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"Triangular solution to the general relativistic three-body problem"

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Triangular solution to the general relativistic three-body problem

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with Ichita-san & Asada-san

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- Introduction
- Equilateral triangular solution in GR
- Triangular solution in GR: general masses
- Summary

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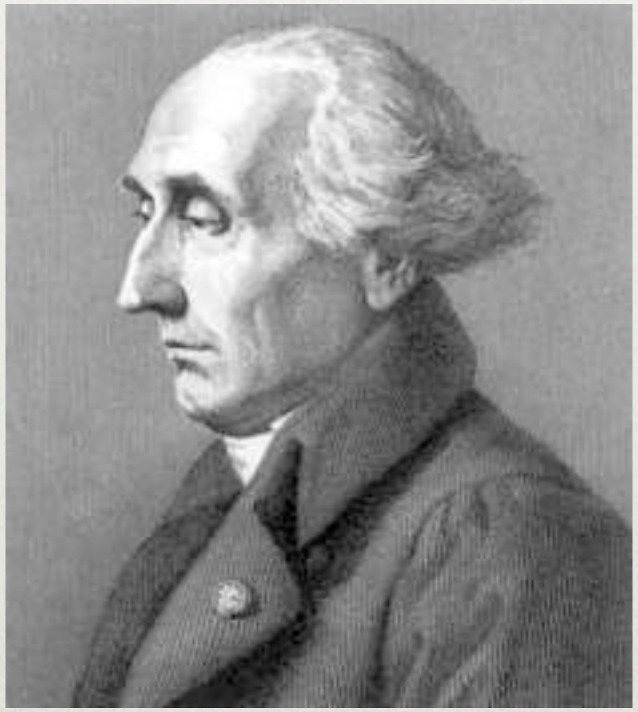
Three-body problem

Particular solutions to the three-body problem

Euler's collinear solution (1765)

