Keiki Saito, JGRG 22(2012)111334

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RESCEU SYMPOSIUM ON

GENERAL RELATIVITY AND GRAVITATION

JGRG 22

November 12-16 2012

Koshiba Hall, The University of Tokyo, Hongo, Tokyo, Japan

High frequency limit for gravitational perturbations of cosmological models in modified gravity theories

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Abstract

We study the effective gravitational stress-energy tensor for short-wavelength perturbations in modified gravity theories in the cosmological context. We consider $f(R)$ gravity theories on the assumptions that (i) the background has the Friedmann-Lemaître-Robertson-Walker symmetry and that (ii) when our $f(R)$ theory reduces to Einstein gravity. We show by explicit computation that the effective stress-energy tensor for a cosmological model in our $f(R)$ theories, as well as that obtained in the corresponding scalar-tensor theory, takes a similar form to that in general relativity and is in fact traceless, hence acting again like a radiation fluid. If the assumption (ii) above is dropped, then an undetermined integration constant appears and the resultant effective stress-energy tensor acquires a term that is in proportion to the background metric, hence being able to describe a cosmological constant. Whether this effective cosmological constant term is positive and whether it has the right magnitude as dark energy depends upon the value of the integration constant.

1 Introduction

Our observable universe appears to be homogeneous and isotropic on large scales, but highly inhomogeneous on small scales. It is therefore considerably interesting to consider whether the local inhomogeneities can have any effects on the global dynamics of our universe, in particular, any effect that corresponds to a positive cosmological constant or dark energy. A number of authors have explored this possibility of explaining the present cosmic accelerating expansion by some backreaction effects of the local inhomogeneities. Such a backreaction effect may be described in terms of an effective stress-energy tensor arising from metric as well as matter perturbations.

In general relativity, a consistent expansion scheme for short-wavelength perturbations and the corresponding effective stress-energy tensor were largely developed by Isaacson [1, 2], in which the small parameter, say ϵ, corresponds to the amplitude and at the same time the wavelength of perturbations. Isaacson's expansion scheme is called the high frequency limit or the short-wavelength approximation. It has been shown that this effective stress-energy tensor is traceless and satisfies the weak energy condition, i.e. acts like radiation [3, 4], and thus cannot provide any effects like dark energy in general relativity.

However, it is far from obvious if this traceless property of the effective stress-energy tensor is a nature specific only to the Einstein gravity or not. The purpose of this paper is to address this question in a simple, concrete model in the cosmological context. Among many, one of the simplest of modified theories so far proposed is the so called $f(R)$ theory. Since $f(R)$ gravity contains higher order derivative terms, we can anticipate the effective stress-energy tensor to be generally modified in the high frequency limit.

It is well-known that $f(R)$ gravity is equivalent to a scalar-tensor theory, which contains the coupling of the scalar curvature R to a scalar field ϕ in a certain way. The Brans-Dicke theory is one of the simplest examples. Therefore, our analysis can be performed, in principle, either (i) by first translating a given $f(R)$ theory into the corresponding scalar-tensor theory and then inspecting the stress-energy tensor for the scalar field ϕ, or (ii) by directly dealing with metric perturbations of the $f(R)$ theory. We may expect that the former approach is much easier than the latter metric approach, as one has to deal with metric perturbations of complicated combinations of the curvature tensors in the latter case. Nevertheless we

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will take the both approaches. In fact, in the metric approach, by directly taking up perturbations of the scalar curvature R, the Ricci tensor R_{ab} and the Riemann tensor $R^a{}_{bcd}$ involved in a given $f(R)$ theory, we can learn how to generalize our present analysis of a specific class of $f(R)$ gravity to analyses of other, different types of modified gravity theories that cannot even be translated into a scalar-tensor theory.

In the next section, we consider the high frequency limit in $f(R)$ gravity theory. Based on the Isaacson's scheme we expand the field equations for $f(R) = R + cR^2$ theory and first derive the general expression of the effective stress-energy tensor for gravitational perturbations in our $f(R)$ gravity. Then, assuming that our background metric has the FLRW symmetry and also that the resulting equations reduce to the corresponding equations for the Einstein gravity in the limit $c \to 0$, we see that the effective stress-energy tensor whose expression is significantly simplified, is in fact traceless as in the Einstein gravity case. As briefly mentioned above, when a given $f(R)$ gravity is translated into the corresponding scalar-tensor theory, the scalar field ϕ, which expresses an extra-degree of freedom in the $f(R)$ theory, possess a non-trivial potential term. In Sec. 3, we will make sure that the effective stress-energy tensor in Brans-Dicke theory is consistent with that in our $f(R)$ gravity. We will also see that in the Einstein frame, the traceless property of the effective stress-energy tensor is shown to hold in more generic circumstances. Section 4 is devoted to a summary and points to future research.

