad \boxed{\nabla^2 \varphi - \mu^2 \sin\varphi = 0} \qquad \varphi \equiv \frac{\Phi}{f_a}$$

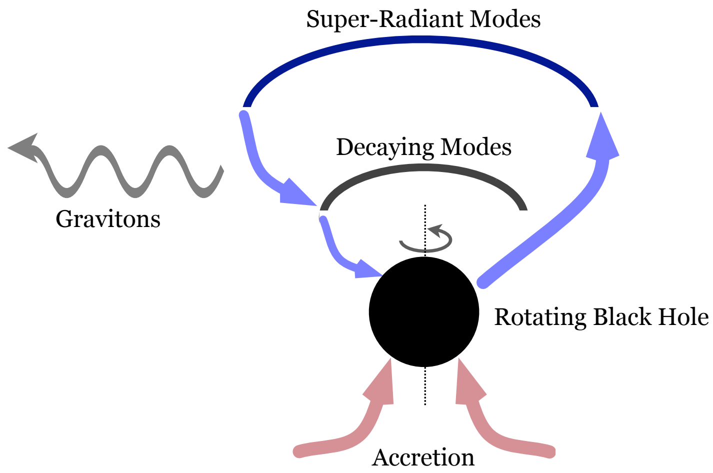
- c.f., QCD axion

PQ phase transition

QCD phase transition

U(1)PQ symmetry ⇒ Potential becomes like a wine bottle ⇒ Z(N) symmetry

BH-axion system

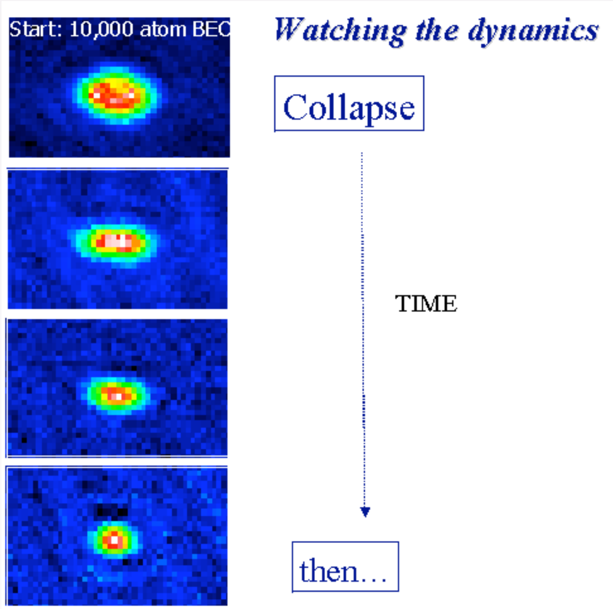


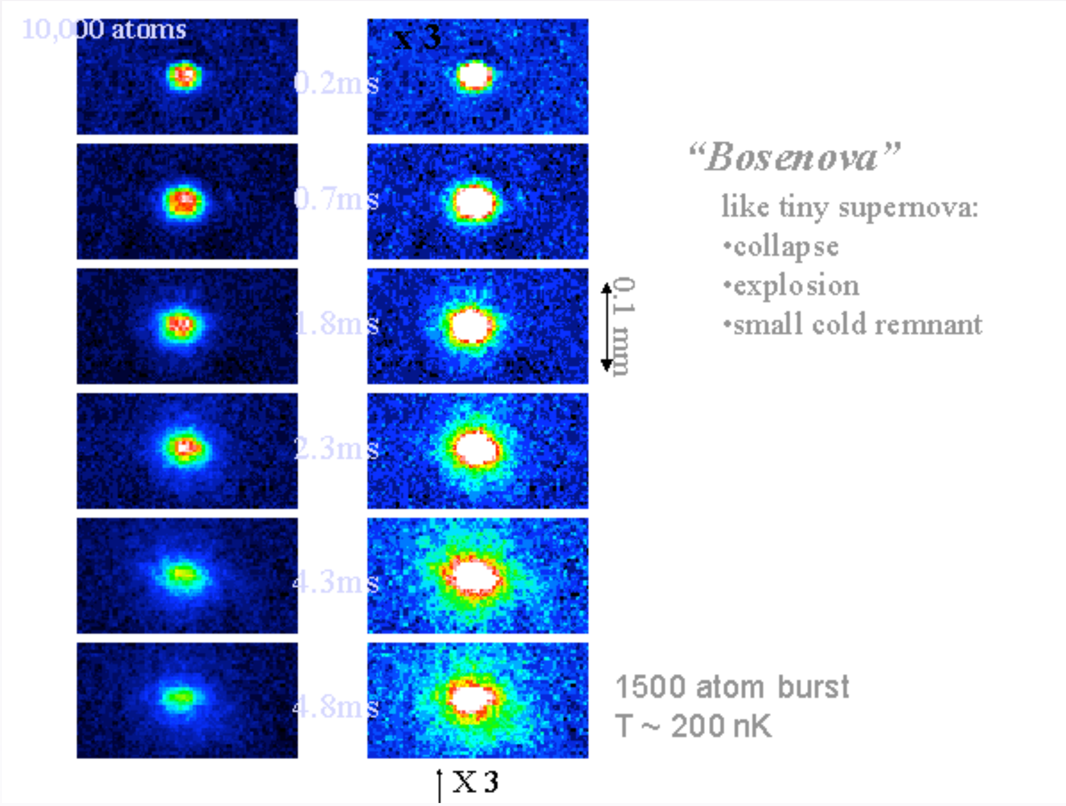
Arvanitaki and Dubovsky, PRD83 (2011), 044026.

- Superradiant instability
  - Emission of gravitational waves
    (Level transition, Pair annihilation of axions)
- Effects of nonlinear self-interaction
  - Bosenova
  - Mode mixing

Bosenova in condensed matter physics

http://spot.colorado.edu/~cwieman/Bosenova.html





BEC state of Rb85 (interaction can be controlled)

Switch from repulsive interaction to attractive interaction

Wieman et al., Nature 412 (2001), 295

What we would like to do

- We would like to study the phenomena caused by axion cloud generated by the superradiant instability around a rotating black hole.

- In particular, we study numerically whether "Bosenova" happens when the nonlinear interaction becomes important.

- We adopt the background spacetime as the Kerr spacetime, and solve the axion field as a test field.

Simulations

- Typical two simulations
- Does the bosenova really happen?

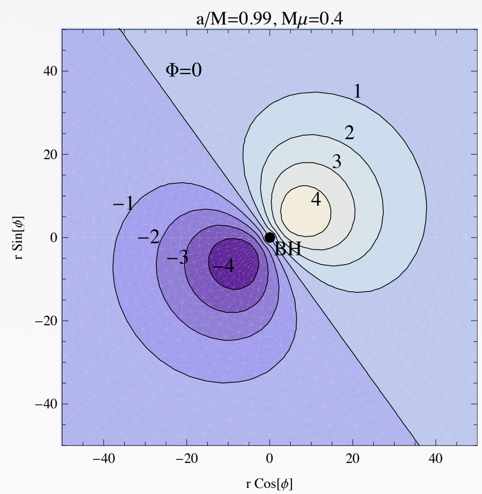
Numerical simulation

- Sine-Gordon equation

$$\nabla^2\varphi - \mu^2 \sin\varphi = 0$$

- Setup $\quad a/M = 0.99, \quad M\mu = 0.4$

As the initial condition, we choose the bound state of the Klein-Gordon field of the $l = m = 1$ mode.



	Initial peak value	$E/[(f_a/M_p)^2 M]$
(A)	0.6	1370
(B)	0.7	1862

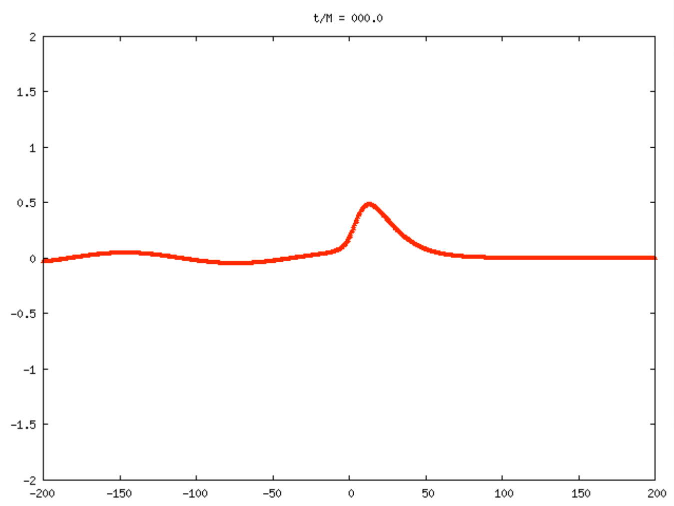
Simulation (A)

$\varphi_{\mathrm{peak}}(0) = 0.6$

- Axion field on the equatorial plane $(\theta = \pi/2)$

$(\phi = 0)$

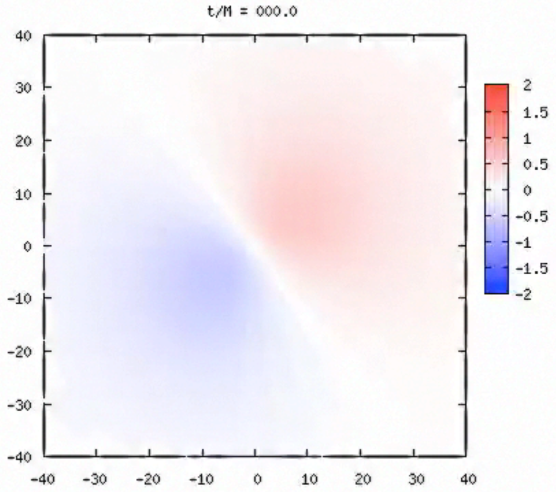
Φ



$-200 \leq r_*/M \leq 200$

(Equatorial plane)