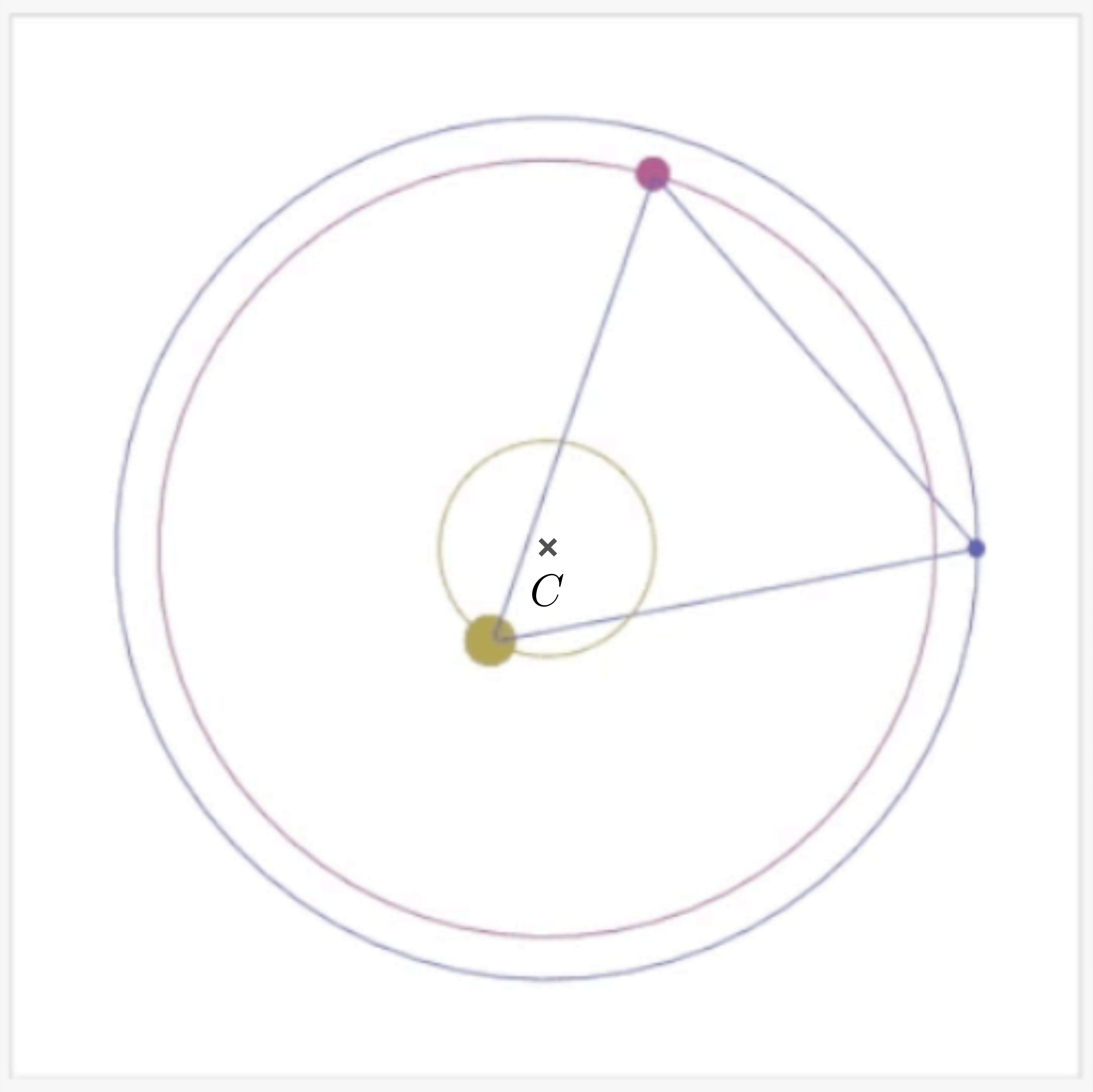
&



J. L. Lagrange

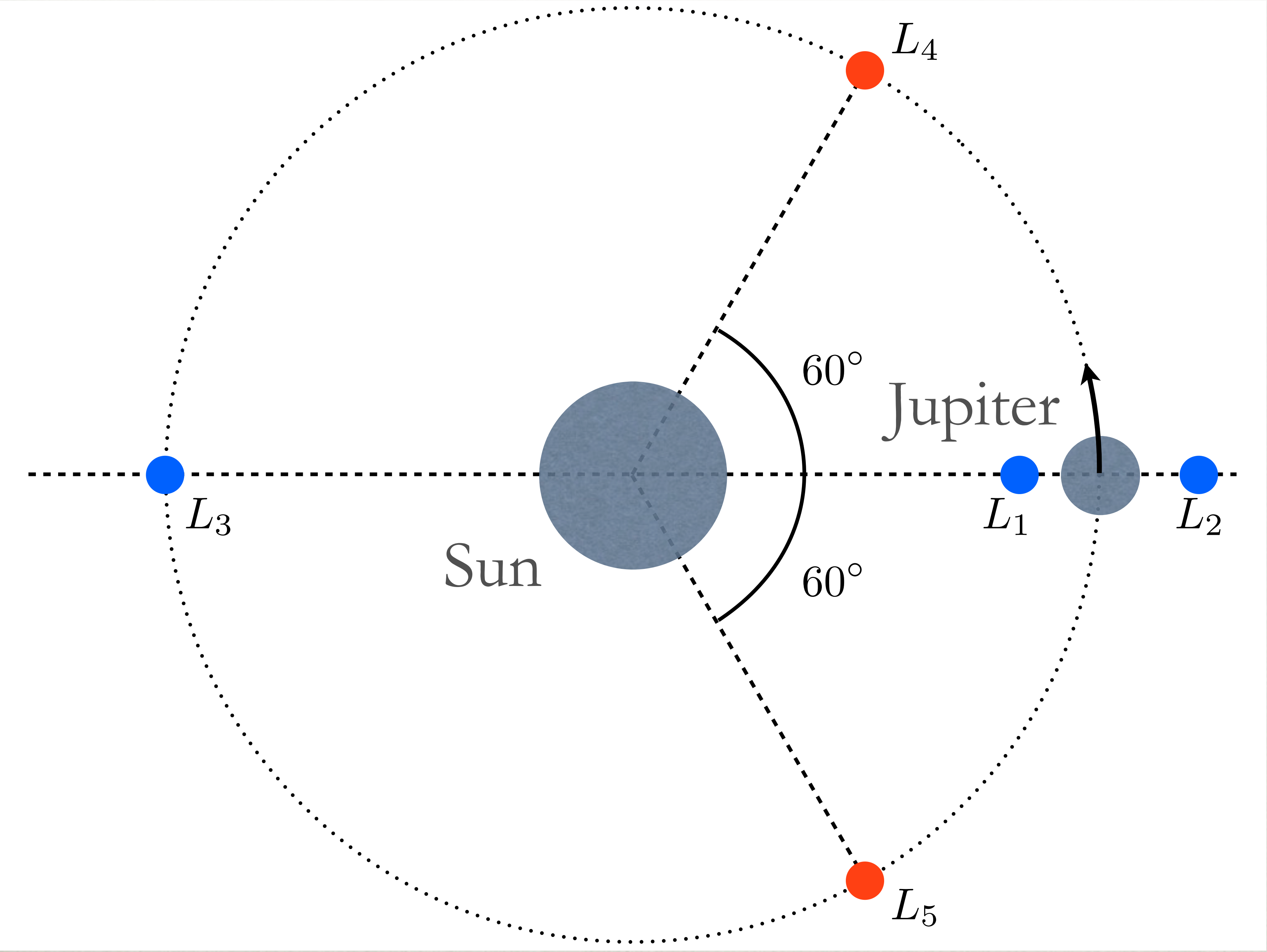
Lagrange's equilateral triangular solution (1772)

