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"Evolution and thermalization of axion dark matter in the condensed regime"

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Evolution and thermalization of axion dark matter in the condensed regime

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Reference: KS and M. Yamaguchi, arXiv:1210.7080 [hep-ph]

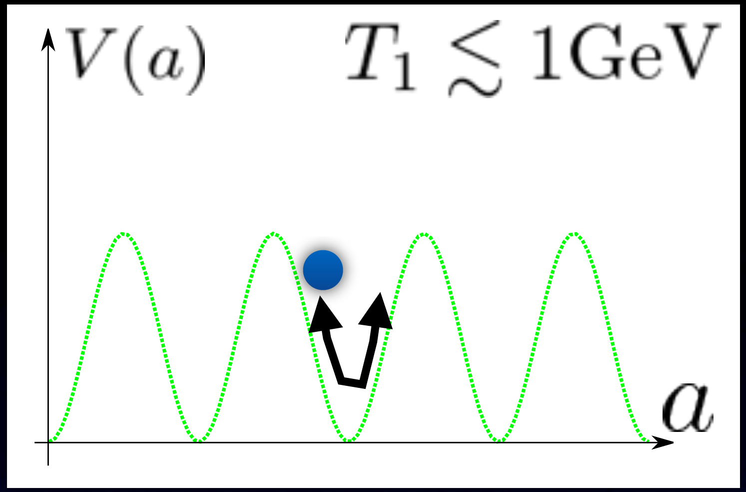
Abstract

- Discuss the possibility that QCD axions form a Bose-Einstein condensate (BEC)
- Calculate time evolution of occupation number of axions in the condensed regime
  - Derive a formula for thermalization rate
  - Revisit axion cosmology

Peculiarities of axion dark matter

- Non-thermal production

$$H \lesssim m_a \quad (t = t_1)$$



$$\Rightarrow \quad t_1 \sim 10^{-7}\mathrm{sec}$$

$$\delta v \sim \frac{\delta p}{m_a} \sim \frac{R(t_1)}{R(t_0)} \frac{1}{m_a t_1} \sim 3 \times 10^{-17} \left(\frac{F_a}{10^{12}\mathrm{GeV}} \right)^{0.81}$$

"cold" dark matter ($\delta v < 10^{-8}$)

- Large occupation number

$$\mathcal{N} \sim n_a \frac{(2\pi)^3}{\frac{4\pi}{3}(m_a \delta v)^3} \sim 10^{61} \left(\frac{F_a}{10^{12}\mathrm{GeV}} \right)^{2.75}$$

($n_a \sim m_a F_a^2 (R(t_1)/R(t_0))^3$: number density of axions)

c.f. $\mathcal{N} \sim 10^{-18} \left(\frac{100\mathrm{GeV}}{m_{\mathrm{wimp}}} \right)^4$ for WIMPs

Do axions form a BEC ?

- Bose-Einstein condensate
  - Large fraction of bosons are in the lowest-energy state
  - Critical temperature

$$T_c = \left(\frac{\pi^2 n_a}{\zeta(3)}\right)^{1/3} \simeq 2 \times 10^2\mathrm{GeV}\left(\frac{F_a}{10^{12}\mathrm{GeV}}\right)^{0.54}\left(\frac{R(t_1)}{R(t)}\right)$$

$$\overset{!}{\gg} \delta\omega \sim \frac{1}{2} m_a (\delta v)^2 \sim 4 \times 10^{-13}\mathrm{eV}\left(\frac{F_a}{10^{12}\mathrm{GeV}}\right)^{0.25}\left(\frac{R(t_1)}{R(t)}\right)^2$$

- Assumptions

	For axions
1. Particles are bosons →	satisfied
2. Number is conserved →	satisfied
3. Large occupation number →	satisfied
4. In thermal equilibrium →	**???**

Axions vs WIMPs

- Thermalize if $\Gamma \sim \dot{\mathcal{N}}(p)/\mathcal{N}(p) > H$
- WIMPs : classical particle limit

  $\hbar \to 0$ while $E = \hbar\omega,\ \vec{p} = \hbar\vec{k}$ fixed

  $\omega,\ \vec{k} \to \infty$ collection of classical "point particles"

  evolution : use Boltzmann eq.

