BH perturbations & gauge dof in the near-horizon limit

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Contents of the Talk

- Introduction
- Secular growth of mass & spin
- Singular behavior of gauge dog
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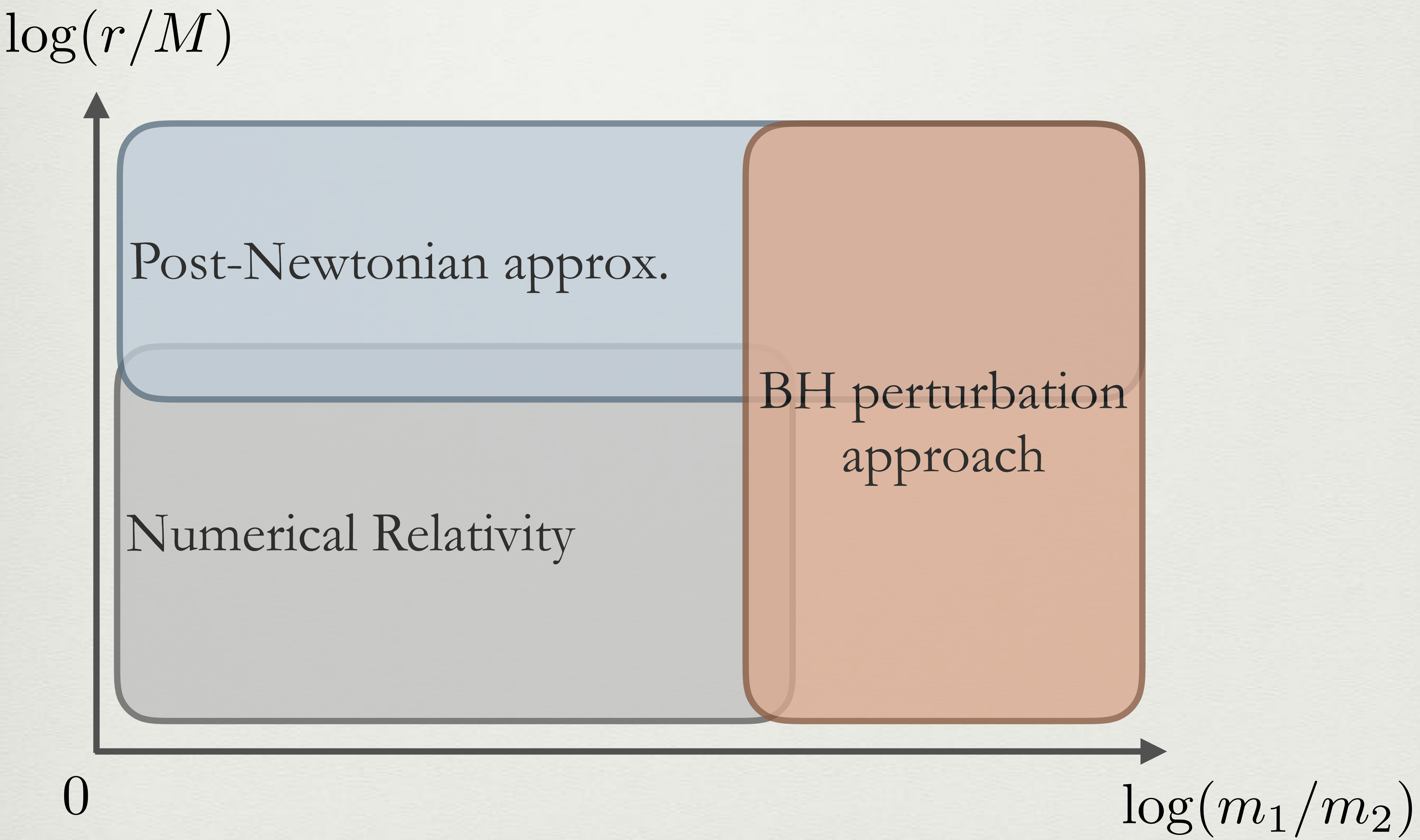
Methods for the 2-Body Problem