Equilateral triangular solution

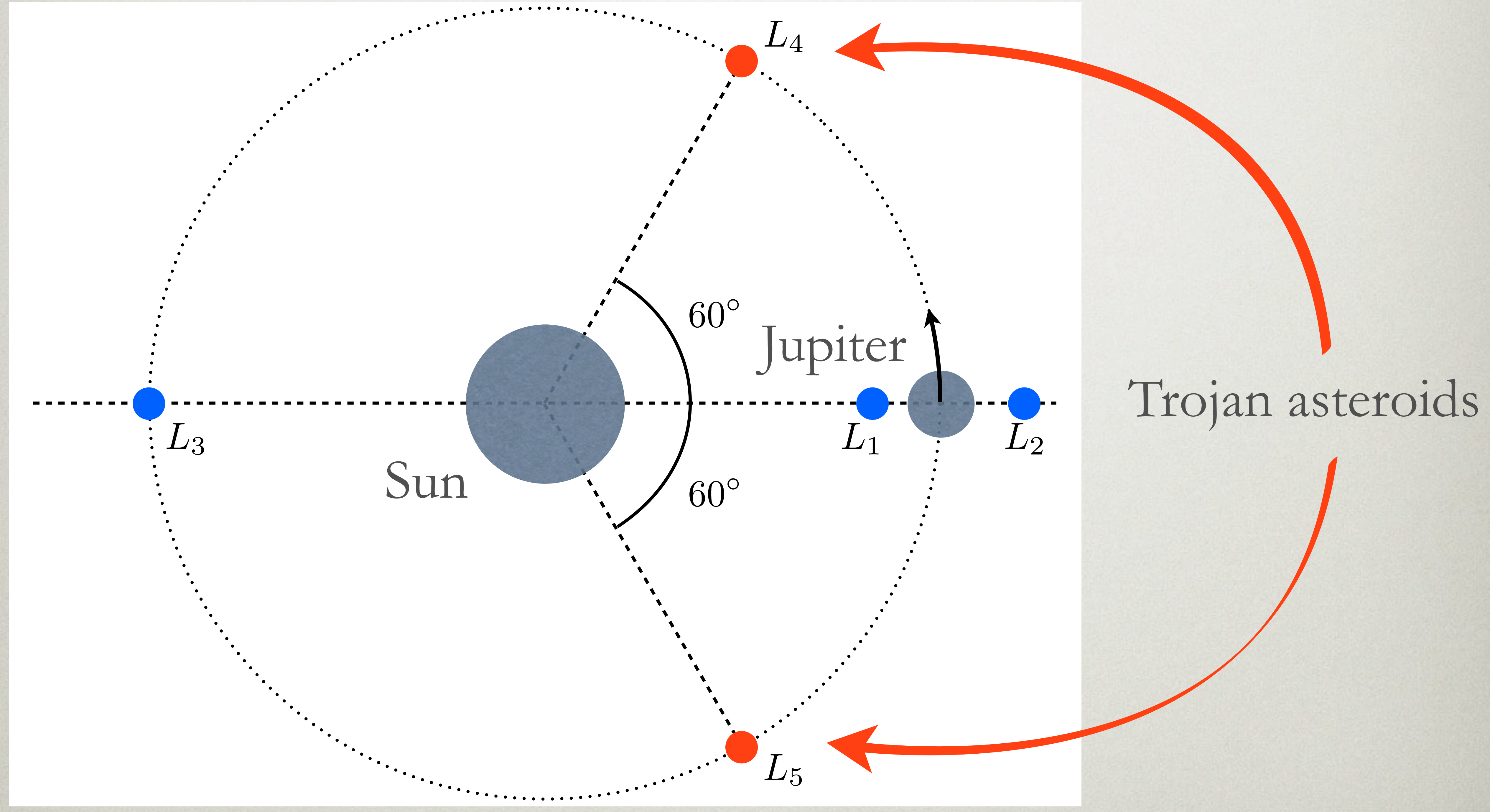


C : the center of mass

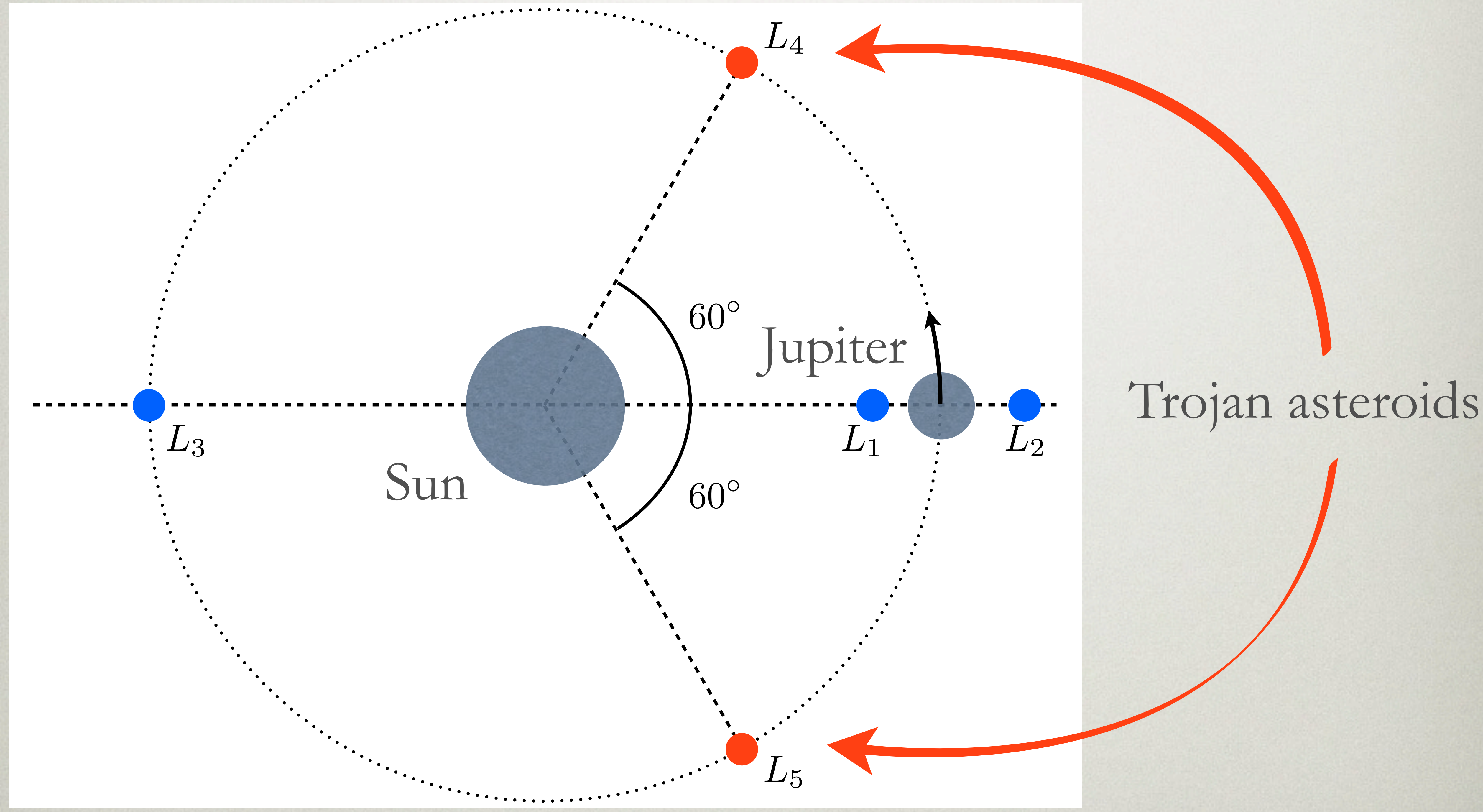
Lagrange points



Lagrange points



Lagrange points