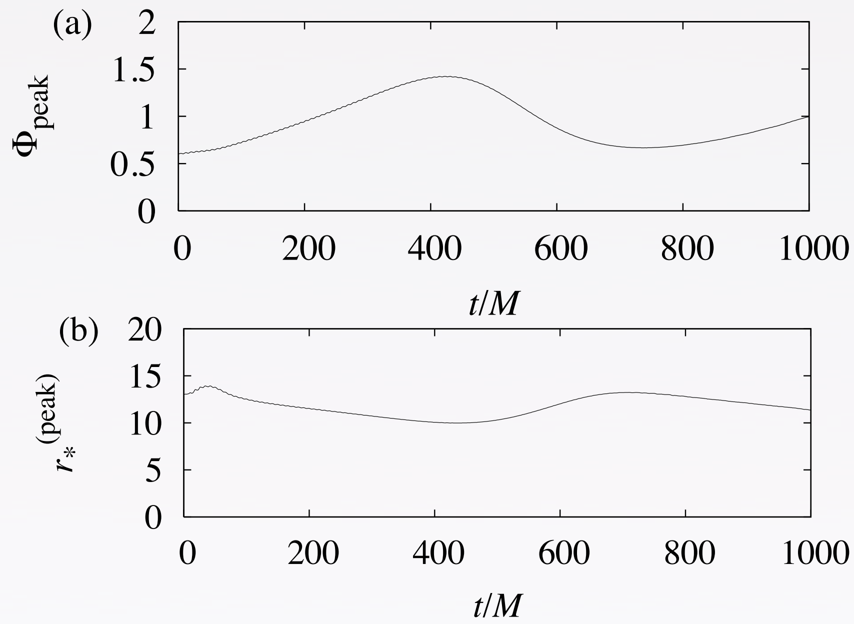
$r \sin\phi$



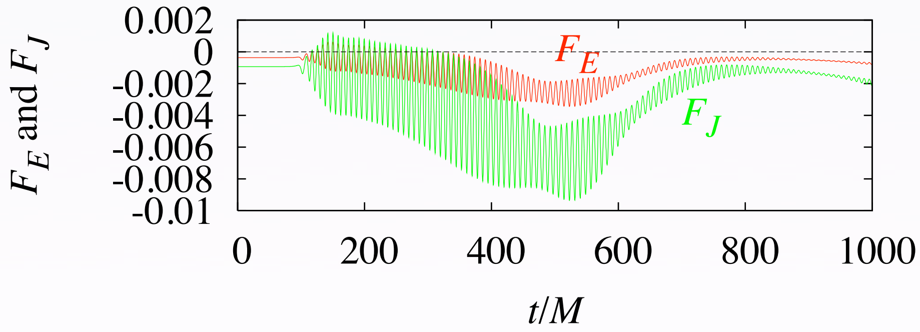
$r \cos\phi$

Simulation (A)

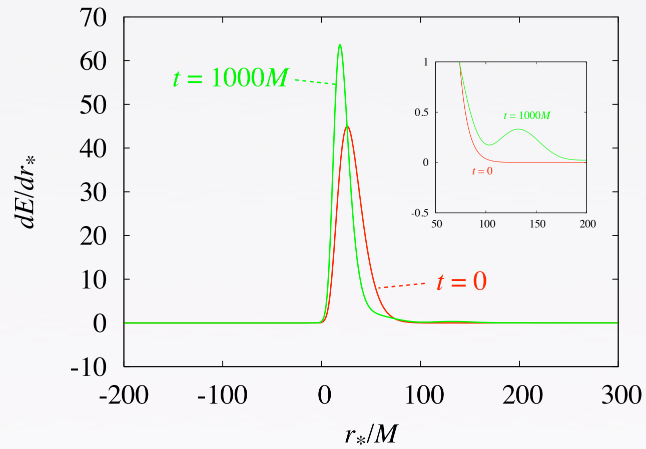
- Peak value and peak location

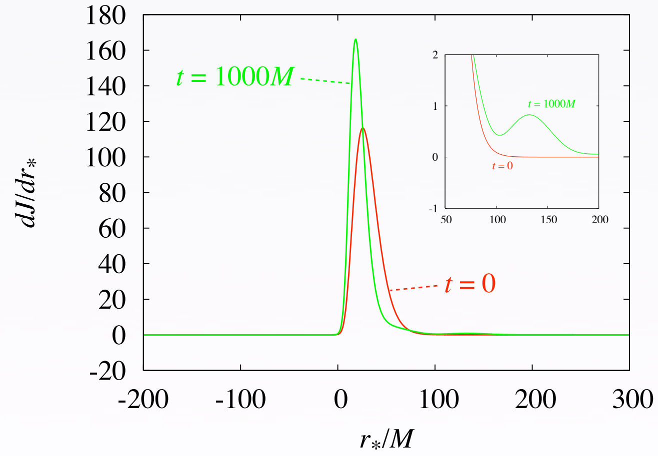


- Fluxes toward the horizon



- Energy and angular momentum distribution





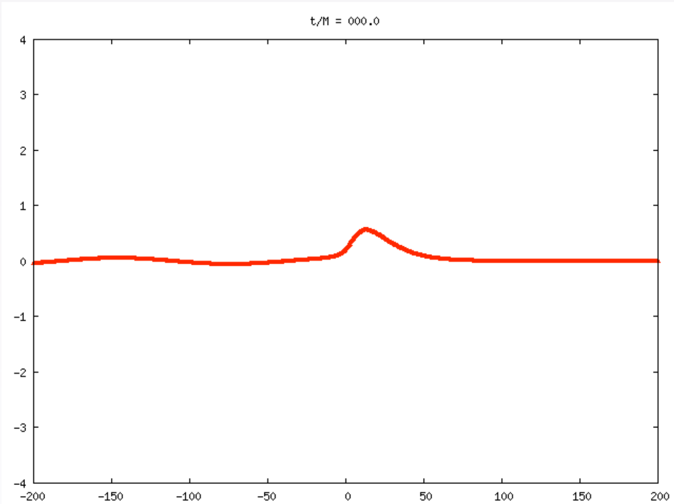
Simulation (B)

$\varphi_{\mathrm{peak}}(0) = 0.7$

- Axion field on the equatorial plane $(\theta = \pi/2)$

$(\phi = 0)$

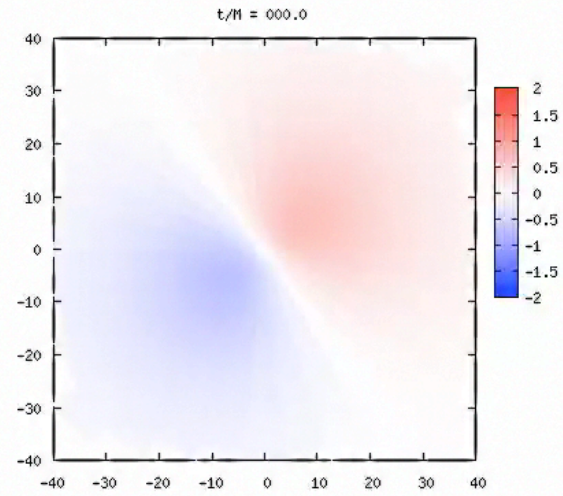
Φ



$-200 \leq r_*/M \leq 200$

(Equatorial plane)

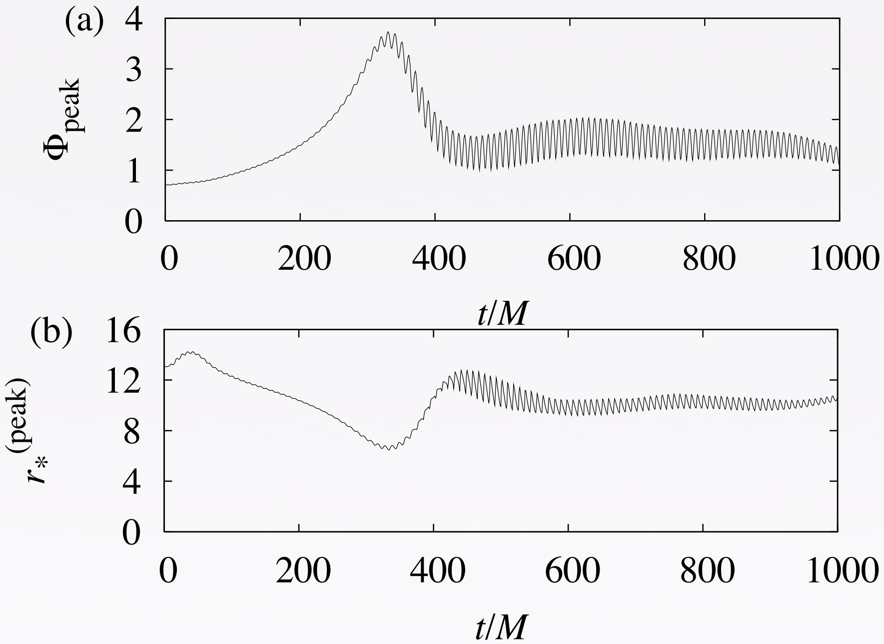
$r \sin \phi$



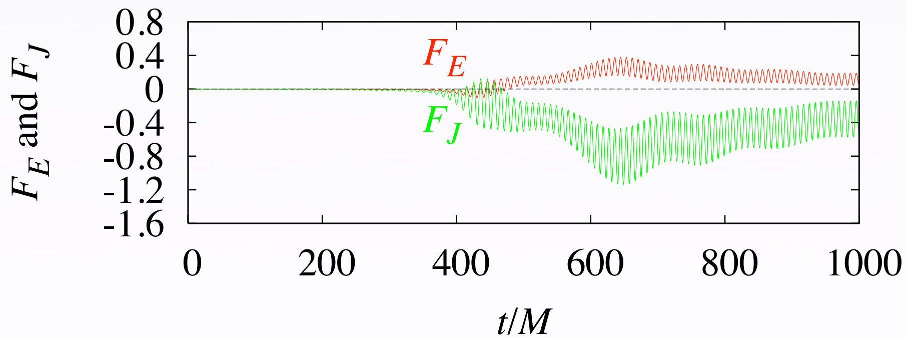
$r \cos \phi$

Simulation (B)