Extreme Mass Ratio Inspiral (EMRI)

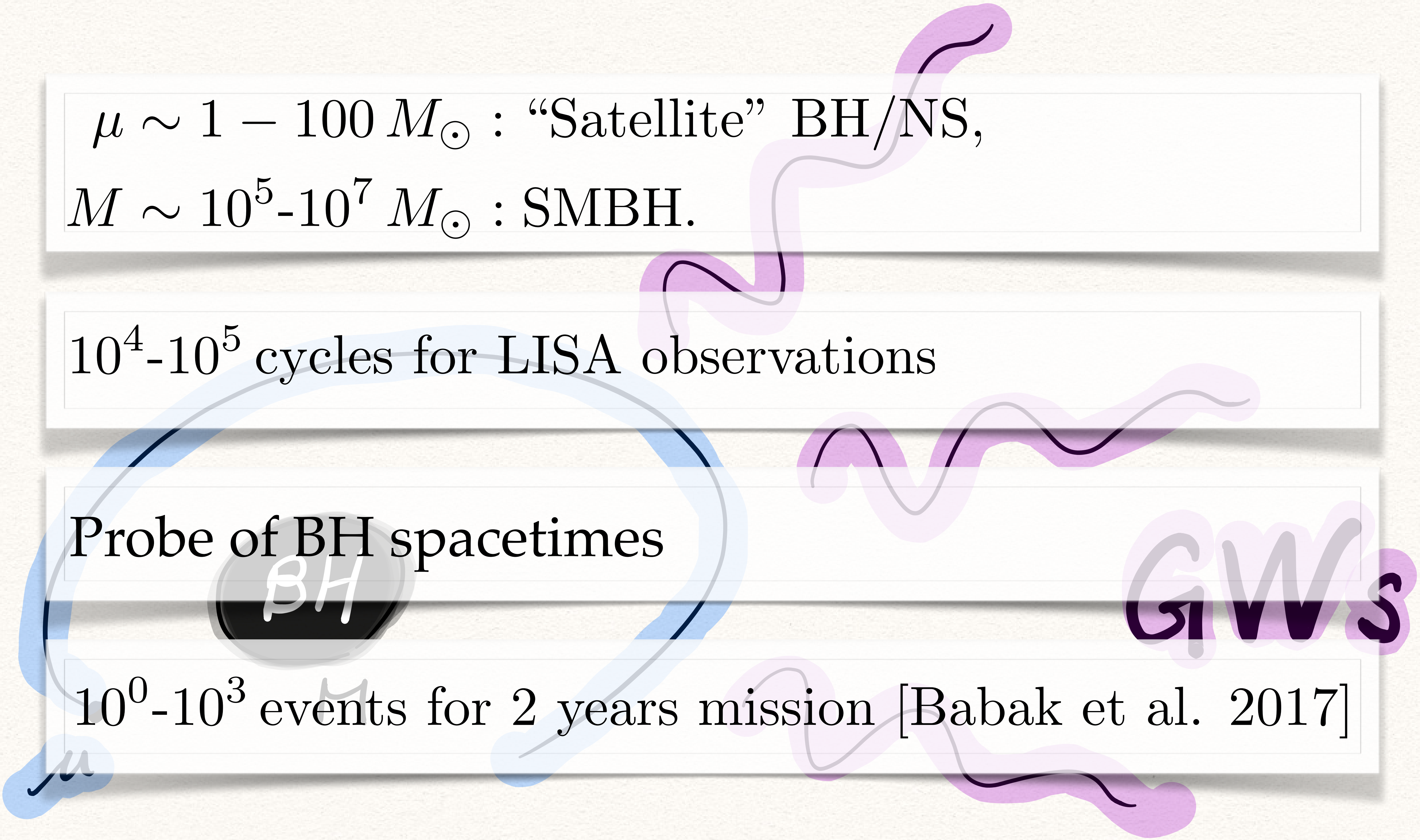
$\mu \sim 1 - 100\, M_\odot$: "Satellite" BH/NS,

$M \sim 10^5$-$10^7\, M_\odot$: SMBH.