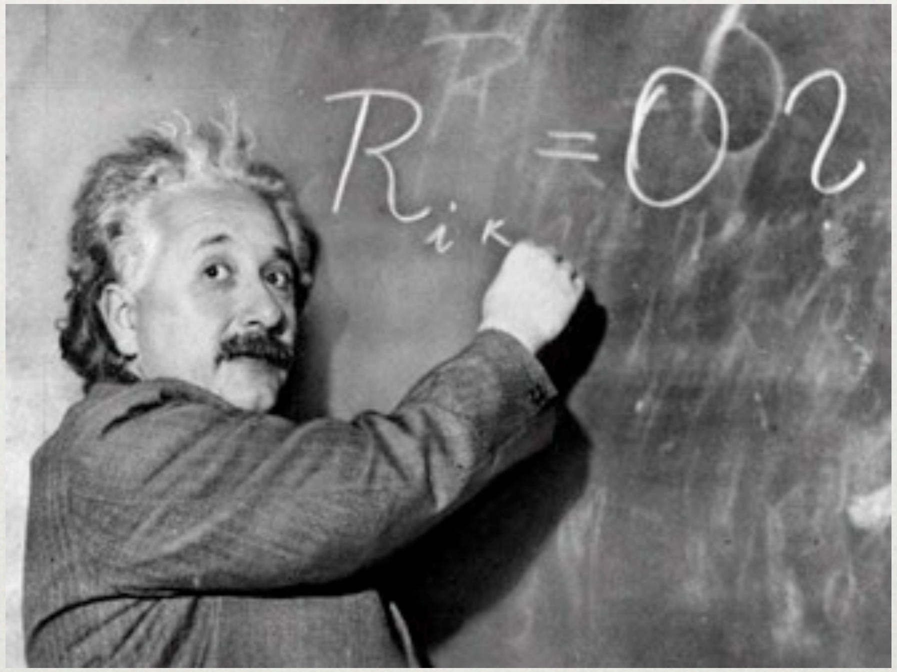


What happens in the general relativity (GR)?

GR effects of Solar system

Two-body systems: ○

(e.g. the perihelion precession of Mercury)



Three-body systems: ?