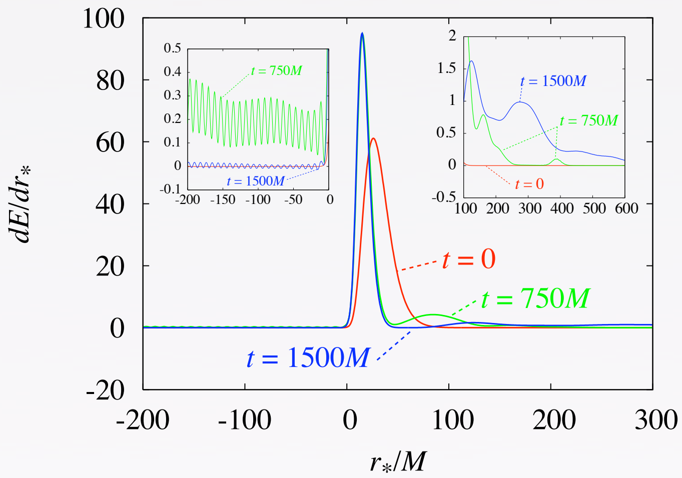
Peak value and peak location

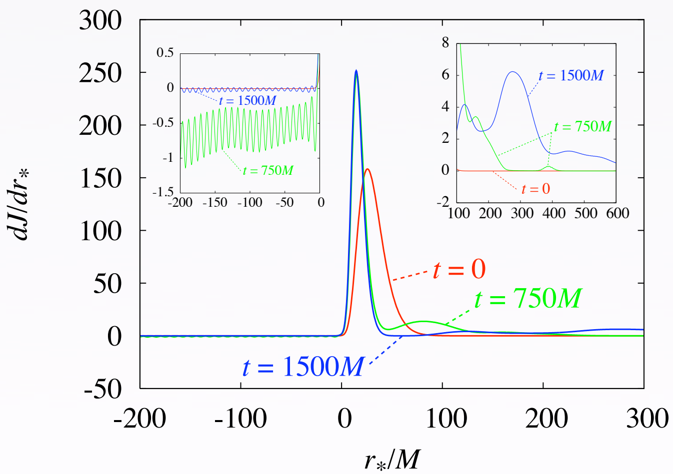


Fluxes toward the horizon



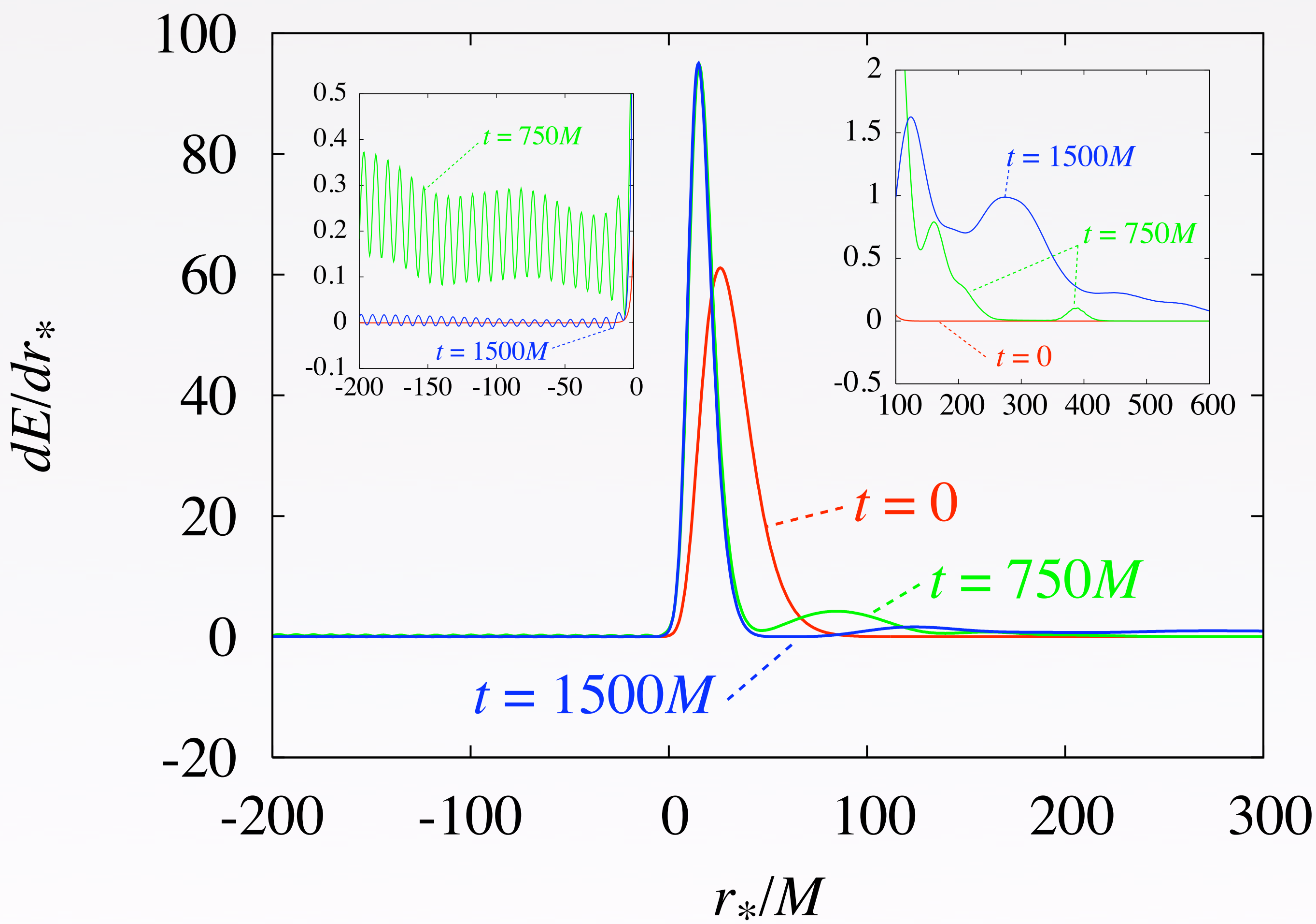
Energy and angular momentum distribution





Simulation (B)

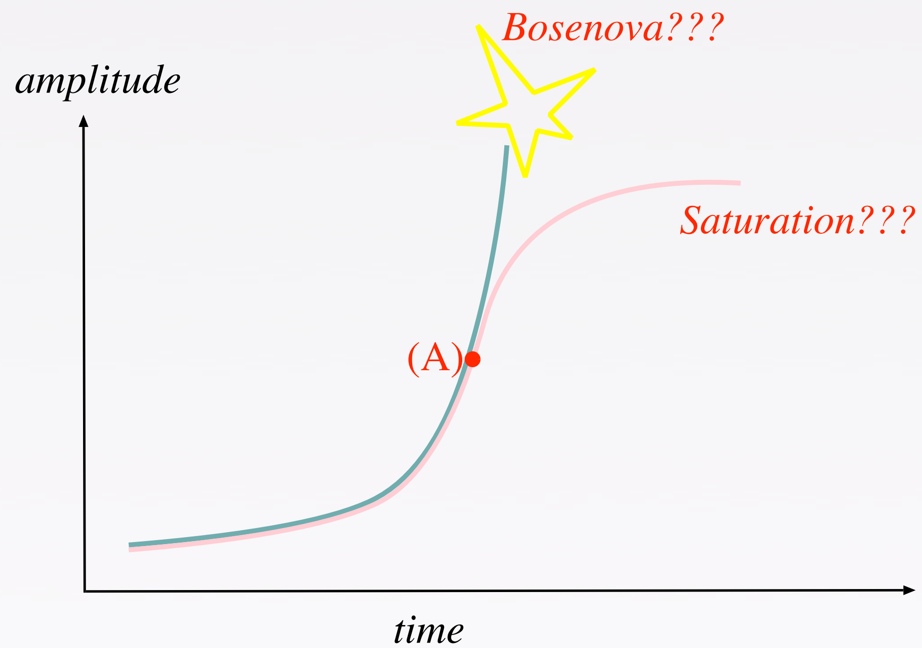
- Energy distribution



Simulations

- Typical two simulations
- Does the bosenova really happen?

Does bosenova really happen?



- Additional simulation:

$$\varphi(0) = C\varphi^{(\mathrm{A})}(1000M)$$
$$\dot{\varphi}(0) = C\dot{\varphi}^{(\mathrm{A})}(1000M)$$

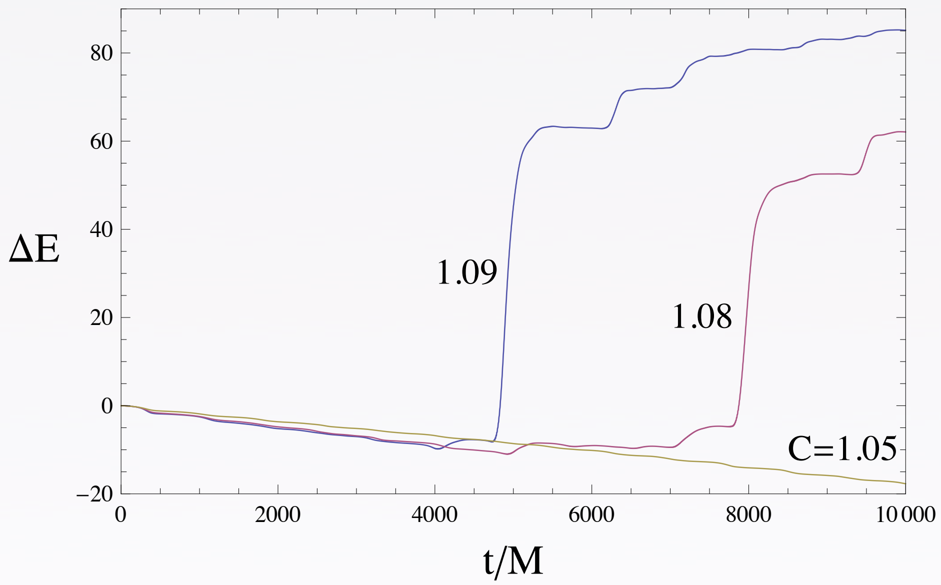
$$C = \begin{cases} 1.05 \\ 1.08 \\ 1.09 \end{cases}$$

Supplementary simulation

$$\varphi(0) = C\varphi^{(\mathrm{A})}(1000M)$$
$$\dot\varphi(0) = C\dot\varphi^{(\mathrm{A})}(1000M)$$

- Energy absorbed by the black hole $\Delta E := \int_0^t F_E dt$

$$C = \begin{cases} 1.05 \\ 1.08 \\ 1.09 \end{cases}$$

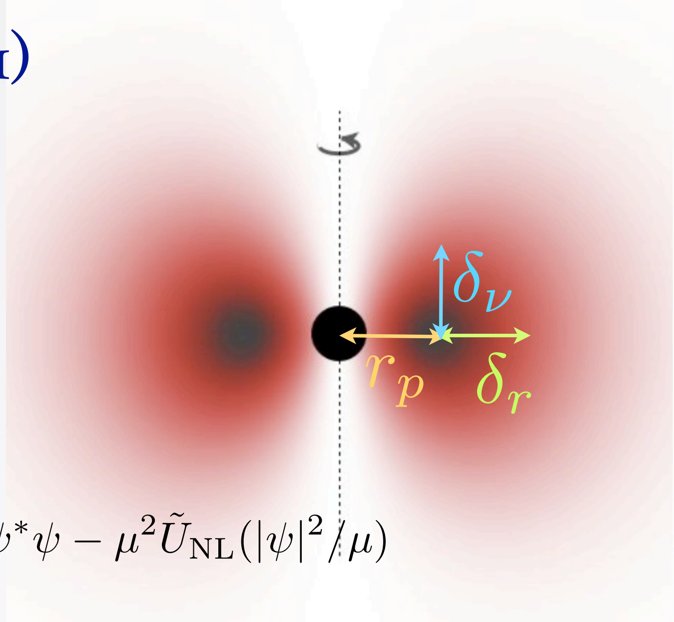


The bosenova happens when $E \simeq 1600 \times (f_a/M_p)^2 M$

Discussions

- Effective theory
- Gravitational waves

Effective theory (1)



- Action $\hat{S} = \int d^4x\sqrt{-g}\left[-\frac{1}{2}(\nabla\varphi)^2 - \mu^2\left(\frac{\varphi^2}{2} + \hat{U}_{\rm NL}(\varphi)\right)\right],$

