Ryotaku Suzuki, JGRG 22(2012)111224

"Analysis of Gregory-Laflamme mode in large D limit"

RESCEU SYMPOSIUM ON

GENERAL RELATIVITY AND GRAVITATION

JGRG 22

November 12-16 2012

Koshiba Hall, The University of Tokyo, Hongo, Tokyo, Japan

Analysis of Gregory–Laflamme mode in large D limit

Ryotaku Suzuki (Kyoto University)
with Roberto Emparan (University of Barcelona, ICREA)

JGRG22, November 12–16, 2012, RESCEU SYMPOSIUM

Outline

1. Introduction
2. Large D limit
3. Dispersion relation
4. Summary

Outline

1. Introduction
2. Large D limit
3. Dispersion relation
4. Summary

Gravity in Higher Dimension

Why higher dimension ?

String theory ➔ spacetime dimension > 4

- -Various compactification
- -Large extra dimension → higher dimensional gravity

Black hole in Higher Dimension

→ No uniqueness, No topology theorem

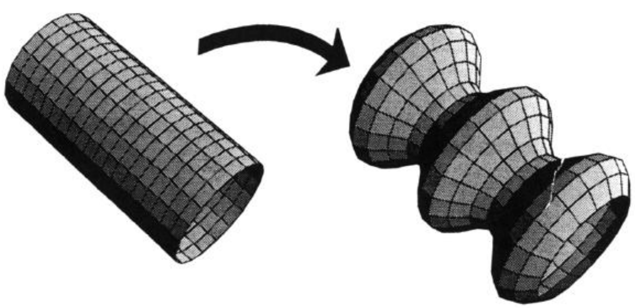
- -Black String, Brane (KK spacetime)
- -Black Ring, Black Saturn, etc…

Gregory-Laflamme Instability

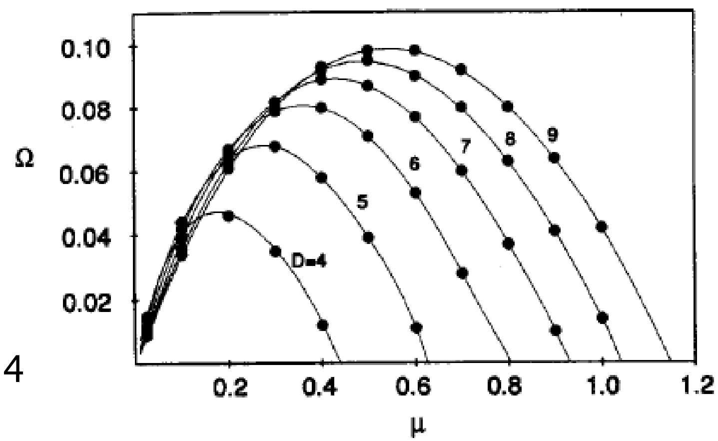
Characteristic in extended black object (sting, brane,…)
Long wave length instability ~ hydrodynamic instability

Gregory–Laflamme Instability

- In 1993, Gregory and Laflamme found a long wave length instability of black string (brane)



Gregory, Laflamme, 1994



threshold : thickness ~ wave length

Importance

–Universal property for extended objects

–Determine Phase Diagram in KK spacetime

UniformBS – NUBS – (caged) Black hole

Outline

1. Introduction
2. Large D limit
3. Dispersion relation
4. Summary

Threshold mode k_GL in large D

Numerical Analysis

Sorkin (2004) studied the threshold mode numerically up to D=50

and observed $\mu_{GL} \propto \gamma^D$ $\gamma \approx 0.686$

$$\mu_{GL} = \frac{(d-2)\Omega_{d-2}}{16\pi}\left(\frac{r_0 k_{GL}}{2\pi}\right)^{d-3}$$

dimensionless mass

$$\mu := G_D M / L^{D-3}$$

Large D limit

Kol, Sorkin(2004), Asnin, et.al.(2007)

solved analytically the threshold mode in large D limit

$$k_{GL} \to \sqrt{d}$$ ➔ $$\tilde{\gamma} = \sqrt{\frac{e}{2\pi}} \approx 0.658$$

agree with Sorkin(2004)