10^4-10^5 cycles for LISA observations

Probe of BH spacetimes

10^0-10^3 events for 2 years mission [Babak et al. 2017]



Gravitational Self-Force (GSF)

- Expand equations in the mass ratio:

$$\varepsilon \equiv \mu/M \ll 1,$$

$$g_{\mu\nu} = g^{\mathrm{BG}}_{\mu\nu} + \varepsilon h^{(1)}_{\mu\nu} + \varepsilon^2 h^{(2)}_{\mu\nu} + \cdots .$$

- Valid even if $v/c \sim 1$ ⟺ PN regime.

- EoM for the "satellite"

$$\ddot{z}^{\mu} = 0 + \varepsilon F^{(1)\mu} + \varepsilon^2 F^{(2)\mu} + \cdots .$$

- Formal expressions of GSF is known up to $\mathrm{O}(\varepsilon^2)$ [Pound, 2012].

Why the Second Order?

- If neglect the second-order self-force $\mathrm{O}(\varepsilon^2)$,
  ➡ error in acceleration is $\delta\ddot{z}^\mu \sim \varepsilon^2/M$.
  - Error in position is $\delta z^\mu \sim \varepsilon^2\tau^2/M$.
  - After inspiral time $\tau \sim M/\varepsilon$,
    error in position becomes $\delta z^\mu \sim M$.
- The second-order perturbation $h^{(2)}$ gives
  detectable effects on GW phase!

Second-Order Vacuum Equations

- We expand equations in the mass ratio:

$$g_{\mu\nu} = g_{\mu\nu}^{\mathrm{BG}} + \varepsilon h_{\mu\nu}^{(1)} + \varepsilon^2 h_{\mu\nu}^{(2)} + \cdots .$$

- The field equations to the second-order are

$$\delta G^{\mu}{}_{\nu}[\varepsilon h^{(1)} + \varepsilon^2 h^{(2)}] = -\delta^2 G^{\mu}{}_{\nu}[\varepsilon h^{(1)}, \varepsilon h^{(1)}],$$

where $\delta G^{\mu}{}_{\nu}[h]$ & $\delta^2 G^{\mu}{}_{\nu}[h, h]$ are linear & quadratic in h .

2 Types of Divergences Near Horizon

- Physical & spurious divergence in frequency domain
  - Secular changes of mass δM & spin δa of BG BH.
- Unphysical pure gauge degrees of freedom
  - Need to identify and
    remove by the boundary conditions.

✓ The singular behavior can be seen
from the l.h.s. of Einstein equations

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Origin of the Spurious Divergence

- The origin of the physical & spurious divergence is slowly growing perturbations in time domain:

$$\hat{h}(\omega) = \int_{-\infty}^{\infty} h(v) e^{-i\omega v}\, dv.$$

- The secular growth would be caused by “constant source”

$$\partial_v h(v) = \text{“const.”} \quad \Rightarrow \quad h(v) \propto v.$$

- Since the source is $\mathrm{O}(\varepsilon^2)$, $h(v) = \mathrm{O}(\varepsilon^2)$ at each time.
  - After inspiral time $v = \mathrm{O}(1/\varepsilon)$, we have $h(v) = \mathrm{O}(\varepsilon)$.
- Consider $h(v) \to h^{(1)}(\tilde{v})$ with the “slow time” $\tilde{v} \equiv \varepsilon v$.

The Eddington-Finkelstein Coordinates

- The Schwarzschild background metric is

$$ds^2 = -f dv^2 + 2dvdr + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

in the ingoing Eddington-Finkelstein coordinates, where $v = t + r_*$.

✓ No singularity appears on the BH horizon.

✓ Ingoing GWs propagate along a null line, on which the "time coordinate" v is constant.