- axions : classical field limit

  $\hbar \to 0$ while $E = \mathcal{N}\hbar\omega,\ \vec{p} = \mathcal{N}\hbar\vec{k}$ fixed

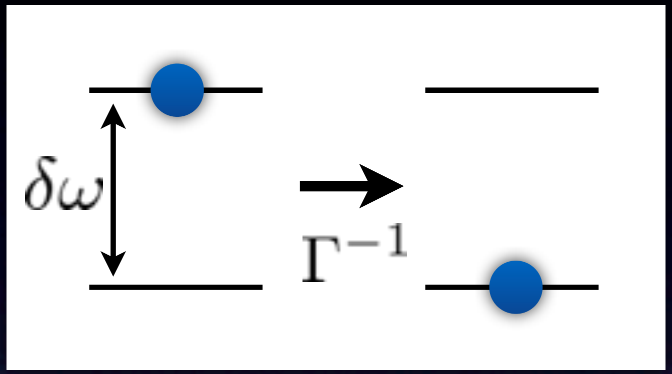
  $\mathcal{N} \to \infty$ $\delta\omega,\ \delta\vec{k} \sim \text{finite}$ "wavy field"

  cannot use Boltzmann eq.

  → consider quantum mechanics

In quantum mechanics...

- We consider transitions between different quantum states.



- Two different regimes
- WIMPs $\omega \to \mathrm{large}$

$$\delta\omega \gg \Gamma$$

energy exchanged in the transitions — transition rate

"particle kinetic regime"

- axions $\omega \to \mathrm{small}$

$$\delta\omega \ll \Gamma$$

"condensed regime"

A transition makes sense if $\mathcal{N}\,\delta\omega \gg \Gamma$

Previous study

Erken, Sikivie, Tam, Yang, PRD85, 063520 (2012)

- Time evolution of quantum operators in the Heisenberg picture

l : label of the state (momentum)

$$H = \sum_i \omega_i a_i^\dagger a_i + \sum_{i,j,k,l} \frac{1}{4} \Lambda_{kl}^{ij} a_k^\dagger a_l^\dagger a_i a_j \qquad \mathcal{N}_l = a_l^\dagger a_l$$

$$\dot{\mathcal{N}}_l = i[H, \mathcal{N}_l]$$

$$= i \sum_{i,j,k} \frac{1}{2} (\Lambda_{ij}^{kl} a_i^\dagger a_j^\dagger a_k a_l e^{-i\Omega_{ij}^{kl} t} - \text{H.c.})$$

$$+ \sum_{k,i,j} \frac{1}{2} |\Lambda_{ij}^{kl}|^2 [\mathcal{N}_i \mathcal{N}_j (\mathcal{N}_l + 1)(\mathcal{N}_k + 1)$$

$$-\mathcal{N}_l \mathcal{N}_k (\mathcal{N}_i + 1)(\mathcal{N}_j + 1)] \frac{2}{\Omega_{ij}^{kl}} \sin(\Omega_{ij}^{kl} t) + \ldots$$

Leading contribution in the condensed regime

$$\Omega_{ij}^{kl} t \ll 1$$

$$\dot{\mathcal{N}}_l \sim \mathcal{O}(\Lambda_{ij}^{kl})$$

reduce to Boltzmann eq. in the particle kinetic regime

$$\Omega_{ij}^{kl} t \gg 1$$

$$\dot{\mathcal{N}}_l \sim \mathcal{O}(|\Lambda_{ij}^{kl}|^2)$$

$$\Omega_{ij}^{kl} \equiv \omega_k + \omega_l - \omega_i - \omega_j$$

- What about the quantum-mechanical averages $\langle \dot{\mathcal{N}}_l(t) \rangle$?