2 High frequency limit in $f(R)$ gravity

The field equations in the $f(R)$ gravity have terms consisting of higher order derivatives of R, and the order of those derivatives are higher than that of R: $\nabla_{a_1}\nabla_{a_2}\cdots\nabla_{a_m}R^{(n)}[h] \sim \mathrm{O}\left(\epsilon^{n-2-m}\right)$. Therefore it is expected that the effect of the short-wavelength approximation would be enhanced. In order to see whether this is the case, from now on we restrict our attention to the following concrete model $f(R) = R + cR^2$, where c is a constant. The field equations are

$$G^{f(R)}_{ab} \equiv G_{ab} + 2c\left(RR_{ab} - \frac{1}{4}g_{ab}R^2 - \nabla_a\nabla_b R + g_{ab}g^{cd}\nabla_c\nabla_d R\right) = \kappa^2 T^{(0)}_{ab}\,. \tag{1}$$

As in Isaacson's formula, we expand the above equations with respect to the small parameter ϵ. From now on, we consider the cosmological context. We assume that our background is spatially homogeneous and isotropic, that is, our background metric possesses the FLRW symmetry. Then, thanks to this background symmetry we can explicitly solve equations of the form $\nabla_a\nabla_b S(t,\vec{x}) = 0$. The equations of motion of $\mathrm{O}\left(\epsilon^{-3}\right)$ is solved to yield $R^{(1)}[h] = const.$ Taking the average, we find $R^{(1)}[h] = const. = \langle const.\rangle = \left\langle R^{(1)}[h]\right\rangle = 0$. Then, those of $\mathrm{O}\left(\epsilon^{-2}\right)$ become $\nabla_a\nabla_b R^{(2)}[h] = 0$. We find $R^{(2)}[h] \equiv S_1 =$ const. By using above equations, the equation of $\mathrm{O}\left(\epsilon^{-1}\right)$ immediately yields

$$\Box R^{(3)}[h] - R_{ab}[g^{(0)}]\Box h^{ab} = 0\,, \tag{2}$$

$$\left(1 + 2cR[g^{(0)}] + 2cS_1\right)R^{(1)}_{ab}[h] = 2c\left(\nabla_a\nabla_b R^{(3)}[h] - R_{cd}[g^{(0)}]\nabla_a\nabla_b h^{cd}\right)\,. \tag{3}$$

The effective stress-energy tensor is then expressed as

$$\begin{aligned}\kappa^2 T^{\rm eff}_{ab} = -\Bigg\langle &\left(1 + 2cR[g^{(0)}]\right)R^{(2)}_{ab}[h] - \frac{1}{2}g^{(0)}_{ab}S_1 \\ &+2c\left\{\left(R^{(3)}[h] - h^{cd}R_{cd}[g^{(0)}]\right)R^{(1)}_{ab}[h] + S_1\left(R_{ab}[g^{(0)}] + R^{(2)}_{ab}\right)\right\} \\ &-\frac{c}{2}g^{(0)}_{ab}\left(2R[g^{(0)}] + S_1\right)S_1 \\ &+2c\left(-g^{(0)}_{ab}h^{cd}\right)\left(\nabla_c\nabla_d R^{(3)}[h] - R_{ef}[g^{(0)}]\nabla_c\nabla_d h^{ef}\right)\Bigg\rangle\,.\end{aligned} \tag{4}$$

This is the most general expression of our effective stress-energy tensor.

We immediately notice that our expression (4) contains the integration constant S_1. There does not seem to be a definite way to determine S_1 within the framework of the present $f(R)$ theory itself. As a sensible way to specify S_1, let us assume in the following that the effective stress-energy tensor (4) in the R^2 model should reduce to that in general relativity when $c = 0$, and accordingly choose S_1 ($= R^{(2)}[h]$) to be 0. Then, (4) becomes

$$\kappa^2 T^{\rm eff}_{ab} = -\left\langle \left(1 + 2cR[g^{(0)}]\right) R^{(2)}_{ab}[h] + 2c\left(R^{(3)}[h] - h^{cd}R_{cd}[g^{(0)}]\right) R^{(1)}_{ab}[h] \right\rangle , \tag{5}$$