- Non-relativistic approximation

$$\varphi = \frac{1}{\sqrt{2\mu}}\left(e^{-i\mu t}\psi + e^{i\mu t}\psi^*\right)$$

$$\hat{S}_{\rm NR} = \int d^4x\left[\frac{i}{2}\left(\psi^*\dot{\psi} - \psi\dot{\psi}^*\right) - \frac{1}{2\mu}\partial_i\psi\partial_i\psi^* + \frac{\alpha_g}{r}\psi^*\psi - \mu^2\tilde{U}_{\rm NL}(|\psi|^2/\mu)\right)$$

$$\tilde{U}_{\rm NL}(x) = -\sum_{n=2}^{\infty}\frac{(-1/2)^n}{(n!)^2}x^n.$$

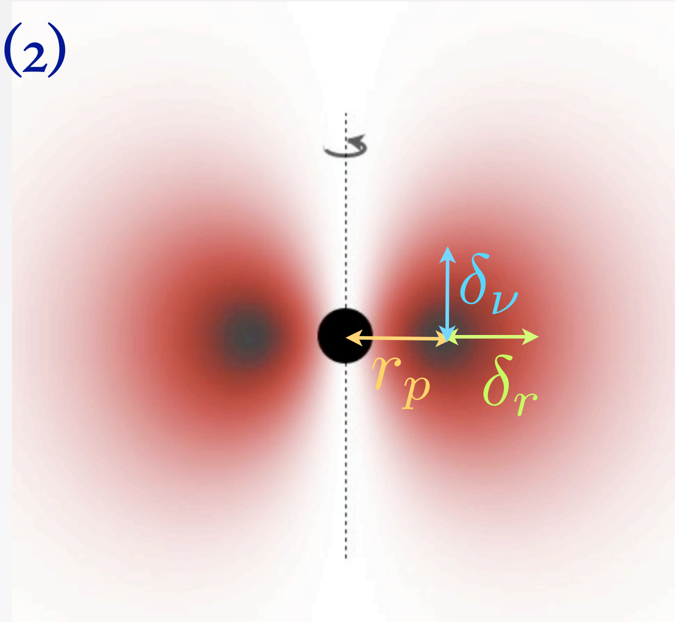
- Approximate the axion cloud as a Gaussian wavepacket

$$\psi = A(t,r,\nu)e^{iS(t,r,\nu)+im\phi} \qquad (\nu = \cos\theta)$$

$$A(t,r,\nu) \approx A_0\exp\left[-\frac{(r-r_p)^2}{4\delta_r r_p^2} - \frac{(\nu-\nu_p)^2}{4\delta_\nu}\right],$$

$$S(t,r,\nu) \approx S_0(t) + p(t)(r-r_p) + P(t)(r-r_p)^2 + \pi_\nu(t)(\nu-\nu_p)^2 + \cdots,$$

Effective theory (2)



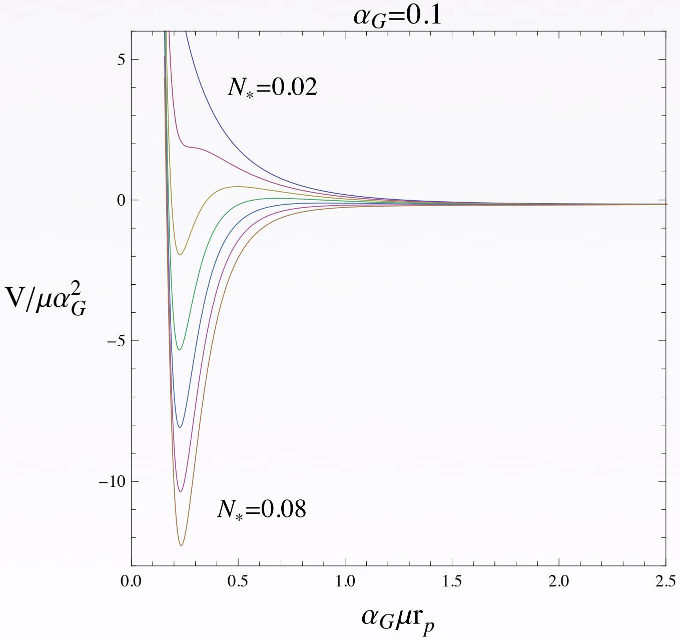
- Effective Lagrangian: $L = T - V$

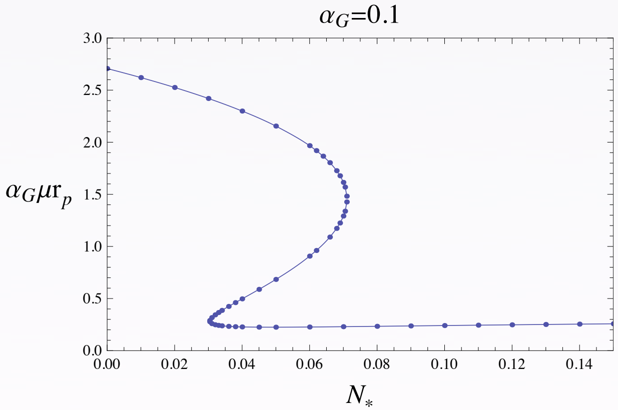
$$T = \frac{1}{2}A\dot{\delta}_r^2 + B\dot{\delta}_r\dot{r}_p + \frac{1}{2}C\dot{r}_p^2 + \frac{1}{2}D\dot{\delta}_\nu^2,$$

$$\frac{V}{\mu\alpha_g^2} = \frac{1}{2(\alpha_g\mu r_p)^2(1+\delta_r)}\left(1+\delta_\nu+\frac{1}{4\delta_r}+\frac{1}{4\delta_\nu}\right) - \frac{1}{(\alpha_g\mu r_p)(1+\delta_r)}$$
$$-\alpha_g^{-2}\sum_{n=2}^{\infty}\frac{(-1/2)^n}{(n!)^2 n}\left[\frac{N_*}{\sqrt{\delta_r\delta_\nu}(\alpha_g\mu r_p)^3(1+\delta_r)}\right]^{n-1}.$$

- Potential

$\alpha_g = 0.1$





Small oscillations

- Oscillation around a equilibrium point $\Delta q_i = (\Delta\delta_r, \Delta\delta_\nu, \alpha_g \mu \Delta r_p)$

$$\Rightarrow \quad \frac{d^2(\Delta q_i)}{dt^2} = -\sum_j \omega_{ij} \Delta q_j$$

- Oscillation frequencies
  - $\alpha_g = 0.4, N_* = 1.1$

$$\omega^2 = 1.141, \quad 0.249, \quad 0.0166, \qquad \boxed{\Delta t \approx 761M}$$

$$\Rightarrow \quad \delta q = \begin{pmatrix} 0.110 \\ -0.027 \\ 0.994 \end{pmatrix}, \quad \begin{pmatrix} 0.075 \\ 0.724 \\ 0.686 \end{pmatrix}, \quad \begin{pmatrix} -0.378 \\ -0.005 \\ 0.925 \end{pmatrix}.$$

  - $\alpha_g = 0.4, N_* = 1.3$

$$\omega^2 = 14.06, \quad 5.59, \quad 0.175, \qquad \boxed{\Delta t \approx 26M}$$

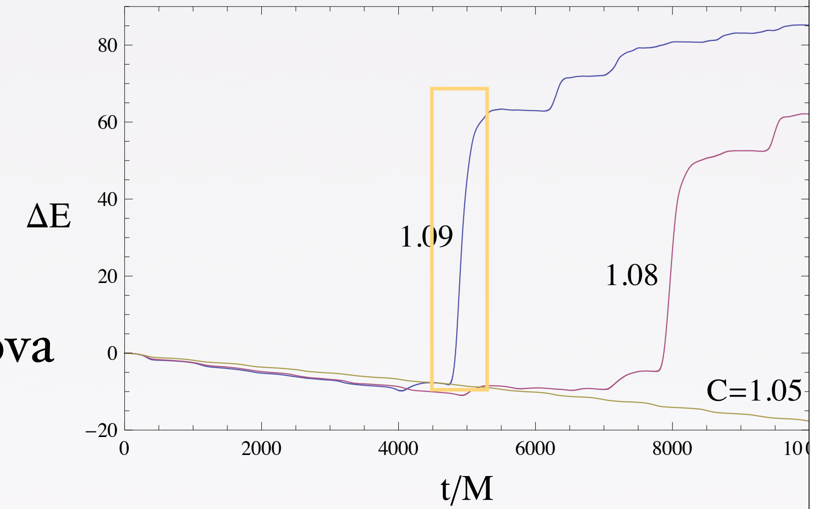
$$\Rightarrow \quad \Delta q = \begin{pmatrix} 0.218 \\ -0.030 \\ 0.975 \end{pmatrix}, \quad \begin{pmatrix} 0.070 \\ 0.927 \\ 0.367 \end{pmatrix}, \quad \begin{pmatrix} -0.640 \\ -0.085 \\ 0.763 \end{pmatrix}.$$

Discussions

- Axion cloud model
- Gravitational waves

GWs emitted in the bosenova (rough estimate)

- Assumption:
  - Quadrupole approximation
  - Change in the quadrupole moment by the infall of energy in the bosenova



$$Q_{ij} \sim r_p^2 E$$

r_p: $10M$

E: $E_0 + \frac{1}{2}(\Delta E)\left[\cos\left(\pi \frac{t}{\Delta t}\right) - 1\right]$

E_0: $10^{-3}M$, ΔE: $0.05E_0$, Δt: $500M$

- Amplitude of generated GWs $$h \sim \frac{\ddot{Q}_{ij}}{r_{\rm obs}} \sim 10^{-7}\frac{M}{r_{\rm obs}}$$

Detectability

$$h \sim \frac{\ddot{Q}_{ij}}{r_{\rm obs}} \sim 10^{-7}\frac{M}{r_{\rm obs}}$$

- Supermassive BH of our galaxy (Sagittarius A*) $(10^{-4}\ \mathrm{Hz})$

$$h_{\rm rss} := \left[\int |h|^2 dt\right]^{1/2} \sim 10^{-16}(\mathrm{Hz})^{-1/2}$$

Angular frequency of GW

⇨ *Detectable by the eLISA*

- Solar-mass BH (e.g., Cygnus X-1) $(10^{2}\ \mathrm{Hz})$

$$h_{\rm rss} \sim 10^{-24}(\mathrm{Hz})^{-1/2}$$

⇨ *below the sensitivity of the KAGRA, Advanced LIGO, Advanced Virgo, etc.*

Summary

Summary

- We developed a reliable code and numerically studied the behaviour of axion field around a rotating black hole.
- The nonlinear effect enhances the rate of superradiant instability when the amplitude is not very large.
- The bosenova collapse would happen as a result of superradiant instability.

Ongoing studies

- Calculation of the gravitational waves emitted in bosenova.
- The case where axions couple to magnetic fields.