Near-Horizon Expansion

- Expand the perturbations near the horizon

$$h^{(i)\mathrm{sta}}_{\mu\nu}(\tilde{v}, r, \theta) = h^{(i)\mathrm{sta}}_{0\mu\nu}(\tilde{v}, \theta) + f h^{(i)\mathrm{sta}}_{1\mu\nu}(\tilde{v}, \theta) + \mathrm{O}(f^2).$$

- Since $\Box \bar{h}_{\mu\nu} = g^{\alpha\beta}\nabla_\alpha \nabla_\beta \bar{h}_{\mu\nu}$ and $g^{rr} = 0$ on $r = 2M$, d^2/dr^2 does not appear in the Lorenz gauge.
  - $d^2/dr^2(f^2) = \mathrm{O}(1)$ does not exist near the horizon.

✓ $\mathrm{O}(f^2)$ in $h^{(i)\mathrm{sta}}_{\mu\nu}(\tilde{v}, r, \theta)$ is not necessary.

Second-Order Einstein Tensor

- At the second order in ε,
  4 components of $\delta G^{\mu}{}_{\nu}$ NOT containing $h_1^{(2)}$ are

$$\sin\theta\,\delta G^{(2)\mathrm{sta}\,r}{}_{v} = \frac{\sin\theta}{8M}\partial_{\tilde{v}}\left(4h^{(1)\mathrm{sta}}_{0vv} + h^{(1)\mathrm{sta}}_{0\theta\theta} + h^{(1)\mathrm{sta}}_{0\phi\phi}\right) - \partial_\theta\left[\frac{\sin\theta}{8M^2}\left(h^{(2)\mathrm{sta}}_{0v\theta} + \partial_\theta h^{(2)\mathrm{sta}}_{0vv} - 4M\partial_{\tilde{v}} h^{(1)\mathrm{sta}}_{0v\theta}\right)\right],$$

$$\sin\theta\,\delta G^{(2)\mathrm{sta}\,r}{}_{\phi} = \frac{\sin^2\theta}{2}\partial_{\tilde{v}}\left(4h^{(1)\mathrm{sta}}_{0v\phi} + h^{(1)\mathrm{sta}}_{0r\phi}\right) + \partial_\theta\left[-\frac{\sin^3\theta}{4M}\partial_\theta\left(\frac{1}{\sin\theta}h^{(2)\mathrm{sta}}_{0v\phi}\right) + \sin^2\theta\,\partial_{\tilde{v}} h^{(1)\mathrm{sta}}_{0\theta\phi}\right],$$

$$\vdots$$

Abbott & Deser's Quantities

- We find

$$\frac{1}{8\pi}\int \sqrt{-g}\sin\theta\,\delta G^{(2)\mathrm{sta}\,r}{}_{v}\,d\theta\,d\phi\bigg|_{r=r_{\mathrm{h}}} = -\partial_{\tilde{v}}\left(M^{\mathrm{AD}}\right)\Big|_{r=r_{\mathrm{h}}},$$

$$\frac{1}{8\pi}\int \sqrt{-g}\sin\theta\,\delta G^{(2)\mathrm{sta}\,r}{}_{\phi}\,d\theta\,d\phi\bigg|_{r=r_{\mathrm{h}}} = -\partial_{\tilde{v}}\left(L^{\mathrm{AD}}\right)\Big|_{r=r_{\mathrm{h}}},$$

where

$$M^{\mathrm{AD}} = \frac{1}{2}\int F^{(v)\alpha\beta}\,d\Sigma_{\alpha\beta}, \quad L^{\mathrm{AD}} = \frac{1}{2}\int F^{(\phi)\alpha\beta}\,d\Sigma_{\alpha\beta},$$

$$F^{(v/\phi)}_{\mu\nu} \equiv -\frac{1}{8\pi}\left[\xi^{(v/\phi)\alpha}\bar{h}^{(1)\mathrm{sta}}_{\alpha[\mu;\nu]} + \xi^{(v/\phi)\alpha}{}_{;[\mu}\bar{h}^{(1)\mathrm{sta}}_{\nu]\alpha} + \xi^{(v/\phi)}_{\mu}\bar{h}^{(1)\mathrm{sta};\alpha}_{\nu]\alpha}\right].$$