Effects on cosmological parameters ?

- Thermalization rate is enhanced in the condensed regime → leads to axion BEC
- Thermalization rate with other species is also enhanced (?) Erken, Sikivie, Tam, Yang, PRD85, 063520 (2012); PRL108, 061304 (2012)
    - axions and photons have thermal contact

$$\rho_{\gamma i} = \frac{\pi^2}{15} T_{\gamma i}^4 = \rho_{\gamma f} + \rho_{af} = \frac{\pi^2}{30} T_{\gamma f}^4 (2+1)$$

$$\Rightarrow \quad T_{\gamma f} = (2/3)^{1/4} T_{\gamma i}$$

baryon-to-photon ratio at BBN $\eta_{\rm BBN} = (2/3)^{3/4} \eta_{\rm std.}$

effective # of neutrino d.o.f. $N_{\rm eff} = 6.77$ (obs. $N_{\rm eff} \simeq 3-4$)

- Is it true ? Does axion BEC conflict with standard cosmology?

In-in formalism

Weinberg, PRD72, 043514 (2005)

- Calculate expectation value of a quantum operator via perturbative expansion

$$\langle \mathcal{O}(t) \rangle = \langle \mathrm{in} | \mathcal{O}(t) | \mathrm{in} \rangle$$
$$= \langle \mathcal{O} \rangle + i \int_{t_0}^{t} dt_1 \langle [H_I(t_1), \mathcal{O}] \rangle$$
$$+ (i)^2 \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 \langle [H_I(t_1), [H_I(t_2), \mathcal{O}]] \rangle + \dots$$

$\mathcal{O} = \mathcal{N}_n \equiv \frac{a_n^\dagger a_n}{V}$: number operator

$\int \frac{d^3p}{(2\pi)^3} \to \frac{1}{V} \sum_{n_x, n_y, n_z}$ (label becomes discrete)

$H_I(t) = \frac{1}{V^4} \sum_{ijkl} \frac{1}{4} \Lambda^{ij}_{kl} e^{-i\Omega^{ij}_{kl} t} a_k^\dagger a_l^\dagger a_i a_j$ (ignore axion # violating process)

(1) scalar self-coupling $\Lambda^{ij}_{kl} = -\frac{\lambda}{4m_a^2} V \delta_{i+j,k+l}$ ⬅ $H_I = -\int d^3x \frac{\lambda}{4!}\phi^4$

(2) gravity $\Lambda^{ij}_{kl} = -4\pi G m_a^2 \left(\frac{1}{|\mathbf{p}_k - \mathbf{p}_i|^2} + \frac{1}{|\mathbf{p}_k - \mathbf{p}_j|^2} \right) V \delta_{i+j,k+l}$ ⬅ $H_I = -\frac{G}{2} \int d^3x d^3x' \frac{\rho(\mathbf{x}, t)\rho(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|}$

In state

- $|\mathrm{in}\rangle$ = a state which represents the coherent oscillation of axions
- Use a coherent state

$$|\alpha_i\rangle = e^{-\frac{1}{2}|\alpha_i|^2} \sum_{n=0}^{\infty} \frac{\alpha_i^n}{n!\sqrt{V^n}} (a_i^\dagger)^n |0\rangle$$

with $a_i|0\rangle = 0$

$$a_i|\alpha_i\rangle = V^{1/2}\alpha_i|\alpha_i\rangle$$

- Field amplitude

(assuming a mode with $|\mathbf{p}_n| \lesssim H \sim t^{-1}$)

$$\phi = \frac{1}{V}\sum_n \frac{1}{\sqrt{2E_{p_n}}}(e^{ip_n\cdot x}a_n + e^{-ip_n\cdot x}a_n^\dagger)$$

$$\langle\alpha_i|\phi|\alpha_i\rangle \simeq \frac{1}{\sqrt{2m_aV}}(e^{-im_at}\alpha_n + e^{im_at}\alpha_n^*) = \sqrt{\frac{2}{m_aV}}|\alpha_n|\cos(m_at - \beta)$$