Appendix

Superradiant instability

Massive scalar fields around a Kerr BH

- Metric

$$ds^2 = -\left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)dt^2 - \frac{2a\sin^2\theta(r^2+a^2-\Delta)}{\Sigma}dtd\phi + \left[\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right]\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2$$

$$\Sigma = r^2 + a^2\cos^2\theta,$$
$$\Delta = r^2 + a^2 - 2Mr.$$

- Massive scalar field
  - Lagrangian density

$$\mathcal{L} = -\sqrt{-g}\left[\frac{1}{2}g^{ab}\nabla_a\Phi\nabla_b\Phi + U(\Phi)\right], \quad U(\Phi) = \frac{1}{2}\mu^2 f_a^2\sin^2(\Phi/f_a) \simeq \frac{1}{2}\mu^2\Phi^2$$

  - Klein-Gordon equation $\nabla^2\Phi - U'(\Phi) = 0$

Massive scalar field around a Kerr BH

- Separation of variables $\Phi = e^{-i\omega t} R(r) S(\theta) e^{im\phi}$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{dS}{d\theta} + \left[-k^2a^2\cos^2\theta - \frac{m^2}{\sin^2\theta} + E_{lm}\right]S = 0$$

$$\frac{d}{dr}\Delta\frac{dR}{dr} + \left[\frac{K^2}{\Delta} - \lambda_{lm} - \mu^2 r^2\right]R = 0$$

$$K = (r^2 + a^2)\omega - am$$

$$k^2 = \mu^2 - \omega^2$$

$$\lambda_{lm} = E_{lm} + a^2\omega^2 - 2am\omega$$

Massive scalar field around a Kerr BH

$$\frac{d}{dr}\Delta\frac{dR}{dr}+\left[\frac{K^2}{\Delta}-\lambda_{lm}-\mu^2r^2\right]R=0$$

- distant region $R\sim r^{-1\pm\frac{\mu^2-2\omega^2}{k}}\exp(\pm kr)$ $k=\sqrt{\mu^2-\omega^2}$

- near horizon $R\sim e^{\pm i\omega_* r_*}$ (outgoing / ingoing) $\boxed{\omega_*=\omega-\Omega_H m}$

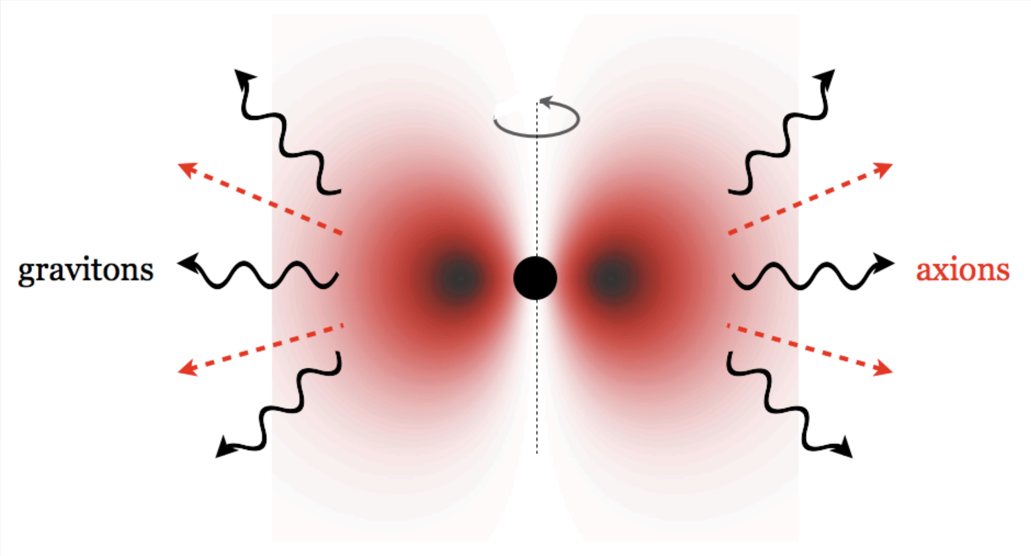
(tortoise coordinate) r_* $dr_*=\frac{r^2+a^2}{\Delta}dr$

- Energy flux $P^\mu=-T^\mu_\nu\xi^\nu$

$$-P_{r_*}=\nabla_{r_*}\Phi\nabla_t\Phi\propto(\omega-\Omega_H m)\omega$$

If $\omega<\Omega_H m$, negative energy falls into the black hole ⇨ superradiance

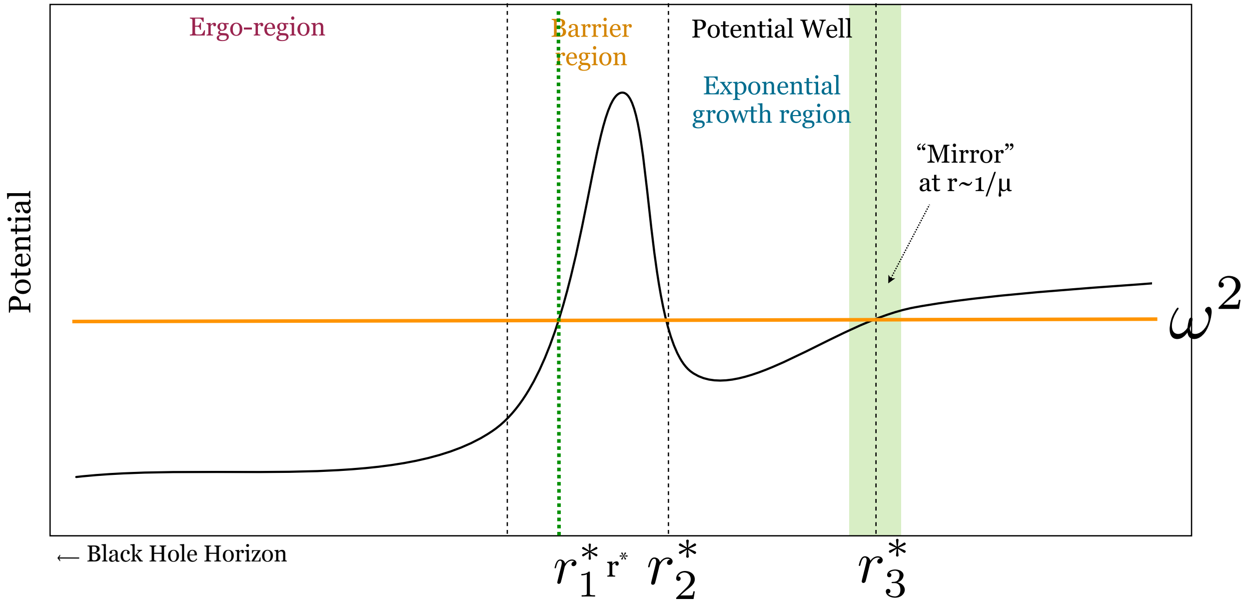
Bound state



- Matching method — Detweiler, PRD22 (1980), 2323. — $M\mu \simeq M\omega \ll 1$
- WKB method — Zouros and Eardley, Ann. Phys. 118 (1979), 139. — $M\mu \gg 1$
- Numerical analysis — Dolan, PRD76 (2007), 084001.

Bound state

Zouros and Eardley, Ann. Phys. 118 (1979), 139.



$$R = \frac{u}{\sqrt{r^2 + a^2}} \quad \Rightarrow \quad \frac{d^2 u}{dr_*^2} + \left[\omega^2 - V(\omega)\right] u = 0$$

$$V(\omega) = \frac{\Delta \mu^2}{r^2 + a^2} + \frac{4Mam\omega r - a^2 m^2 + \Delta\left[E_{lm} + (\omega^2 - \mu^2)a^2\right]}{(r^2 + a^2)^2} + \frac{\Delta}{(r^2 + a^2)^3}\left(3r^2 - 4Mr + a^2 - \frac{3\Delta r^2}{r^2 + a^2}\right)$$

Qualitative discussion on "Bosenova"

Arvanitaki and Dubovsky, PRD83 (2011), 044026.

- action

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} g^{ab} \nabla_a \Phi \nabla_b \Phi + U(\Phi) \right]$$

$$U(\Phi) = \frac{1}{2} \mu^2 f_a^2 \sin^2(\Phi/f_a)$$

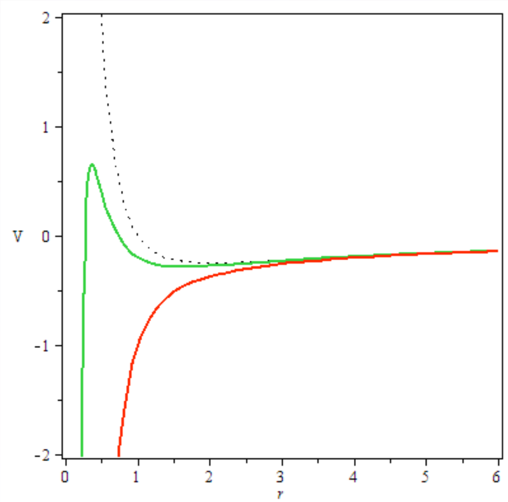
- Nonrelativistic approximation

$$\Phi = \frac{1}{\sqrt{2\mu}} \left(e^{-i\mu t}\psi + e^{i\mu t}\psi^* \right)$$

$$S = \int d^4x \left[i\Psi^* \partial_t \psi - \frac{1}{2\mu} \partial_i \psi \partial_i \psi - \mu V_N \psi^* \psi + \frac{1}{16 f_a^2} (\psi^* \psi)^2 \right]$$

- Effective potential

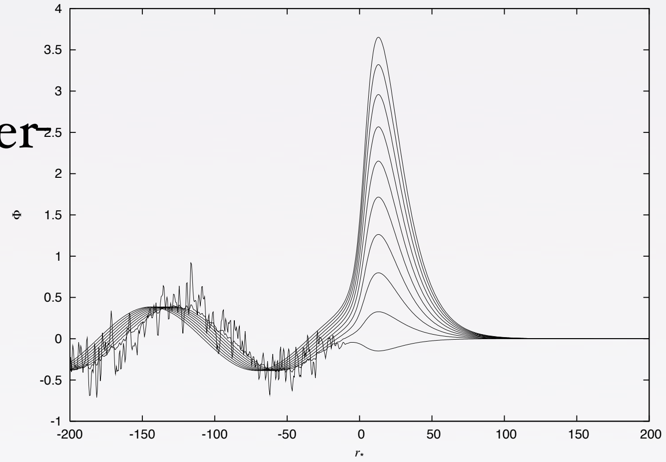
$$V(r) \approx N \frac{l(l+1)+1}{2\mu r^2} - \frac{NM\mu}{r} - \frac{N^2}{32\pi f_a^2 r^3}$$



Code

First difficulty

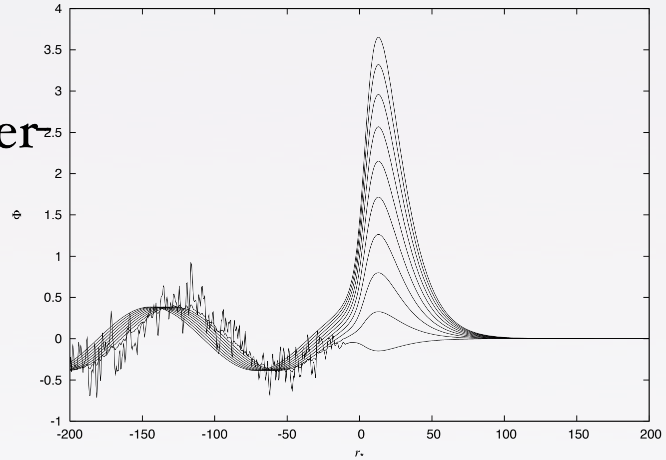
- Stable simulation cannot be realized in Boyer-Lindquist coordinates.