Energy & Angular Momentum Fluxes

- Ingoing GW's $\dot{E}$ & $\dot{L}$ across the horizon are

$$\dot{E} \equiv \frac{1}{8\pi} \int \sqrt{-g} \left(\delta^2 G^r{}_\alpha \xi^\alpha_{(v)} \right) d\theta \, d\phi,$$

$$\dot{L} \equiv \frac{1}{8\pi} \int \sqrt{-g} \left(\delta^2 G^r{}_\alpha \xi^\alpha_{(\phi)} \right) d\theta \, d\phi.$$

- $\dot{E}$ & $\dot{L}$ are "stationary" for the first-order GWs, which calculated from a geodesic motion.

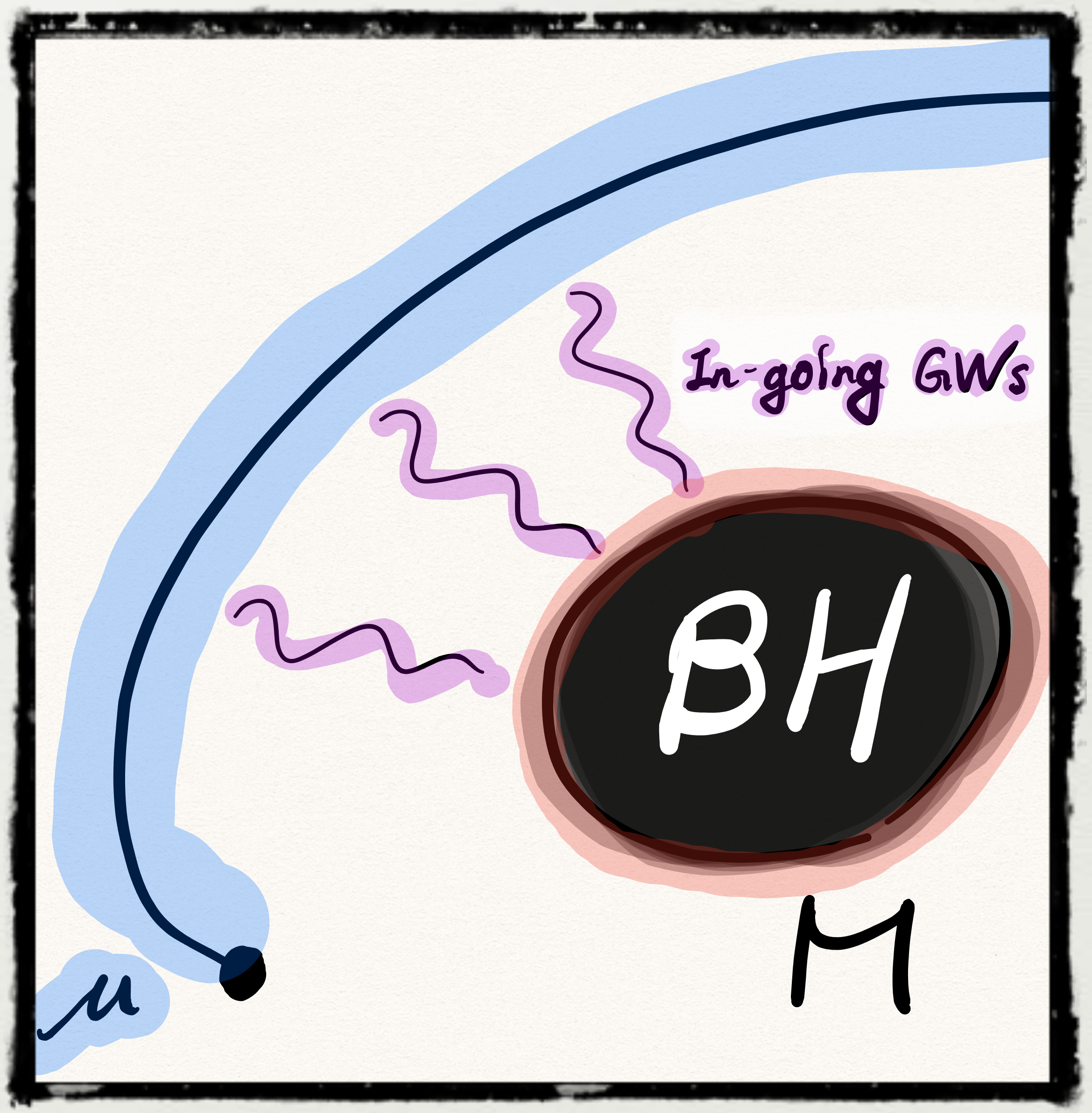
Physical Secular Growth

- $\{rv\}$ & $\{r\phi\}$ components of

$$\int \sqrt{-g}\,\delta G^{\mu}{}_{\nu}\,d\theta\,d\phi = \int \sqrt{-g}\,\left(-\delta^2 G^{\mu}{}_{\nu}\right)\,d\theta\,d\phi,$$

  determine the secular growth.

- The secular growth
  ➡the secular change of the BH's mass/spin $\delta M = \dot{E}\,\tilde{v}$ & $\delta a = \dot{L}\,\tilde{v}$.



Counter Term of Secular Growth

- We have found the secular growth δM & δa

$$h_{\mu\nu}^{(1)\delta M,\delta L} = \frac{\partial g_{\mu\nu}^{\mathrm{BG}}}{\partial M}\delta M + \frac{\partial g_{\mu\nu}^{\mathrm{BG}}}{\partial a}\delta a,$$

which reproduces the spurious divergence.

- Therefore, the effective source term,

$$S_{\mu\nu}^{\mathrm{eff}} = -\delta^2 G_{\mu\nu}[\varepsilon\, h^{(1)}, \varepsilon\, h^{(1)}] - \delta G_{\mu\nu}[\varepsilon\, h^{(1)\delta M,\delta L}]$$

is "regularized."

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2nd-Order Field Eq. in EF Coordinates

- In this coordinates, we obtain the field eq. as

$$\begin{aligned}