(inside the horizon $\mathbf{p}_n \cdot \mathbf{x} \ll 1$)

classical field trajectory

- Mean square deviation

$$\Delta\phi = \sqrt{\langle\alpha_i|\phi^2|\alpha_i\rangle - \langle\alpha_i|\phi|\alpha_i\rangle^2} = \sqrt{\frac{1}{V}\sum_n \frac{1}{2E_n}}$$

vacuum fluctuation

“Zero modes”

- Assume plural (say K) oscillating modes

$$|\{\alpha\}\rangle = \prod_i^K e^{-\frac{1}{2}|\alpha_i|^2} \sum_{n=0}^{\infty} \frac{\alpha_i^n}{n!\sqrt{V^n}} (a_i^\dagger)^n |0\rangle$$

$$|\mathbf{p}_i| \lesssim H(t_1) \sim m_a(t_1)$$

for $i = 1, \ldots, K$

- number density

$$n_a = \frac{1}{V} \sum_n \langle \{\alpha\} | \mathcal{N}_n | \{\alpha\} \rangle = \frac{1}{V} \sum_i^K |\alpha_i|^2 \equiv \sum_i^K n_{c,i}$$

- Question : how these plural oscillating modes (“zero modes”) reach thermal equilibrium ?
  - decoupled axions
    = each of K modes oscillates independently
  - thermalized axions
    = transition between plural modes becomes significant

Evolution of occupation number

$$\langle \mathrm{in}|\mathcal{N}_p(t)|\mathrm{in}\rangle = \langle \mathcal{N}_p\rangle + i\int_{t_0}^{t} \langle [H_I(t_1), \mathcal{N}_p]\rangle + \mathcal{O}(H_I^2) + \dots$$

$$i\int_{t_0}^{t} dt_1 \langle [H_I(t_1), \mathcal{N}_p]\rangle \xrightarrow{t-t_0\to\infty} -\frac{1}{2V^2}\sum_j^K\sum_k^K\sum_l^K \left[\Lambda_{kl}^{pj}\frac{e^{-i\Omega_{kl}^{pj}t}}{\Omega_{kl}^{pj}}\alpha_k^*\alpha_l^*\alpha_j\alpha_p + \mathrm{c.c.}\right]$$

for $$|\mathrm{in}\rangle = \prod_i^K e^{-\frac{1}{2}|\alpha_i|^2}\sum_{n=0}^{\infty}\frac{\alpha_i^n}{n!\sqrt{V^n}}(a_i^\dagger)^n|0\rangle$$

coherent state

$$i\int_{t_0}^{t} dt_1 \langle [H_I(t_1), \mathcal{N}_p]\rangle = 0$$ for $$|\mathrm{in}\rangle = \prod_k \frac{1}{\sqrt{\mathcal{N}_k! V^{\mathcal{N}_k}}}(a_k^\dagger)^{\mathcal{N}_k}|0\rangle$$

number state

- First order term is relevant if

  (1) condensed regime $\Omega_{kl}^{pj}t \ll 1$

  (c.f. $e^{-i\Omega_{kl}^{pj}t} \approx 0$ for particle kinetic regime $\Omega_{kl}^{pj}t \gg 1$)

  (2) coherent state representation $|\mathrm{in}\rangle = |\{\alpha\}\rangle$

Thermalization rate

$$\Gamma \equiv \frac{1}{\mathcal{N}_p(t)}\frac{d\mathcal{N}_p(t)}{dt} \simeq \frac{1}{\mathcal{N}_p V^2}\sum_{j,k,l}^{K} \mathrm{Im}[\Lambda_{pj}^{kl}\alpha_k\alpha_l\alpha_j^*\alpha_p^*]$$