Matched asymptotic expansion

near horizon — new coordinate $X = r^n$ — expand with 1/n

asymptotic — expand with r_0^n / r^n

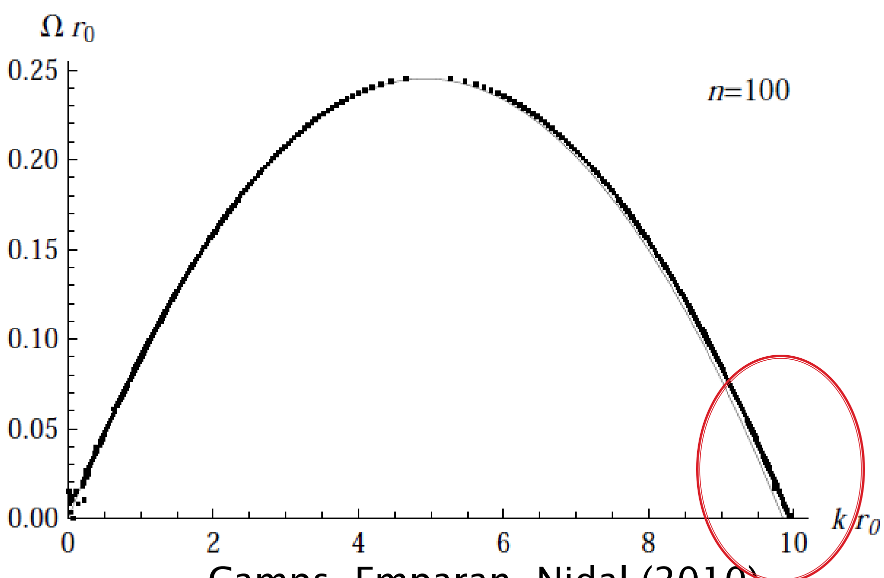
Hydrodynamical approximation

- Camps et.al(2010) studied the instability in long wave length limit ➔ Narvier–Stokes Eq. +viscosity

dispersion relation

$$\Omega = \frac{k}{\sqrt{n+1}}\left(1 - \frac{n+2}{n\sqrt{n+1}} k r_0\right)$$

valid up to k^3



Camps, Emparan, Nidal (2010)

highly coincident with numerical data even k is large !

they proposed at large D

$$\Omega \to \frac{k}{\sqrt{n}}\left(1 - \frac{k}{\sqrt{n}}\right)$$

Question
Can we prove this dispersion relation analytically ?

Outline

1. Introduction
2. Large D limit
3. Dispersion relation
4. Summary

Set up and Master equation

Black String background

n = D - 4 Large D → Large n

$$ds_0^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + dz^2 + r^2 d\Omega_{n+1}^2 \qquad f(r) = 1 - r_0^n/r^n$$

Scalar Perturbation with Transverse-Traceless gauge

$$h_{\mu\nu} = \Re \exp\left[\left(\frac{\Omega t}{r_0} + \frac{ikz}{r_0}\right) P_{\mu\nu}\right]$$

$$P_{tt} = -f\psi(r), \quad P_{tr} = \eta(r), \quad P_{rr} f^{-1}\chi(r), \quad P_\Omega = r^2\kappa(r)\gamma_\Omega$$