where we have used (3) in the first equality above, and $\left\langle h^{cd}R^{(1)}_{cd}[h]\right\rangle = 0$ in the second equality so as to make the above expression compatible with that of general relativity in the $c = 0$ case. Then using this and $R^{(1)}[h] = 0$, we can find that $\kappa^2 T^{\rm eff}_{ab}$ is in fact traceless:

$$\kappa^2 T^{{\rm eff}\,a}{}_a = 0\,. \tag{6}$$

It should be stressed that as mentioned above, there is a prior no way to determine S_1 by the theory itself. If we choose S_1 to be, instead, a non-zero constant, then the effective stress-energy tensor, (4), has a term proportional to the background metric, that is, a cosmological-constant-looking term, even in the limit to the Einstein gravity.

3 The high frequency limit in scalar-tensor theory

In the previous section, the scalar curvature R and the Ricci tensor R_{ab} are taken up directly in the metric formalism of the $f(R)$ gravity. It is well-known that any $f(R)$ gravity theory is included in Brans-Dicke theory, which is one of the simplest examples of scalar-tensor theory. In this section, we will see that the results obtained in the previous section are indeed consistent with those obtained within the corresponding scalar-tensor theory.

In the Jordan frame, the $f(R)$ gravity of the metric formalism can be cast into the form of the above Brans-Dicke theory by setting $\phi = F(R) \equiv df(R)/dR$, $\omega_{\rm BD} = 0$ and $V = (FR - f)/2$. In this case, as one can find $R - 2\partial_\phi V = 0$, the equations of motion for ϕ and g_{ab} just given above become, respectively

$$\Box\phi - \frac{1}{2\phi}\nabla^a\phi\nabla_a\phi \;=\; \kappa^2 T^{(0)}_{\phi\ ab}\,, \tag{7}$$

$$G^{\rm ST}_{ab} \equiv \phi\left(R_{ab} - \frac{1}{2}g_{ab}R\right) - \nabla_a\nabla_b\phi + g_{ab}\left(g^{cd}\nabla_c\nabla_d\phi + V(\phi)\right) \;=\; \kappa^2 T^{(0)}_{ab}\,. \tag{8}$$

From now we consider short-wavelength perturbations for ϕ: $\phi = \phi_0 + \delta\phi$. We also assume that there is no coupling of matter fields with the second-order derivatives of ϕ, so that there are no non-vanishing terms of order $O(\epsilon^{-1})$ in the stress-energy tensor for matter fields. Then, the equation of motion for ϕ of $\mathrm{O}\left(\epsilon^{-1}\right)$ is $\Box\delta\phi = 0$. and the equations of motion for g_{ab} of $\mathrm{O}\left(\epsilon^{-1}\right)$ are

$$\phi_0\left(R^{(1)}_{ab}[h] - \frac{1}{2}g^{(0)}_{ab}R^{(1)}[h]\right) = \nabla_a\nabla_b\delta\phi - g^{(0)}_{ab}\Box\delta\phi\,. \tag{9}$$

Contracting with $g^{(0)ab}$, we have $R^{(1)}[h] = 3\Box\delta\phi/\phi_0 = 0$. From this, we can immediately find $R^{(1)}_{ab}[h] = 1\nabla_a\nabla_b\delta\phi/\phi_0$. The equations of motion in $\mathrm{O}\left(1\right)$ are given by $G^{\rm ST}_{ab}[g^{(0)}, \phi_0] = \kappa^2 T^{(0)}_{ab} + \kappa^2 T^{\rm eff}_{ab}$, where

$$\kappa^2 T^{\rm eff}_{ab} \equiv -\left\langle \phi_0 R^{(2)}_{ab}[h] + \delta\phi R^{(1)}_{ab}[h] - g^{(0)}_{ab}h^{cd}\nabla_c\nabla_d\delta\phi \right\rangle . \tag{10}$$

Here we would like to emphasise that so far we have made no assumptions concerning the form of $f(R)$ or the symmetry of our background metric $g^{(0)}_{ab}$; the above expression, (10), applies to the generic $f(R)$ theory with an arbitrary background metric.