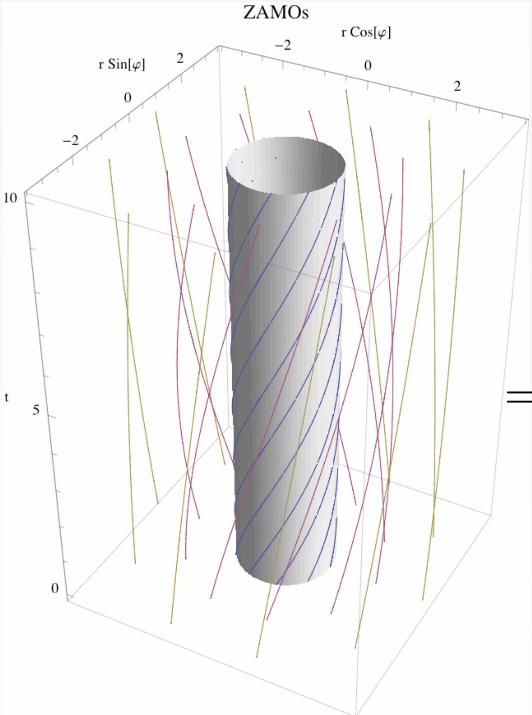


First difficulty

- Stable simulation cannot be realized in Boyer-Lindquist coordinates.

➡ We use ZAMO coordinates.

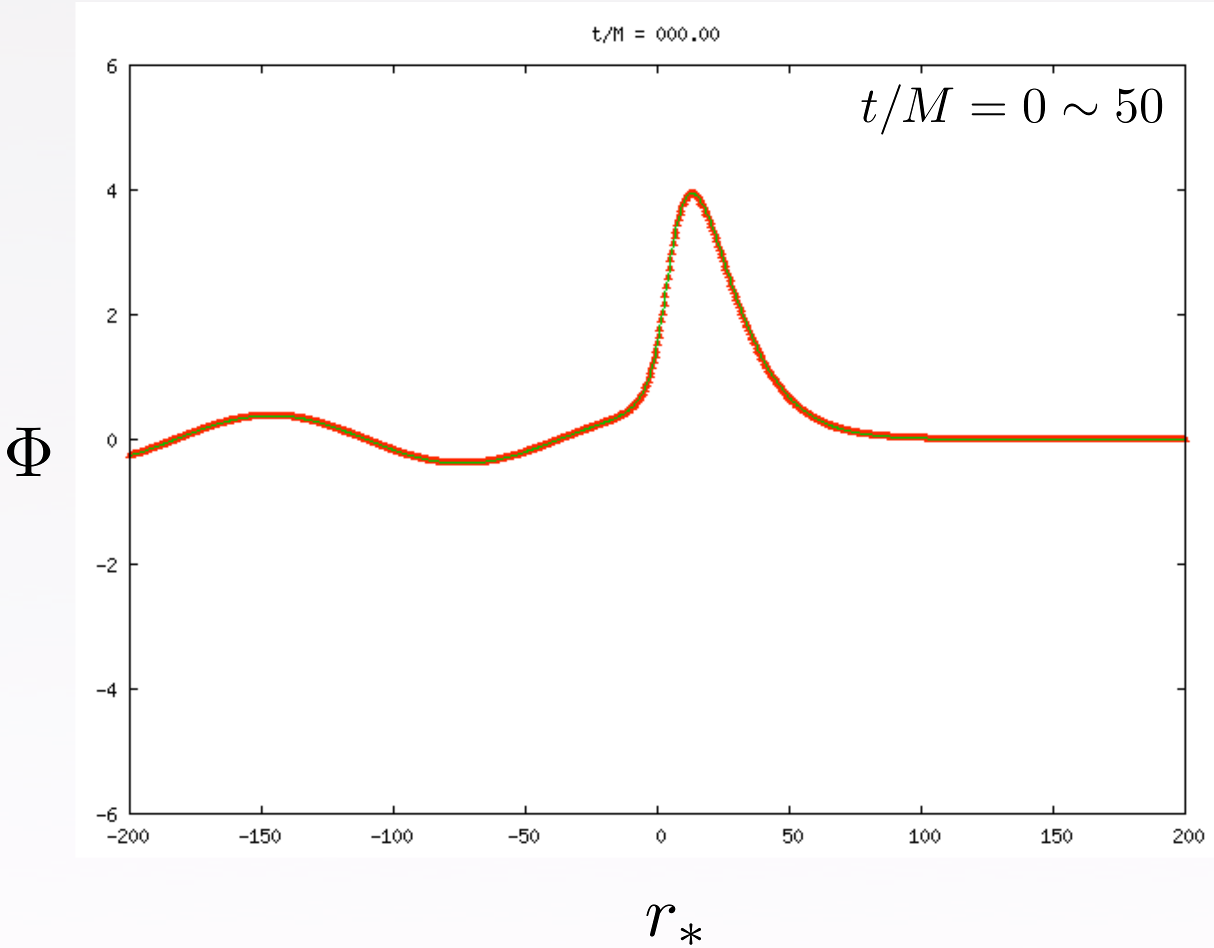




$$\Omega = \frac{d\phi}{dt} = \frac{u^\phi}{u^t} = -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{2Mar}{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}$$

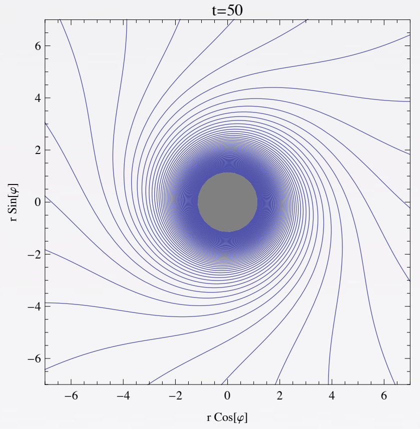
$$\begin{aligned} \tilde{t} &= t, \\ \tilde{\phi} &= \phi - \Omega(r,\theta)t, \\ \tilde{r} &= r, \\ \tilde{\theta} &= \theta, \end{aligned}$$

Numerical solution in the ZAMO coordinates

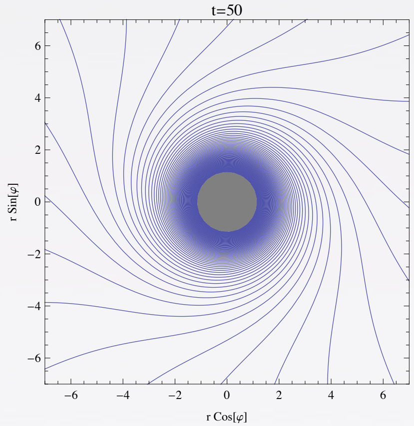


Second difficulty

- ZAMO coordinates become more and more distorted in the time evolution



Second difficulty

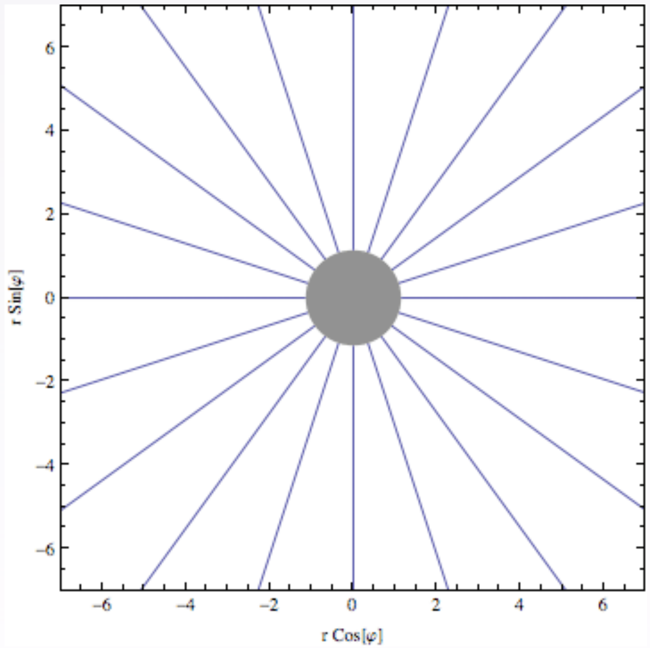


- ZAMO coordinates become more and more distorted in the time evolution

➡ We "pull back" the coordinates

$$nT_P \leq t \leq (n+1)T_P :$$

$$\begin{aligned} t^{(n)} &= t, \\ \phi^{(n)} &= \phi - \Omega(r,\theta)(t - nT_P), \\ r^{(n)} &= r, \\ \theta^{(n)} &= \theta. \end{aligned}$$

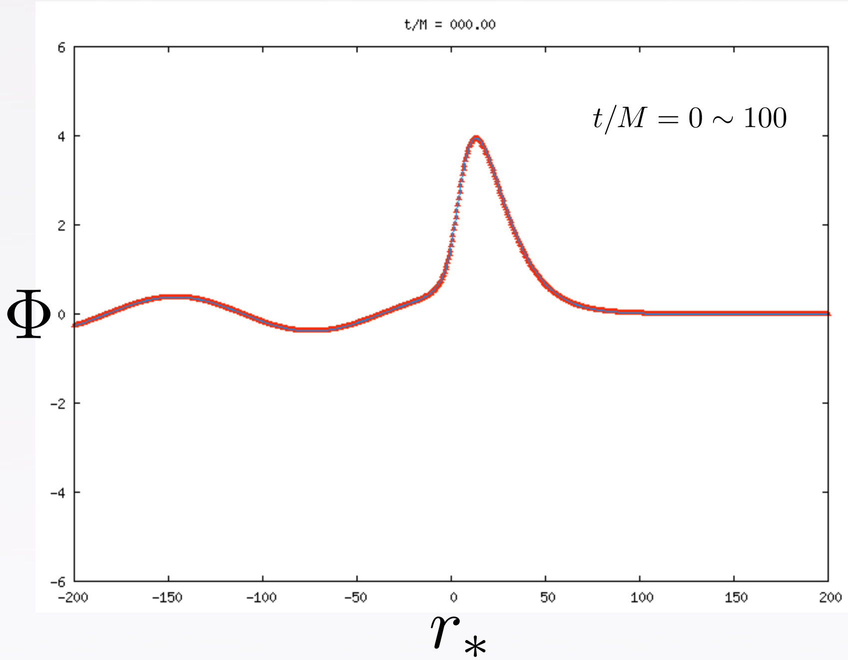


Our 3D code

- Space direction : 6th-order finite discretization
- Time direction : 4th-order Runge-Kutta
- Grid size: $$\Delta r_* = 0.5 \quad (M = 1)$$ $$\Delta\theta = \Delta\phi = \pi/30$$
- Courant number: $$C = \frac{\Delta t}{\Delta r_*} = \frac{1}{20}$$
- Pure ingoing BC at the inner boundary,
  Fixed BC at the outer boundary
- Pullback: 7th-order Lagrange interpolation

Code check (I)