Using $\Lambda_{pj}^{kl} = \Lambda V \delta_{k+l,p+j}$ and $\mathcal{N}_p \simeq |\alpha_p|^2 \simeq \mathcal{N}/K$

we obtain

$$\Gamma_{\mathrm{condensed}} \simeq \Lambda\frac{\mathcal{N}}{V} = \Lambda n_a$$

n_a : number density of axions

- Recover the previous estimation Erken, Sikivie, Tam, Yang, PRD85, 063520 (2012)

scalar phi^4 $\quad \Gamma_{\mathrm{condensed},s} \simeq \dfrac{\lambda n_a}{4m_a^2} \quad \propto 1/R^3(t)$

gravity $\quad \Gamma_{\mathrm{condensed},g} \simeq \dfrac{4\pi G m_a^2 n_a}{(\delta p)^2} \quad \propto 1/R(t)$

$$\delta p \sim m_a \delta v \propto 1/R(t)$$

Formation of axion BEC

- Axions form a BEC when $\Gamma_{\mathrm{condensed},g} \gtrsim H$

  corresponding to the photon temperature

$$T_{\mathrm{BEC}} \simeq 2 \times 10^3 \mathrm{eV} \left(\frac{F_a}{10^{12}\mathrm{GeV}} \right)^{0.56}$$

- At this time, axions enter into thermal equilibrium with temperature

$$T_a(t_{\mathrm{BEC}}) \sim \frac{\delta p^2(t_{\mathrm{BEC}})}{3m_a} \sim 2.5 \times 10^{-37} \mathrm{eV} \left(\frac{F_a}{10^{12}\mathrm{GeV}} \right)^{3.25}$$

  thermally excited modes

$$n_T(T_a(t_{\mathrm{BEC}})) \simeq \left(\frac{m_a T_a(t_{\mathrm{BEC}})}{2\pi} \right)^{3/2}$$

$$\frac{n_T(T_a(t_{\mathrm{BEC}}))}{n_a(t_{\mathrm{BEC}})} \simeq 7.5 \times 10^{-80} \left(\frac{F_a}{10^{12}\mathrm{GeV}} \right)^{-0.005}$$

- Almost all axions stay in the lowest energy state.

No photon cooling

- Interaction with other species b

$$H_{I,b}(t) = \frac{1}{V^4}\sum_{ijkl}\frac{1}{4}{\Lambda_b}^{ij}{}_{kl}e^{-i\Omega^{ij}_{kl}t}a^\dagger_k b^\dagger_l a_i b_j$$

- Assume b particles are represented as a number state

$$|\mathrm{in}\rangle = \prod_k \frac{1}{\sqrt{\mathcal{N}_k! V^{\mathcal{N}_k}}}(b^\dagger_k)^{\mathcal{N}_k}|\{\alpha\}\rangle$$

while $$|\{\alpha\}\rangle = \prod_i^K e^{-\frac{1}{2}|\alpha_i|^2}\sum_{n=0}^{\infty}\frac{\alpha_i^n}{n!\sqrt{V^n}}(a^\dagger_i)^n|0\rangle$$

- First order term exactly vanishes

$$\langle[H_{I,b}(t), \mathcal{N}_p]\rangle = 0$$

- Thermalization with other species is second order effect.
  - BEC axions do not have thermal contact with photons
    → does not affect cosmological parameters

Summary

- Derive the formula for thermalization rate in the condensed regime by using
  - in-in formalism
  - coherent state representation
- Formation of axion BEC occurs at $T_{\rm BEC} \sim \mathcal{O}(1){\rm keV}$
- It does not conflict with standard cosmology
- Future directions
  - Extend the formalism including general relativistic corrections
  - Seek for other observable effects