\delta G_{vv}^{\mathrm{EFC}(2)} &= \frac{1}{4M}\partial_{\tilde{v}}\left(2h_{vv}^{\mathrm{EFC}(1)} + h_{\theta\theta}^{\mathrm{EFC}(1)}\right) - f\left[\partial_{\tilde{v}}\left\{\frac{3}{2M}h_{vv}^{\mathrm{EFC}(1)} + \frac{1}{M}h_{vr}^{\mathrm{EFC}(1)} + \left(2\partial_{r_*} + \frac{5}{4M}\right)h_{rr}^{\mathrm{EFC}(1)}\right\}\right. \\
&\quad \left. + \frac{1}{2}\left(\partial_{r_*}^2 + \frac{5}{4M}\partial_{r_*} + \frac{1}{4M^2}\right)h_{rr}^{\mathrm{EFC}(2)} + \frac{1}{2M}\partial_{r_*}h_{vr}^{\mathrm{EFC}(2)} + \frac{1}{4M^2}h_{vv}^{\mathrm{EFC}(2)} + \mathrm{O}(f)\right] + S_{vv}^{\mathrm{EFC}}, \\
\delta G_{vr}^{\mathrm{EFC}(2)} &= \partial_{\tilde{v}}\left[\frac{1}{M}h_{vr}^{\mathrm{EFC}(1)} + \frac{3}{2}\left(\partial_{r_*} + \frac{1}{2M}\right)h_{rr}^{\mathrm{EFC}(1)} - \frac{1}{2M}h_{\theta\theta}^{\mathrm{EFC}(1)}\right] \\
&\quad + \frac{1}{2}\left(\partial_{r_*}^2 + \frac{5}{4M}\partial_{r_*} + \frac{1}{4M^2}\right)h_{rr}^{\mathrm{EFC}(2)} + \frac{1}{2M}\partial_{r_*}h_{vr}^{\mathrm{EFC}(2)} + \frac{1}{4M^2}h_{vv}^{\mathrm{EFC}(2)} + S_{vr}^{\mathrm{EFC}} + \mathrm{O}(f), \\
\delta G_{rr}^{\mathrm{EFC}(2)} &= -\frac{1}{f}\partial_{r_*}\left[\partial_{\tilde{v}}h_{rr}^{\mathrm{EFC}(1)} + \frac{1}{2}\left(\frac{1}{M} + \partial_{r_*}\right)h_{rr}^{\mathrm{EFC}(2)} + \mathrm{O}(f)\right] + S_{rr}^{\mathrm{EFC}}, \\
\delta G^{\mathrm{EFC}(2)}{}_I{}^I &= \frac{4M^2}{f}\partial_{r_*}\left(2\partial_{\tilde{v}}h_{vr}^{\mathrm{EFC}(1)} + \partial_{r_*}h_{vr}^{\mathrm{EFC}(2)} + \mathrm{O}(f)\right) + S^{\mathrm{EFC}}{}_I{}^I,
\end{aligned}$$