It is interesting as a new test of GR

EIH equation of motion

Einstein-Infeld-Hoffman (EIH) equation of motion for N bodies

Newtonian term

GR correction by mass

$$
\begin{aligned}
m_K \frac{d^2 \boldsymbol{r}_K}{dt^2} = \sum_{A \neq K} \boldsymbol{r}_{AK} \frac{G m_A m_K}{r_{AK}^3} \Bigg[1 &- 4 \sum_{B \neq K} \frac{G m_B}{c^2 r_{BK}} - \sum_{C \neq A} \frac{G m_C}{c^2 r_{CA}} \left(1 - \frac{\boldsymbol{r}_{AK} \cdot \boldsymbol{r}_{CA}}{2 r_{CA}^2} \right) \\
&+ \left(\frac{\boldsymbol{v}_K}{c} \right)^2 + 2 \left(\frac{\boldsymbol{v}_A}{c} \right)^2 - 4 \left(\frac{\boldsymbol{v}_A}{c} \right) \cdot \left(\frac{\boldsymbol{v}_K}{c} \right) - \frac{3}{2} \left(\frac{\left(\frac{\boldsymbol{v}_A}{c} \right) \cdot \boldsymbol{r}_{AK}}{r_{AK}} \right)^2 \Bigg] \\
&- \sum_{A \neq K} \left[\left(\frac{\boldsymbol{v}_A}{c} \right) - \left(\frac{\boldsymbol{v}_K}{c} \right) \right] \frac{G m_A m_K}{r_{AK}^3} \boldsymbol{r}_{AK} \cdot \left[3 \left(\frac{\boldsymbol{v}_A}{c} \right) - 4 \left(\frac{\boldsymbol{v}_K}{c} \right) \right] \\
&+ \frac{7}{2} \sum_{A \neq K} \sum_{C \neq A} \boldsymbol{r}_{CA} \frac{G m_C m_K}{r_{CA}^3} \frac{G m_A}{c^2 r_{AK}}
\end{aligned}
$$

GR correction by velocity

Triple product

We look for an equilibrium solution in a circular motion

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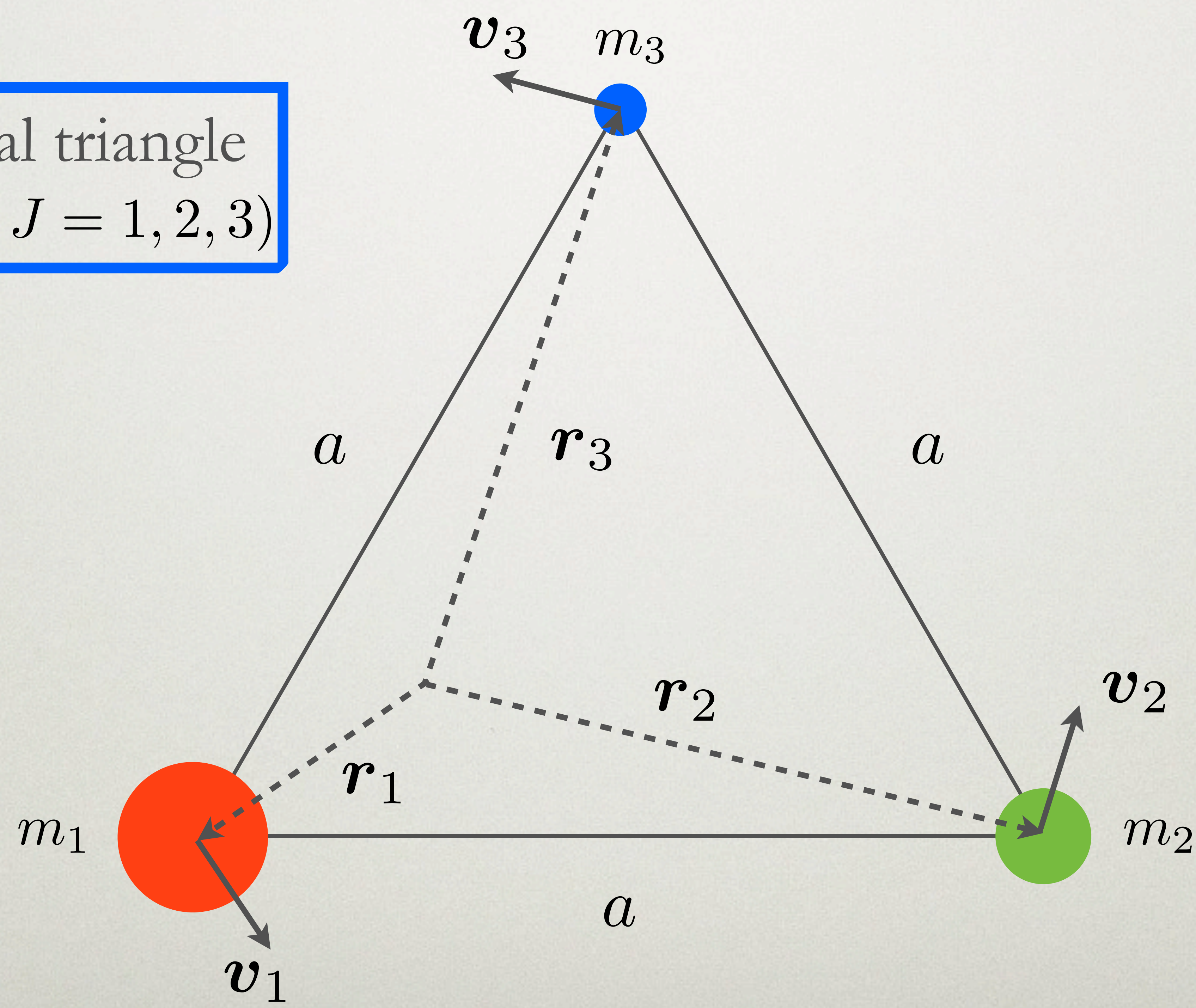
Equilateral triangular configuration