- Comparison with semianalytic solution of the Klein-Gordon case



- Growth rate

$$\omega_I = \frac{\dot{E}}{2E} \simeq \frac{E(100M) - E(0)}{200ME(0)}$$

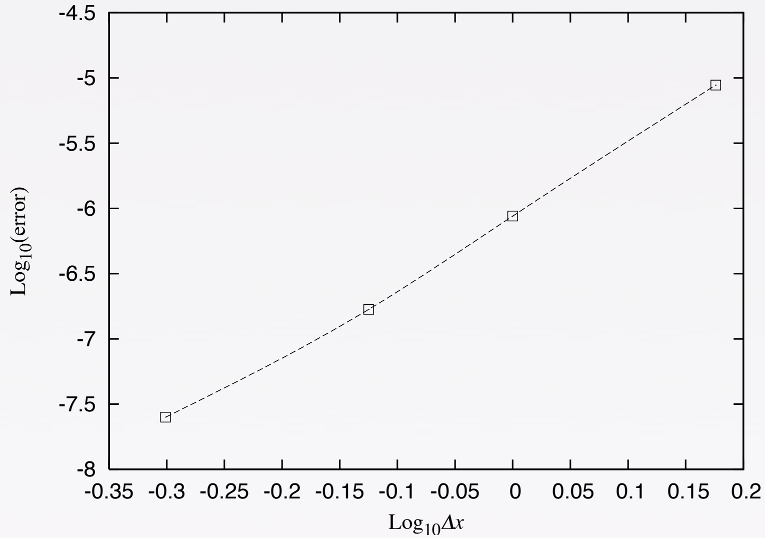
$$\omega_I^{(\mathrm{CF})}/\mu = 3.31 \times 10^{-7}$$

$$\omega_I^{(\mathrm{Numerical})}/\mu = 3.26 \times 10^{-7}$$

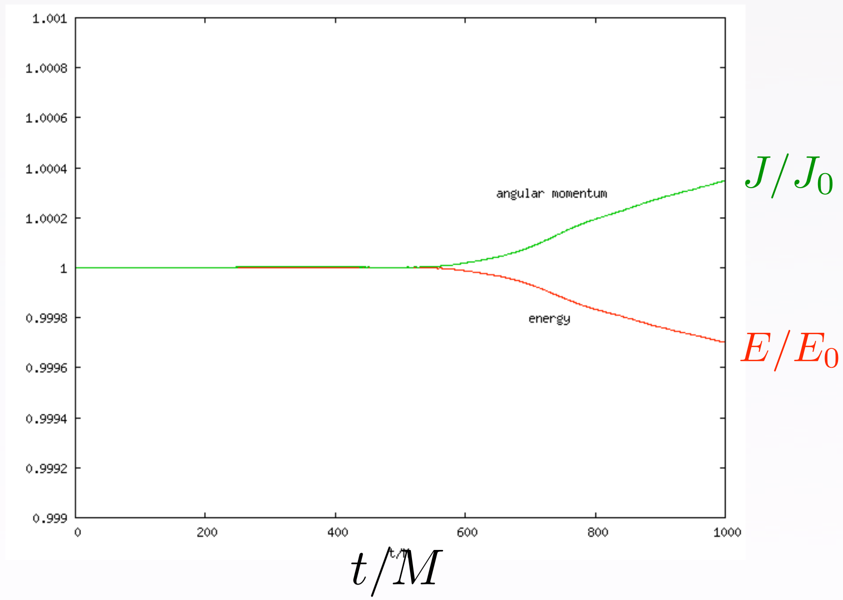
Code check (2)

- Convergence

$(t = 12.5M)$



- Conserved quantities



Comparison with BEC

Action

Saito and Ueda, PRA63 (2001), 043601

BEC

- Action

$$S = N\hbar \int d^3x dt \left[i\psi^* \dot{\psi} + \frac{1}{2}\psi^* \nabla^2 \psi - \frac{r^2}{2}\psi^*\psi - \frac{g}{2}(\psi^*\psi)^2 \right]$$

$$i\dot{\psi} = -\frac{1}{2}\nabla^2\psi + \frac{r^2}{2}\psi + g|\psi|^2\psi$$

Gross-Pitaevskii equation

BH-axion

- Action

$$\hat{S} = \int d^4x \sqrt{-g} \left[-\frac{1}{2}(\nabla\varphi)^2 - \mu^2 \left(\frac{\varphi^2}{2} + \hat{U}_{\rm NL}(\varphi) \right) \right],$$

- Non-relativistic approximation

$$\varphi = \frac{1}{\sqrt{2\mu}} \left(e^{-i\mu t}\psi + e^{i\mu t}\psi^* \right)$$

$$\hat{S}_{\rm NR} = \int d^4x \left[\frac{i}{2}\left(\psi^*\dot{\psi} - \psi\dot{\psi}^* \right) - \frac{1}{2\mu}\partial_i\psi\partial_i\psi^* + \frac{\alpha_g}{r}\psi^*\psi - \mu^2 \tilde{U}_{\rm NL}(|\psi|^2/\mu) \right]$$

$$\tilde{U}_{\rm NL}(x) = -\sum_{n=2}^{\infty} \frac{(-1/2)^n}{(n!)^2} x^n.$$

Effective theory

Saito and Ueda, PRA63 (2001), 043601

$(\nu = \cos\theta)$

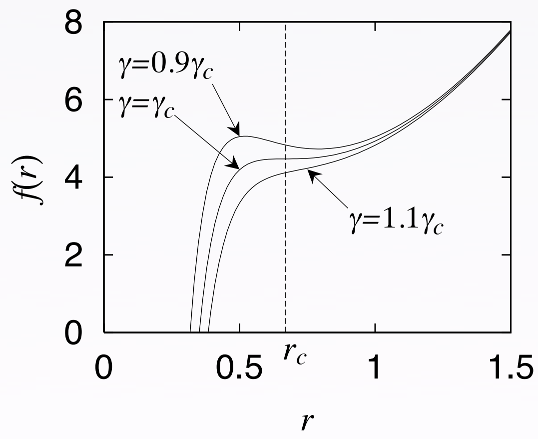
- BEC $\quad \psi = A(x,y,z,t)e^{i\phi(x,y,z,t)}$

$$A = \frac{\exp\left[-\left(\frac{x^2}{2d_x^2(t)} + \frac{y^2}{2d_y^2(t)} + \frac{z^2}{2d_z^2(t)}\right)\right]}{\sqrt{\pi^{3/2}d_x(t)d_y(t)d_z(t)}}$$

$$\phi = \frac{\dot{d}_x(t)}{2d_x(t)}x^2 + \frac{\dot{d}_y(t)}{2d_y(t)}y^2 + \frac{\dot{d}_z(t)}{2d_z(t)}z^2$$

- Spherical case $\quad d_x = d_y = d_z = r(t)$

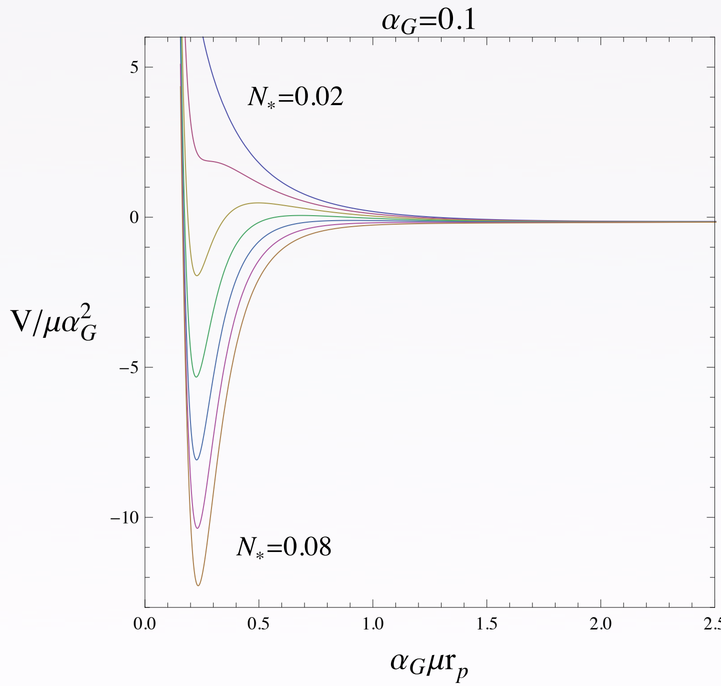
$$S = \frac{N\hbar}{4}\int dt\left[3\dot{r}^2 + 3\dot{r} - f(r)\right]$$



- BH-axion $\quad \psi = A(t,r,\nu)e^{iS(t,r,\nu)+im\phi}$

$$A(t,r,\nu) \approx A_0 \exp\left[-\frac{(r-r_p)^2}{4\delta_r r_p^2} - \frac{(\nu-\nu_p)^2}{4\delta_\nu}\right],$$

$$S(t,r,\nu) \approx S_0(t) + p(t)(r-r_p) + P(t)(r-r_p)^2 + \pi_\nu(t)(\nu-\nu_p)^2 + \cdots,$$

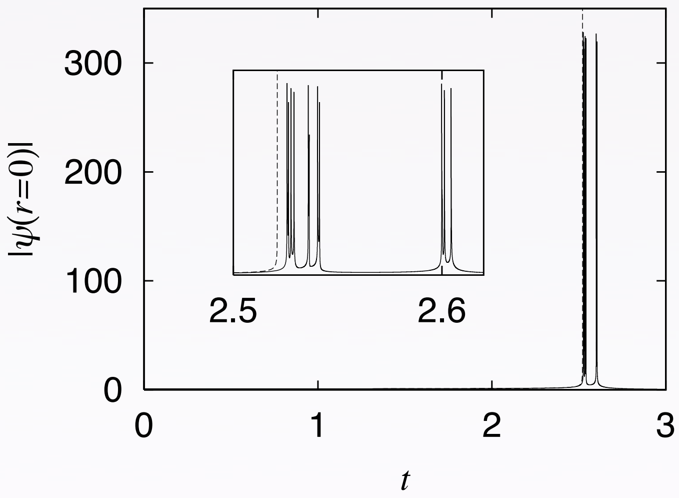


Simulation results

Saito and Ueda, PRA63 (2001), 043601

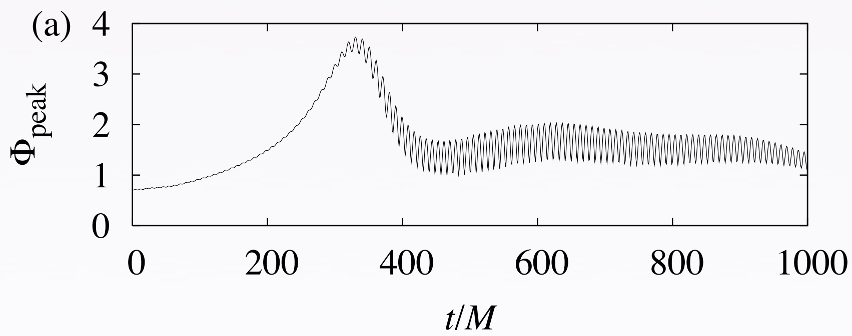
- BEC

$$i\dot{\psi} = -\frac{1}{2}\nabla^2\psi + \frac{r^2}{2}\psi + g|\psi|^2\psi - \frac{i}{2}\left(\frac{L_2}{2}|\psi|^2 + \frac{L_3}{6}|\psi|^4\right)\psi$$