Singular Behavior of Homogeneous Sols.

- First, we obtain singular asymptotic sols. as

$$h_{vr}^{\mathrm{EFC}(2)} = -\frac{c_{1rr}^{(2)}}{2f} + c_{1vr}^{(2)} r_* + c_{1rr}^{(2)} + c_{2vr}^{(2)} + \mathrm{O}(f),$$

$$h_{rr}^{\mathrm{EFC}(2)} = \frac{c_{1rr}^{(2)}}{f^2} - \frac{2c_{1rr}^{(2)}}{f} + c_{1rr}^{(2)} + c_{2rr}^{(2)} + \mathrm{O}(f),$$

where each $c_{\cdots}^{(2)}$ is a constant.

- We can remove such singularities by an appropriate gauge choice $c_{1rr}^{(2)} = c_{1vr}^{(2)} = 0$.

Residual Gauge Degrees of Freedom

- The residual gauge degrees of freedom

$$\xi^\mu = (\xi^v(\varepsilon; \tilde{v}, r_*), \xi^r(\varepsilon; \tilde{v}, r_*), 0, 0)\,,$$

which must satisfy

$$\Box \xi^\mu = 0.$$

- We obtain the asymptotic solutions as

$$\frac{\xi^v}{2M} = \frac{c_1^v}{f} - 2c_1^v + \frac{\int c_2^v \delta M\, dv}{M^2} + c_1^r f + \mathrm{O}(f^2),$$
$$\frac{\xi^r}{2M} = c_1^r + 2\frac{\int c_2^v \delta M\, dv}{\int \delta M\, dv}\frac{\delta M}{M} + c_2^v \frac{\delta M}{M} f r_* + f\left(c_2^r + 2\frac{\int c_2^v \delta M\, dv}{\int \delta M\, dv}\frac{\delta M}{M}\right) + \mathrm{O}(f^2).$$

of DoF

- # of d.o.f regarding the singularities is 2:

$$c^{(2)}_{1vr} \ \ \& \ \ c^{(2)}_{1rr}.$$

- # of the residual gauge d.o.f is 4:

$$\frac{\xi^v}{2M} = \frac{c_1^v}{f} - 2c_1^v + \frac{\int c_2^v \delta M \, dv}{M^2} + c_1^r f + \mathrm{O}(f^2),$$

$$\frac{\xi^r}{2M} = c_1^r + 2\frac{\int c_2^v \delta M \, dv}{\int \delta M \, dv}\frac{\delta M}{M} + c_2^v \frac{\delta M}{M} f r_* + f\left(c_2^r + 2\frac{\int c_2^v \delta M \, dv}{\int \delta M \, dv}\frac{\delta M}{M}\right) + \mathrm{O}(f^2).$$

→ use 2 of them to remove the singularities.

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Summary

- Need the second-order metric perturbations
  for EMRI observations by LISA.
  - IR & gauge divergences appear near the BH horizon.
- We have
  - identified the IR divergence even for the Kerr BG.
  - found the appropriate gauge choice for SSS pert.
- What about gauge d.o.f for general pert. & Kerr?

THANK YOU FOR YOUR ATTENTION