Equilateral triangle
$r_{IJ} = a \ (I, J = 1, 2, 3)$



Center of mass at 1PN

$$\boxed{\boldsymbol{r}_G = \frac{\sum_A \nu_A \boldsymbol{r}_A \left[1 + \frac{1}{2}\left(\left(\frac{v_A}{c}\right)^2 - \sum_{B\neq A}\frac{Gm_B}{c^2 r_{AB}}\right)\lambda\right]}{\sum_C \nu_C \left[1 + \frac{1}{2}\left(\left(\frac{v_C}{c}\right)^2 - \sum_{D\neq C}\frac{Gm_D}{c^2 r_{CD}}\right)\lambda\right]}}, \quad \lambda \equiv \frac{GM}{c^2 a} \ll 1$$

In general, this is different from the Newtonian one

Center of mass at 1PN

$$\boldsymbol{r}_G = \frac{\sum_A \nu_A \boldsymbol{r}_A \left[1 + \frac{1}{2}\left(\left(\frac{v_A}{c}\right)^2 - \sum_{B \neq A} \frac{G m_B}{c^2 r_{AB}} \right) \lambda \right]}{\sum_C \nu_C \left[1 + \frac{1}{2}\left(\left(\frac{v_C}{c}\right)^2 - \sum_{D \neq C} \frac{G m_D}{c^2 r_{CD}} \right) \lambda \right]}, \quad \lambda \equiv \frac{GM}{c^2 a} \ll 1$$

In general, this is different from the Newtonian one

But

In this case, this agrees with the Newtonian case

Equilateral triangular solution at the 1PN

At 1PN order, EOM for m_1 becomes

$$-\omega^2 \boldsymbol{n}_1 = -\frac{M}{a^3}\boldsymbol{n}_1 + g_{PN1}\boldsymbol{n}_1 + \frac{\sqrt{3}}{16}\frac{M}{a^3}\frac{\nu_2\nu_3(\nu_2-\nu_3)}{\nu_2^2+\nu_2\nu_3+\nu_3^2}\left[6+9\big(\nu_2+\nu_3\big)\right]\lambda\boldsymbol{n}_{\perp 1}$$

$$g_{PN1} = \frac{\lambda}{16(\nu_2^2+\nu_2\nu_3+\nu_3^2)}\frac{M}{a^3} \times \Big[48(\nu_2^2+\nu_2\nu_3+\nu_3^2) - 2(8\nu_2^3+7\nu_2^2\nu_3+7\nu_2\nu_3^2+8\nu_3^3) + (16\nu_2^4+41\nu_2^3\nu_3+84\nu_2^2\nu_3^2+41\nu_2\nu_3^3+16\nu_3^4)\Big]$$

ω : angular velocity, $\nu_I \equiv m_I/M,\ M = \sum_I m_I\ (I=1,2,3)$

$\boldsymbol{n}_1 \equiv \boldsymbol{r}_1/|\boldsymbol{r}_1|,\ \boldsymbol{n}_{\perp 1} \equiv \boldsymbol{v}_1/|\boldsymbol{v}_1|,\ \boldsymbol{n}_{\perp 1}$ is normal to $\boldsymbol{n}_1$

Equilateral triangular solution at the 1PN

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In only 2 cases, bodies satisfy EOM;

- mass ratio 1 : 1 : 1
- mass ratio 0 : 0 : 1

Equilateral triangular solution at the 1PN

$$-\omega^2 \boldsymbol{n}_1 = -\frac{M}{a^3}\boldsymbol{n}_1 + g_{PN1}\boldsymbol{n}_1 + \frac{\sqrt{3}}{16}\frac{M}{a^3}\frac{\nu_2\nu_3(\nu_2-\nu_3)}{\nu_2^2+\nu_2\nu_3+\nu_3^2}\left[6+9\left(\nu_2+\nu_3\right)\right]\lambda\boldsymbol{n}_{\perp 1}$$

In only 2 cases, bodies satisfy EOM;

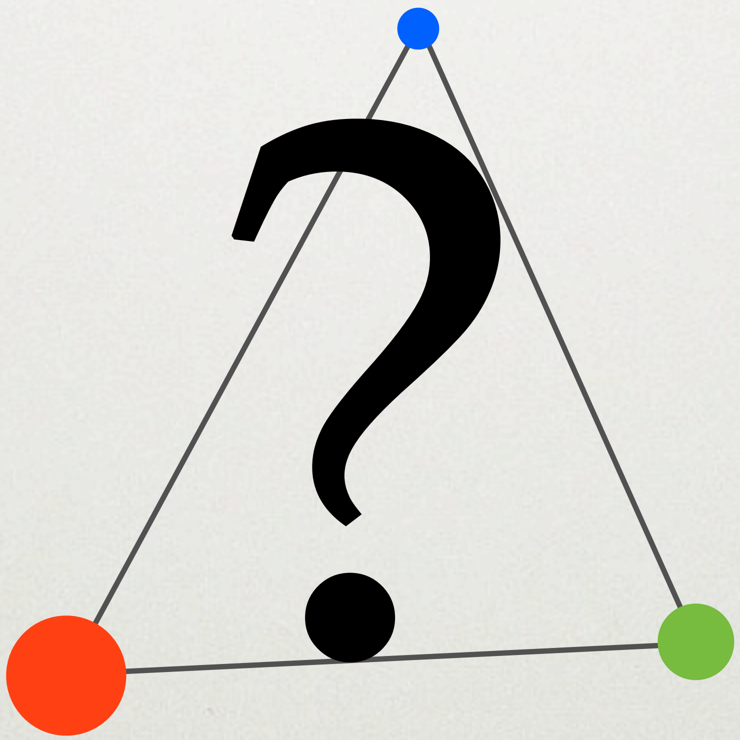
- mass ratio 1 : 1 : 1
- mass ratio 0 : 0 : 1