- BH-axion

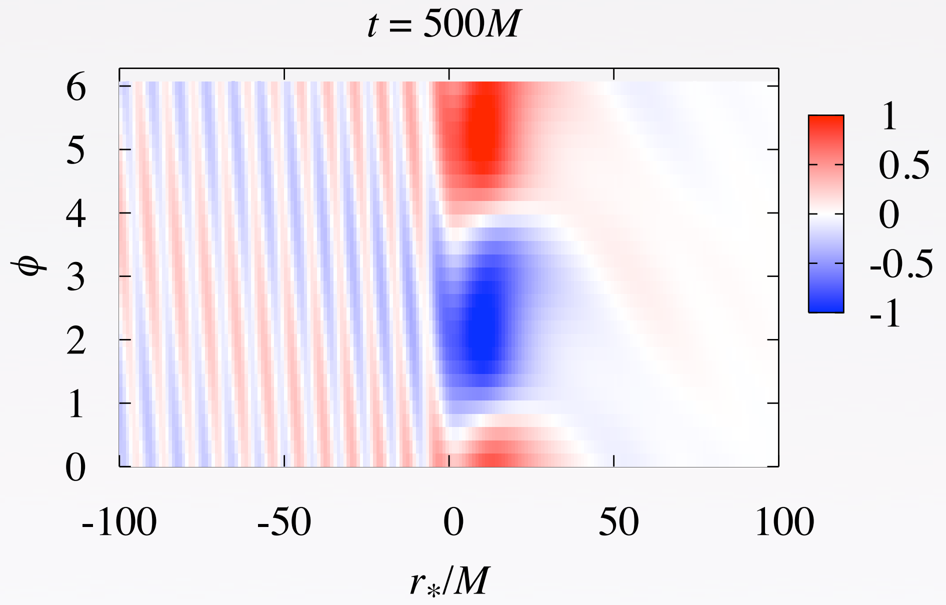
- Our simulation results

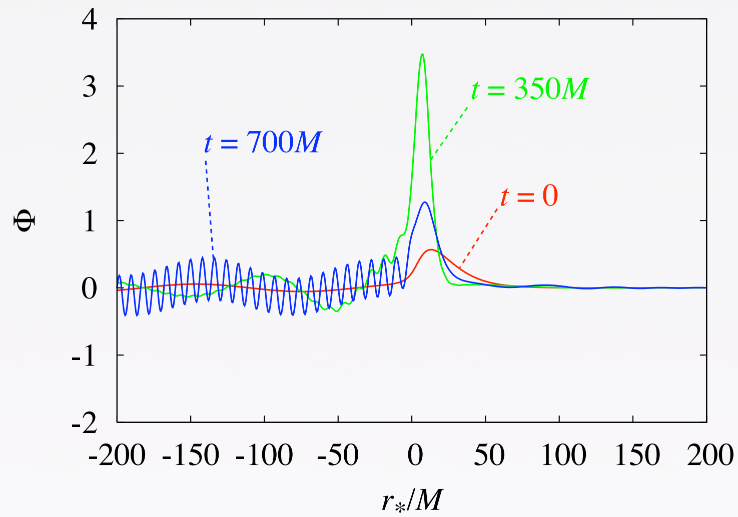


Green's function analysis

Simulation (B)

- Snapshots





m=-1 mode is generated!

(Near the horizon) $\Phi \sim e^{-i\omega t} e^{-i\tilde{\omega} r_*}$

$\tilde{\omega} = \omega - m\Omega_H$

$M\omega_{\mathrm{KG}} = 0.39$	$M\omega_{\mathrm{NL}} = 0.35$
$M\tilde{\omega}_{\mathrm{KG}} = -0.04$	$M\tilde{\omega}_{\mathrm{NL}} = 0.87$

Green's function approach (1)

- Approximation

$$\varphi(x) = \varphi_0(x) + \Delta\varphi, \qquad \varphi_0 = 2\mathrm{Re}\left[e^{(\gamma - i\omega_0)t} P(r) S_1^1(\cos\theta) e^{i\phi}\right],$$

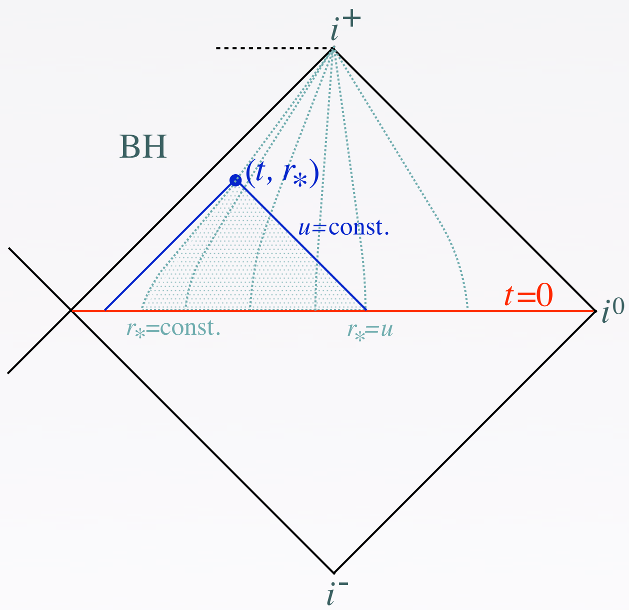
$O(\varphi_0^4)$ is ignored

- Equation

$$(\nabla^2 - \mu^2)\Delta\varphi = J(\varphi_0) := -\frac{\mu^2}{6}\varphi_0^3$$

- Green's function

$$(\nabla^2 - \mu^2)_{x'} G(x, x') = \delta^4(x, x')$$



- Formal solution

$$\Delta\varphi(x) = \int_{D'} d^4x' \sqrt{-g(x')}\, G(x, x') J(\varphi_0(x'))$$

Green's function approach (2)

- Constructing the Green's function

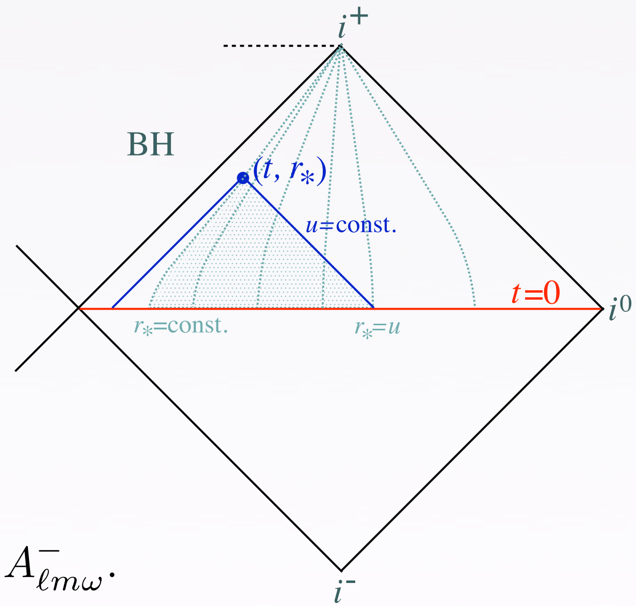
$$G(x,x') = \frac{1}{(2\pi)^2}\sum_{\ell,m}\int_{-\infty}^{\infty} d\omega G^{\omega}_{\ell m}(r,r') e^{-i\omega(t-t')+im(\phi-\phi')} S^m_\ell(\cos\theta)\bar{S}^m_\ell(\cos\theta'),$$

$$G^{\omega}_{\ell m}(r,r') = \frac{1}{W_{\ell m\omega}}\left[\theta(r-r')R^+_{\ell m\omega}(r)R^-_{\ell m\omega}(r') + \theta(r'-r)R^-_{\ell m\omega}(r)R^+_{\ell m\omega}(r')\right],$$

- Radial function $\quad k=\sqrt{\omega^2-\mu^2}, \mathrm{Im}[k]\geq 0$

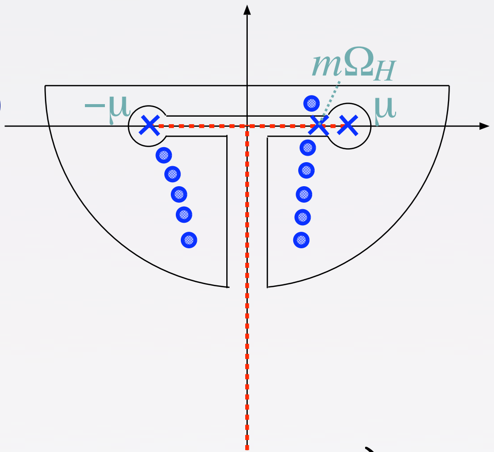
$$R^+_{\ell m\omega} \simeq \begin{cases} C^+_{\ell m\omega}e^{ikr}/r, & r\to\infty; \\ A^+_{\ell m\omega}e^{i\tilde{\omega}r_*} + B^+_{\ell m\omega}e^{-i\tilde{\omega}r_*}, & r\simeq r_+, \end{cases}$$

$$R^-_{\ell m\omega} \simeq \begin{cases} A^-_{\ell m\omega}e^{-ikr}/r + B^-_{\ell m\omega}e^{ikr}/r, & r\to\infty; \\ C^-_{\ell m\omega}e^{-i\tilde{\omega}r_*}, & r\simeq r_+, \end{cases}$$



$$W(R^-,R^+) = 2i\tilde{\omega}(r_+^2+a^2)C^-_{\ell m\omega}A^+_{\ell m\omega} = 2ikC^+_{\ell m\omega}A^-_{\ell m\omega}.$$

Green's function approach (3)



- Near-horizon solution

$$\Delta\varphi = \sum_{\ell,m} e^{im\phi} S_\ell^m(\cos\theta) \frac{e^{im\Omega_H r_*}}{2i(r_+^2 + a^2)}$$
$$\times \left\{ e^{(3\gamma - im\omega_0)u} D_{\ell m}(u, r_*) - \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega u}}{\tilde{\omega} A_{\ell m}^{+(\omega)} [3\gamma + i(\omega - m\omega_0)]} E_{\ell m}^{(\omega)}(u, r_*) \right\},$$

  - First term $\sim e^{-i(m\omega_0 + 3i\gamma)t}$
  - Second term
    - Pole $\omega = m\omega_0 + 3i\gamma$ ⇨ $\sim e^{-i(m\omega_0 + 3i\gamma)t}$
    - Pole $A_{\ell m}^{+(\omega_{\mathrm{BS}}^{(\ell m n)})} = 0$ ⇨ $\sim \sum_n (\cdots) e^{-i\omega_{\mathrm{BS}}^{(\ell m n)} t}$

$$\omega_{\mathrm{BS}}^{(n)} \simeq \pm\mu \simeq \pm\omega_0$$

Nonlinear term makes transfer from growing bound state to decaying bound state with negative frequency.