Scalar-tensor cosmologies: attractor mechanisms and dualities.

Cosmo/CosPA – Tokyo, Japan – September 26 - October 1

José P. Mimoso[§] and Tiago Charters[†]

[§] Departamento de Física, Faculdade de Ciências da Universidade de Lisboa and
Centro de Astronomia e Astrofísica da Universidade de Lisboa,
Campo Grande, Edifício C8
P-1749-016 Lisbon, Portugal

[†] Área Departamental de Matemática, Instituto Superior de Engenharia de Lisboa
Rua Conselheiro Emídio Navarro, 1, P-1949-014 Lisbon, Portugal
Centro de Astronomia e Astrofísica da Universidade de Lisboa,
Campo Grande, Edifício C8
P-1749-016 Lisbon, Portugal

[§]jpmimoso@cii.fc.ul.pt, [†]tca@cii.fc.ul.pt

Abstract

When approaching the Planck scale of energies the need to consider generalizations of Einstein's theory of general relativity (GR) seems inescapable. Recently, these extended gravity theories have also attracted a lot of interest because of the possibility that they might convey an explanation for the late time acceleration of the universe. Among the generalized gravity theories, scalar-tensor (ST) theories stand up as exhibiting a dynamical scalar-field non-minimally coupled to the space-time geometry. This provides an appropriate theoretical framework for the variation of Newton's fundamental constant and is akin to the archetypal dilaton of the low-energy limit of string theories. In the present work we analyse Friedmann scalar-tensor cosmologies with a perfect fluid, and put forward a new integration method that: On the one hand, permits the derivation of exact solutions in a unified, and simpler way than in previous approaches; And, on the other hand, allows a thorough characterization of the cosmological attractor mechanisms. We find that, in addition to the mechanism of relaxation to GR investigated by Damour and Nordtvedt, Brans-Dicke theory provides another cosmological attractor. Our procedure also enables us to investigate form-invariance transformations that establish dualities between solutions of different scalar-tensor theories, and between different solutions within a single theory.

1 Introduction

In the so-called Jordan-Fierz frame [1], scalar-tensor gravity theories are defined from the lagrangian description [2, 3]

$$L_\Phi = \Phi R - \frac{\omega(\Phi)}{\Phi} g^{ab}\Phi_{,a}\Phi^{,b} + 2U(\Phi) + 16\pi L_m, \tag{1}$$

where R is the usual Ricci curvature scalar of a space-time endowed with the metric g_{ab}, Φ is a scalar field, $\omega(\Phi)$ is a dimensionless coupling function, $U(\Phi)$ is a cosmological potential for Φ, and $\mathscr{L}_m$ represents the Lagrangian for the matter fields (note that we shall use units that set $c = 1$). Since Φ is a dynamical field the trademark of these theories is the variation of the gracitational constant $G = \Phi^{-1}$, and the archetypal theory is Brans-Dicke theory in which $\omega(\Phi)$ is a constant [4].
This class of theories can be given in the so-called Einstein frame by means of an appropriate conformal transformation. Following Damour and Nordvedt's notation [1] , the original metric is rescaled according to ($g_{ab} \to \tilde{g}_{ab} = A^{-2}(\varphi)\, g_{ab}$, where $A^{-2}(\varphi) = (\Phi/\Phi_*)$ with $\Phi_* = G^{-1}$ being a constant that we take to be the inverse of Newton's gravitational constant, and $\frac{\mathrm{d}\ln\Phi}{\mathrm{d}\varphi} = \sqrt{\frac{16\pi}{\Phi_*}}\,\alpha(\varphi)$). The action becomes

$$L_\varphi = \tilde{R} - \tilde{g}^{ab}\varphi_{,a}\varphi^{,b} + 2U(\varphi) + 16\pi \tilde{L}_m(\Psi_m, A^2(\varphi)\tilde{g}_{ab}). \tag{2}$$

Still as in Damour and Nordvedt [1] we introduce $\mathscr{A}(\varphi) = \ln A(\varphi)$, $\alpha(\varphi) = \dfrac{\partial \mathscr{A}(\varphi)}{\partial\varphi}$ and $\mathscr{K}(\varphi) = \dfrac{\partial\alpha(\varphi)}{\partial\varphi}$.