This solution does not always exist in GR

[Ichita, KY & Asada, PRD **83**, 084026 (2011)]

Equilateral triangular solution at the 1PN

For the arbitrary mass ratio,
a solution exists?



cf. [Krefetz, Astron. J. **72**, 471 (1967)]
for restricted 3-body problem,
used by [Seto & Muto, PRD **81**, 103004 (2010)]

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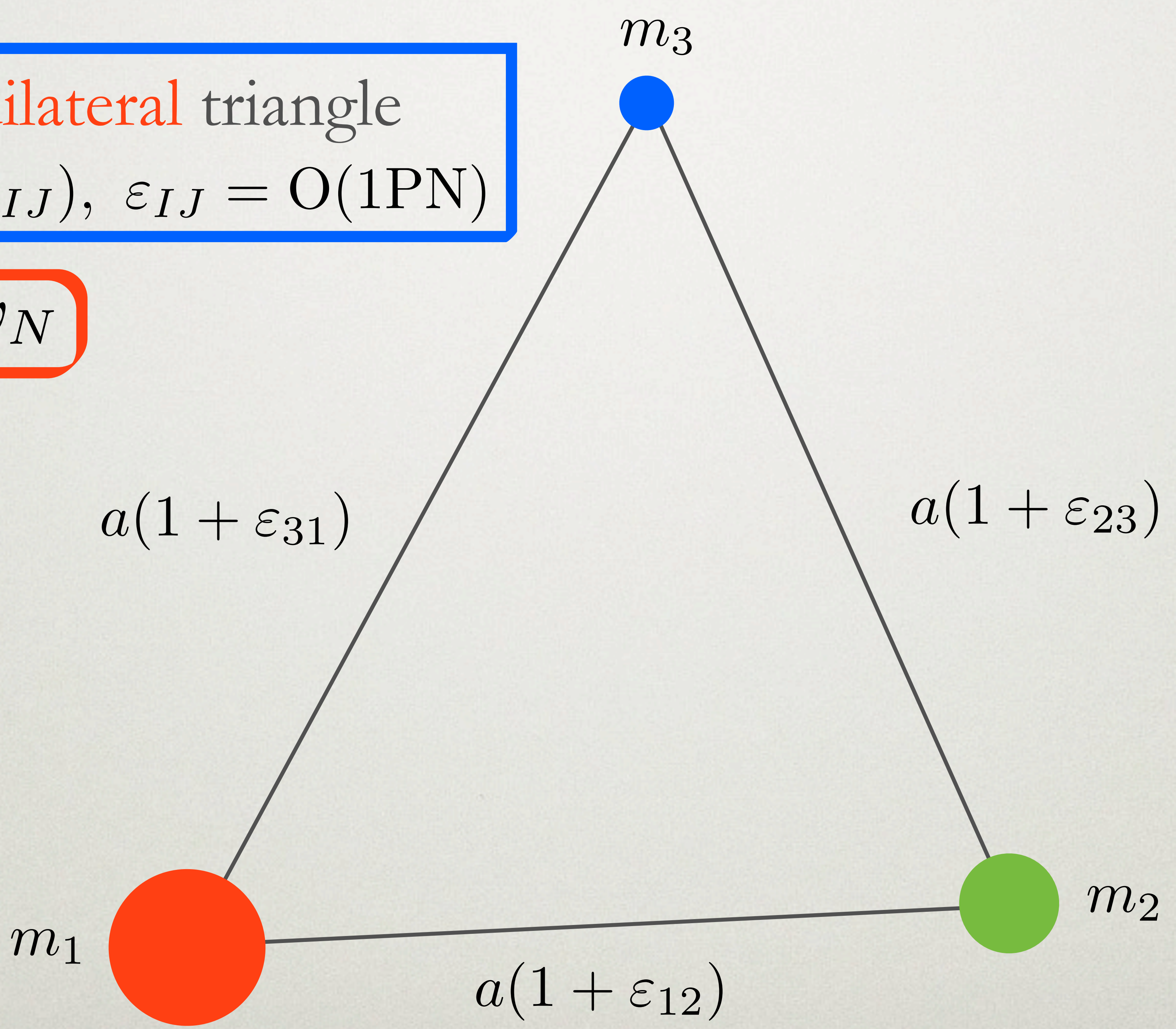
Corrections of distance