Master equation for $\eta(r)$

➔ Same equation in GL94

Assumption

$$k = \sqrt{n}\hat{k}$$

$$\begin{aligned}
&(n^2 - 4r^2\Omega^2 - 2\left(n^2 + 2k^2r^2\right) f + n^2 f^2)\eta'' \\
&+\frac{1}{rf}(3\left(n^3 - 4nr^2\Omega^2\right) + \left(3n^2 - 6n^3 - 4r^2\Omega^2 + 8nr^2\left(-k^2 + \Omega^2\right)\right) f \\
&+\left(-6n^2 + 3n^3 - 4k^2r^2 + 4k^2nr^2\right) f^2 + 3n^2 f^3)\eta' \\
&+\frac{1}{r^2f^2}(n^4 - 5n^2r^2\Omega^2 + 4r^4\Omega^4 + \left(3n^3 - n^4 + 8k^2r^4\Omega^2 + n^2r^2\left(k^2 + 10\Omega^2\right)\right) f \\
&+\left(-6n^3 - n^4 + 4nr^2(\Omega^2 - k^2) + 4r^2\left(k^4r^2 + \Omega^2\right) + n^2\left(1 - 2k^2r^2 - 5r^2\Omega^2\right)\right) f^2 \\
&+\left(3n^3 + n^4 + 4k^2r^2 + 8k^2nr^2 + n^2\left(-2 + k^2r^2\right)\right) f^3 + n^2 f^4)\eta = 0
\end{aligned}$$

Near horizon in Large D

"Good" coordinate in Large D (used in Asnin et. al. (2007))

$X = r^n$ the effect of the horizon correctly incorporated at large D expansion

$$\eta(X)'' + P_X \eta(X)' + Q_X \eta(X) = 0$$

up to 1/n ➔ $X(X-1)^3\eta(X)'' + (X-1)^2(4X-1)\eta(X)' + (X-1)(2X-1)\eta(X) = S_1[\eta]$

$$S_1[\eta] = -\frac{1}{n}(2(X-1)^3(4\hat{k}^2X^2 - 2\hat{k}^2X + 1)\eta(X)' + (X-1)^2(\hat{k}^2(8X^2 - 4X + 1) + 3)\eta(X))$$

leading solution

$$U_1(X) = \frac{1}{X-1},\ U_2(X) = \frac{\ln(X-1) - \ln X}{X-1}$$

$U_1 \simeq X^{-1} + X^{-2}$ and $U_2 \simeq -1/X^2$ when $X \gg 1$,

➔ require up to $1/r^{2n}$ in asymptotic region

Asymptotic region in Large D

Leading order f(r) → 1

Expansion with r_0^n/r^n

$$r^2\eta'' + (n+1)\eta' - (k^2+\Omega^2)\eta = 0.$$

regularity at the infinity

→ $$\eta_{as}(r) = \frac{1}{r^{\frac{n}{2}}} K_\nu(k_\Omega r).$$

modified Bessel of 2nd kind

$\nu = (n+2)/2$ and $k_\Omega = \sqrt{k^2+\Omega^2}$.

Next to Leading

modified Bessel Eq. with source term

$u_1(r) = r^{\frac{n}{2}}\eta_1(r)$,

$x = k_\Omega r$

$$x u_{1,xx} + u_{1,x} - (x + \nu^2 x^{-1})u_1 - \mathcal{S}_1[u_0] = 0$$

$$\mathcal{S}_1[u] \equiv k_\Omega^{2\nu-2} x^{-2\nu+1}[(A_1 x^2 + A_2)u(x) + A_3 x u'(x)]$$

Using Green function

→ $$u_1(x) = C_1 K_\nu(x) - K_\nu(x)\int^x I_\nu(y)\mathcal{S}_1[u_0](y)dy - I_\nu(x)\int_x^\infty K_\nu(y)\mathcal{S}_1[u_0](y)dy$$

just contribute to ovarall scaling

$$\eta_{as,1} \simeq \left(1 - \frac{1}{2n} + \frac{\hat{k}^2}{2n} + \frac{\Omega^2}{2\hat{k}^2 n}\right)\frac{1}{r^n}\frac{K_\nu}{r^{\frac{n+2}{2}}}$$ (up to 1/n)

Intermediate region

Asymptotic solution $X = r^n < n^k (k > 1 : \text{const.})$ so that $n^{-1} \ln X$ is subleading

$$\eta_{as} \propto \frac{1}{X} + \frac{1}{X^2} - \frac{1+\hat{k}^2}{n}\frac{\ln X}{X} + \left(\frac{\Omega^2}{2\hat{k}^2 n} - \frac{1}{2n} + \frac{\hat{k}^2}{2n}\right)\frac{1}{X^2} + \mathcal{O}(X^{-3}, n^{-2})$$