If we restrict the form of $f(R)$ to be $f(R) = R + cR^2$, then by inspecting the expansions $\phi = \phi_0 + \delta\phi + \cdots$ and $F(R) = 1 + 2cR = \left(1 + 2cR[g^{(0)}]\right) + 2c\left(R^{(3)}[h] - h^{cd}R_{cd}[g^{(0)}]\right) + \cdots$, we find $\phi_0 = 1 + 2cR[g^{(0)}]$ and $\delta\phi = 2c\left(R^{(3)}[h] - h^{cd}R_{cd}[g^{(0)}]\right)$. Using these, we have

$$\begin{aligned}\kappa^2 T^{\rm eff}_{ab} \;=\; & -\Big\langle \left(1 + 2cR[g^{(0)}]\right) R^{(2)}_{ab}[h] \\ & +2c\left(R^{(3)}[h] - h^{cd}R_{cd}[g^{(0)}]\right) R^{(1)}_{ab}[h] - g^{(0)}_{ab}\phi_0 h^{cd}R^{(1)}_{cd}[h] \Big\rangle\,. \end{aligned} \tag{11}$$

Now we work on the cosmological situation so that the background metric has the FLRW symmetry. Provided that the limit $c \to 0$ should reproduce results in the case of the Einstein gravity, we finally obtain

$$\kappa^2 T^{\rm eff}_{ab} = -\left\langle \left(1 + 2cR[g^{(0)}]\right) R^{(2)}_{ab}[h] + 2c\left(R^{(3)}[h] - h^{cd}R_{cd}[g^{(0)}]\right) R^{(1)}_{ab}[h] \right\rangle . \tag{12}$$

We see that the expression (12) above is precisely the same as (5) derived within the metric formalism of the $f(R)$ gravity. This verifies our methods of Sec. 2 for dealing with short-wavelength perturbations of the $f(R)$ gravity within the metric formalism.

In the Einstein frame, the action becomes $S_{\rm E} = \int d^4x\sqrt{-\tilde{g}}\{\tilde{R} - (\tilde{\nabla}\tilde{\phi})^2 - \tilde{V}(\tilde{\phi})\}/(2\kappa^2)$, where $\tilde{g}_{ab} \equiv Fg_{ab}$, $\tilde{\phi} \propto \ln\phi$ and $\tilde{V} \equiv V/F^2$. From this action, one can find that the equation of motion of $\mathrm{O}\left(\epsilon^{-1}\right)$ is $\Box\delta\tilde{\phi} = 0$ and the effective stress-energy tensor is $\kappa^2 T^{\rm eff}_{ab} = \left\langle \tilde{\nabla}_a\delta\tilde{\phi}\tilde{\nabla}_b\delta\tilde{\phi}\right\rangle$. It then immediately follows that the effective stress-energy tensor must be traceless, i.e., $T^{{\rm eff}\,a}{}_a = 0$. This can be shown only on the assumption of the FLRW symmetry.

4 Summary

We have addressed the effective gravitational stress-energy tensor for short-wavelength perturbations in the simple class of $f(R)$ gravity of R^2 type in the cosmological context. By imposing that our background has the FLRW symmetry, we have derived our effective stress-energy tensor for short-wavelength metric perturbations in cosmological models. At this point, thanks to the background FLRW symmetry, the spacetime averaging over several wavelengths and our choice of the constant $S_1 = 0$, the expression of our effective stress-energy tensor has been significantly reduced to have the simple form, (5). We have also shown that the obtained effective stress-energy tensor is traceless, so that it acts like a radiation fluid as in the Einstein gravity case and thus, in particular, cannot mimic dark energy.

We would like to stress that in order to obtain the traceless feature of our effective stress-energy tensor, we have set $S_1 = 0$. However, the field equations for the Einstein gravity need not be, a prior, reproduced in the limit to the Einstein gravity: $c \to 0$. In that case S_1 could take a non-vanishing value and give rise to a cosmological-constant looking term in our effective stress-energy (4). It would be interesting to consider the question of whether there exists any sensible way to provide the right sign and magnitude for S_1 so that (4) can mimic dark energy within the framework of our modified gravity theory.

Our formulas derived in Sec. 2 deal directly with the scalar curvature R and the Ricci tensor R_{ab}, and therefore should be able to apply to similar analyses of other modified gravity theories which contain higher order curvature terms composed of R, R_{ab}, and $R^a{}_{bcd}$ and which cannot even be cast in the form of a scalar-tensor theory. It would be interesting to consider an extension of our present work to a wide class of modified gravity theories with high-rank curvatures.

Acknowledgements

We would like to thank Hideo Kodama, Hirotaka Yoshino, Robert M. Wald, and Stephen R. Green for useful discussions and comments. This work was supported in part by a Grant-in-Aid for JSPS Fellows under Grant No. 23-4465 (KS) and by the JSPS Grant-in-Aid for Scientific Research (C) No. 22540299 (AI) and by the Barcelona Supercomputing Center (BSC) under Grant No. AECT-2012-2-0005.

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