PN inequilateral triangle

$r_{IJ} = a(1+\varepsilon_{IJ}),\ \varepsilon_{IJ} = \mathrm{O(1PN)}$

$\omega = \omega_N$



We can ignore the 1PN correction to the center of mass

Triangular solution at the 1PN

EOM for m_1 becomes

$$
\begin{aligned}
-\omega^2 \boldsymbol{r}_1 = &-\omega_N^2 \boldsymbol{r}_1 \\
&+ \nu_2 \left(-3 + \nu_1\nu_2 + \nu_2\nu_3 + \nu_3\nu_1 - \frac{3}{8}\nu_3[5 - 3(\nu_1 + \nu_2)] \right) \lambda \boldsymbol{r}_{21} \\
&+ \nu_3 \left(-3 + \nu_1\nu_2 + \nu_2\nu_3 + \nu_3\nu_1 - \frac{3}{8}\nu_2[5 - 3(\nu_3 + \nu_1)] \right) \lambda \boldsymbol{r}_{31} \\
&- 3(\nu_2 \varepsilon_{12} \boldsymbol{r}_{21} + \nu_3 \varepsilon_{31} \boldsymbol{r}_{31})
\end{aligned}
$$

$$\omega = \omega_N$$

Triangular solution at the 1PN

EOM for m_1 becomes

$$
\begin{aligned}
-\omega^2 \boldsymbol{r}_1 = &- \omega_N^2 \boldsymbol{r}_1 \\
&+ \nu_2 \left(-3 + \nu_1\nu_2 + \nu_2\nu_3 + \nu_3\nu_1 - \frac{3}{8}\nu_3[5 - 3(\nu_1 + \nu_2)] \right) \lambda \boldsymbol{r}_{21} \\
&+ \nu_3 \left(-3 + \nu_1\nu_2 + \nu_2\nu_3 + \nu_3\nu_1 - \frac{3}{8}\nu_2[5 - 3(\nu_3 + \nu_1)] \right) \lambda \boldsymbol{r}_{31} \\
&- 3(\nu_2 \varepsilon_{12} \boldsymbol{r}_{21} + \nu_3 \varepsilon_{31} \boldsymbol{r}_{31})
\end{aligned}
$$

$\omega = \omega_N$ ⇒ [ ] $= 0$

Triangular solution at the 1PN

EOM for m_1 becomes

$$-\omega^2 \boldsymbol{r}_1 = -\omega_N^2 \boldsymbol{r}_1 + \nu_2\left(-3+\nu_1\nu_2+\nu_2\nu_3+\nu_3\nu_1-\frac{3}{8}\nu_3[5-3(\nu_1+\nu_2)]\right)\lambda\boldsymbol{r}_{21} + \nu_3\left(-3+\nu_1\nu_2+\nu_2\nu_3+\nu_3\nu_1-\frac{3}{8}\nu_2[5-3(\nu_3+\nu_1)]\right)\lambda\boldsymbol{r}_{31} - 3(\nu_2\varepsilon_{12}\boldsymbol{r}_{21}+\nu_3\varepsilon_{31}\boldsymbol{r}_{31})$$

$\omega = \omega_N$ ⇒ [ ] $= 0$

Triangular solution at the 1PN

As a result, we could uniquely express ε_{IJ}

$$\varepsilon_{12} = -\left[1 - \frac{1}{3}(\nu_1\nu_2 + \nu_2\nu_3 + \nu_3\nu_1) + \frac{1}{8}\nu_3[5 - 3(\nu_1 + \nu_2)]\right]\lambda,$$

$$\varepsilon_{23} = -\left[1 - \frac{1}{3}(\nu_1\nu_2 + \nu_2\nu_3 + \nu_3\nu_1) + \frac{1}{8}\nu_1[5 - 3(\nu_2 + \nu_3)]\right]\lambda,$$

$$\varepsilon_{31} = -\left[1 - \frac{1}{3}(\nu_1\nu_2 + \nu_2\nu_3 + \nu_3\nu_1) + \frac{1}{8}\nu_2[5 - 3(\nu_1 + \nu_3)]\right]\lambda.$$

Triangular solution for the arbitrary mass ratio at 1PN

[KY & Asada, submitted]

Application for Solar system

Corrections for L4 (L5) of Solar system [m]

Planet	Sun-Planet	Sun-L4 (L5)	Planet-L4 (L5)
Earth	-1477	-1477	-1477 -923
Jupiter	-1477	-1477	-1477 -922

The sign + denotes increase of distance

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- Introduction
- Equilateral triangular solution in GR
- Triangular solution in GR: general masses
- Summary

Summary

- We found a triangular solution at the 1PN order
- The PN triangle is smaller than the Newtonian one (for same mass ratio), and changed from an equilateral triangle
- This solution may also be applied to near SMBHs and compact binaries
- Future observations are needed

Ongoing & Future works

- The Stability
- The Gravitational wave
- Higher order PN approximation
- An elliptical motion
- Four (or more) body systems

Thank you for your attention