Leading at near horizon

$$\eta_0(X) = \frac{1}{X-1}$$

$$\rightarrow \quad S_1[U_1] = -\frac{\hat{k}^2+1}{n}(X-1)$$

$$\frac{K_\nu}{r^{\frac{n+2}{2}}} \propto \frac{1}{X}\left(1 - \frac{1+\hat{k}^2}{n}\ln X\right)$$

Sub-leading at near horizon

$$\begin{aligned}\eta_1(X) &= \frac{C_1}{X-1} + C_2\frac{\ln(X-1) - \ln X}{X-1} - \frac{1+\hat{k}^2}{n}\frac{\ln(X-1)}{X-1} \\ &\simeq \frac{C_1}{X} - \frac{1+\hat{k}^2}{n}\frac{\ln X}{X} + \frac{nC_1 - nC_2 + 1 + \hat{k}^2}{X^2}\end{aligned}$$

$\rightarrow$ $C_1 = 0$ and $nC_2 - 1 - \hat{k}^2 = -\frac{\Omega^2}{2\hat{k}^2} + \frac{1}{2} - \frac{\hat{k}^2}{2}$

Horizon regularity

At X -> 1, the regular solution is

large n limit

$$\eta(X) = (X-1)^{-1+\frac{\Omega}{n}}(1+\mathcal{O}(X-1)) \simeq \frac{1}{X-1}\left(1+\underline{\frac{\Omega}{n}}\ln(X-1)+\dots\right)$$

matched solution

$$\eta(X) \simeq \frac{1}{X-1} + \underline{\frac{nC_2-1-\hat{k}^2}{n}}\frac{\ln(X-1)}{X-1}$$

$$nC_2 - 1 - \hat{k}^2 = -\frac{\Omega^2}{2\hat{k}^2} + \frac{1}{2} - \frac{\hat{k}^2}{2}$$

➔ $$\boxed{\Omega = -\frac{\Omega^2}{2\hat{k}^2} + \frac{1}{2} - \frac{\hat{k}^2}{2}}$$

Leading order matching

➔ $$\Omega = \underline{\hat{k} - \hat{k}^2},\ -\hat{k} - \hat{k}^2$$

Expected growing mode !!

Outline

1. Introduction
2. Large D limit
3. Dispersion relation
4. Summary

Summary

- We analytically solved the scalar perturbation on black brane in large D limit and obtained the expected dispersion relation.

- Our calculation shows that large D expansion should be useful analytic approximation in higher dimension.
  Application to another situation seems possible in the similar way.

Thank you !

Appendix

A. Matching at singular point

Master Eq has a singular point between horizon and the infinity

$$r_s^n = \frac{\sqrt{n}}{2\hat{k}} \exp\left(\frac{\hat{k}}{\sqrt{n}} - \frac{1}{2n}\ln\frac{n}{4\hat{k}^2} - \frac{\Omega^2}{2\hat{k}^2 n} + \frac{1}{2n^{\frac{3}{2}}}\left(-2\hat{k} - \frac{\hat{k}^3}{3} - \frac{\Omega^2}{\hat{k}} + \hat{k}\ln\frac{n}{4\hat{k}^2}\right) + \mathcal{O}(n^{-2})\right)$$

We first attempted to do matching at the singular point as Kol, Sorkin (2004)

$$r_s = 1 + \frac{1}{2n}\ln\frac{n}{4\hat{k}^2} +$$

B.C $$\frac{r_s\eta'(r_s)}{\eta(r_s)} = -\frac{r_s Q_\eta}{P_\eta}\bigg|_{r_s} = \underline{-n - 2\hat{k}\sqrt{n} - 1 - 3\hat{k}^2} - \frac{1}{\sqrt{n}}(2\hat{k}^3 + \hat{k}\ln\frac{n}{4\hat{k}^2} - \hat{k} + \frac{2\Omega^2}{\hat{k}}) + \mathcal{O}(n^{-1})$$

LO $r^{-n/2}K_\nu(k_\Omega r)$ → $r_s\eta'_{as}(r_s)/\eta_{as}(r_s) = -n - \hat{k}^2 - 1 + \ldots$ trivial.

NLO $u_1(r) = r^{\frac{n}{2}}\eta_1(r)$ → $\frac{r_s\eta_{as}(r_s)'}{\eta_{as}(r_s)} = -n - 2\hat{k}\sqrt{n} + \mathcal{O}(1)$ trivial…

$$u_1(x) = C_1 K_\nu(x) - K_\nu(x)\int_{x_s}^{x} I_\nu(y)\mathcal{S}_1[u_0](y)dy - I_\nu(x)\int_x^\infty K_\nu(y)\mathcal{S}_1[u_0](y)dy$$

NNLO

trivial…….

$$u_2(x) = -K_\nu(x)\int_{x_s}^{x} I_\nu(y)[\mathcal{S}_1[u_1](y) + \mathcal{S}_2[u_0](y)dy - I_\nu(x)\int_x^\infty K_\nu(y)[\mathcal{S}_1[u_1](y) + \mathcal{S}_2[u_0](y)dy$$

→ $$\frac{r_s\eta'_{as}(r_s)}{\eta_{as}(r_s)} \sim -n - 2\hat{k}\sqrt{n} - 1 - 3\hat{k}^2$$

B. failure in KS04

As B.C. for asymptotic sols, Kol,Sorkin(2004) used

$$\chi'(r_s)/\chi(r_s) = -\frac{d}{r_s}$$

because master Eq. is singular at $r_s{}^{d-3} = \frac{d-1}{2}$

they say since

$$-d = \frac{r_s \chi'_{as}(r_s)}{\chi_{as}(r_s)} = -d + 1 - k^2/d$$

→ $k_{GL} \to \sqrt{d}$

But, since 1/r^n ~ 1/n at r_s, NLO should affect the matching.

We calculated the next order and …

$$\frac{r_s \chi'_{as}(r_s)}{\chi_{as}(r_s)} = -d + \underline{1 - k^2/d - (1 - k^2/d)}$$

canceled

→ Trivial matching !!

coincidence or
reflecting some physics ?

C. Asymptotic Perturbation

$$u_1(r) = r^{\frac{n}{2}}\eta_1(r)$$

$$xu_{1,xx} + u_{1,x} - (x + \nu^2 x^{-1})u_1 - \mathcal{S}_1[u_0] = 0$$

$$\mathcal{S}_1[u] \equiv \quad k_\Omega^{2\nu-2} x^{-2\nu+1}[(A_1x^2 + A_2)u(x) + A_3 x u'(x)]$$

$$A_1 = k_\Omega^{-4}(k^4 + 3k^2\Omega^2 + 2\Omega^4) \simeq 1$$
$$A_2 = \frac{1}{2}k_\Omega^{-2}(3n^2\Omega^2 + 2n(n-1)k^2) \simeq n^2$$
$$A_3 = -k_\Omega^{-2}n(2k^2 + 3\Omega^2) \simeq -2n$$

$$\mathcal{S}_2[u] \equiv \quad \frac{k_\Omega^{4\nu-4}}{4x^{4\nu-1}}[(B_1x^4 + B_2x^2 + B_3)u(x) + (B_4x^2 + B_5)xu'(x)]$$

$$B_1 = 4(k^2 + 3\Omega^2)k_\Omega^{-2} \simeq 4$$
$$B_2 = (2k^4(3n^2 - 2n) + 2n^2\Omega^4 + 10n^2k^2\Omega^2)k_\Omega^{-4} \simeq 6n^2$$
$$B_3 = n^4 + 3n^3 + 2n^2 \simeq n^4$$
$$B_4 = -2n(4k^4 + 12k^2\Omega^2 + 6\Omega^4)k_\Omega^{-4} \simeq -8n$$
$$B_5 = 2n^2(n+1) \simeq 2n^3$$