2 Equations and procedure

We introduce a new dimensionless variable $x = \dot\varphi/\tilde{H}$, the time coordinate $\tau \propto \ln a$ [5], and $m(\varphi) \propto A^{4-3\gamma}(\varphi)$. We then obtain the following autonomous planar dynamical system

$$x' = -\left(3 - \frac{x^2}{2}\right)\left[\left(1 - \frac{\gamma}{2}\right)x + \frac{m_{,\varphi}}{m}\right], \tag{3}$$

$$\varphi' = x. \tag{4}$$

Dividing the former of these equations by the latter, we reduce the integration to a quadrature. Moreover, specifying $x(\varphi)$ always allows us to obtain the solutions in parametric form

$$a(\varphi) = a_0 \exp\left(\int \frac{d\varphi}{x(\varphi)}\right), \tag{5}$$

$$t(\varphi) = \int dt = \int \frac{d\varphi}{\pm\sqrt{m_0}\,x(\varphi)} \exp\left(\int \left[\left(\frac{2-\gamma}{4}\right)x(\varphi) + \frac{3\gamma}{2x(\varphi)}\right]\mathrm{d}\varphi\right). \tag{6}$$

The choice of the generating function $x(\varphi)$ corresponds to choosing $m(\varphi)$ which effectively defines the scalar-tensor gravity theory under consideration.

3 Asymptotic Behaviour

The system of Eqs.(3) and (4) is a 2-dimensional, autonomous system in the φ, x phase plane. The fixed points correspond to the asymptotic regimes of the scalar-tensor theories.
When $x = 0$, the asymptotic states are such that φ is frozen at φ_*, and occur at extrema of $m(\phi)$. This corresponds to general relativity (GR), and it is an attractor at the minima of m and a repellor at the maxima. This result recovers the attractor mechanism put forward by Damour and Nordtvedt in [1] (see also [7]).
On the other hand, there are fixed points at infinite values of φ, that are analysed by studying the infinity manifold with the change o variable $\chi = 1/\varphi$ [6] : $x = x_1 = +\sqrt{6}$ or $x^2 = -\sqrt{6}$ and

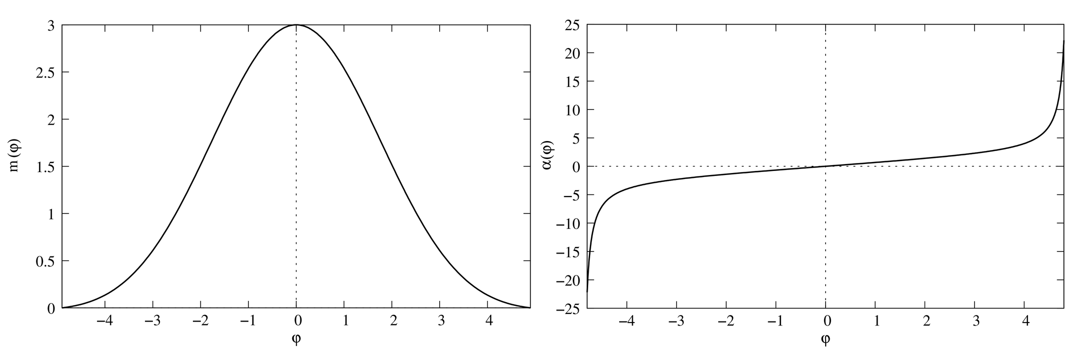
$$\lim_{\varphi\to\infty} \frac{\partial_\varphi m(\varphi)}{m(\varphi)} = \lambda = -\frac{2-\gamma}{2}x_\star. \tag{7}$$

They correspond to a rolling φ with an exponential, or an asymptotically exponential behaviour of $m(\varphi)$ arising from $x(\phi) = \lambda = $ const.. This is what defines Brans-Dicke theory in the Einstein frame and, thus, we realise that the latter theory is also a possible attractor (repellor) of the ST theories.

4 Examples of Exact Solutions

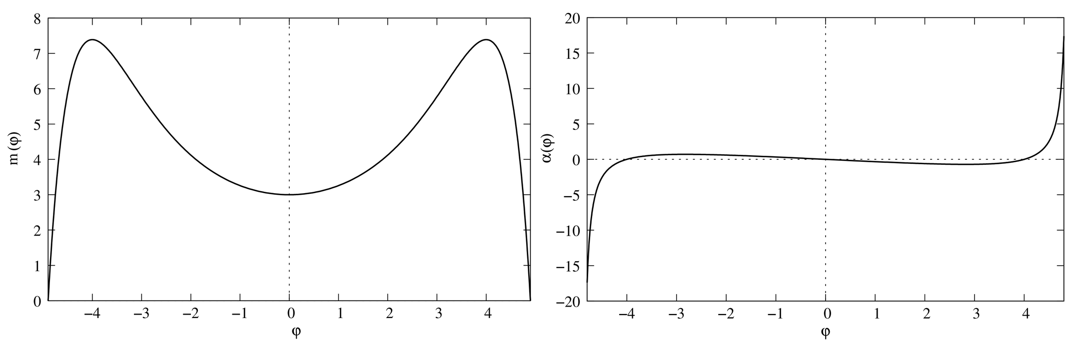
In the following figures we plot $m(\varphi)$ and $\alpha(\varphi)$ for choices of $x(\varphi)$ which yield interesting exact solutions [8].

$x = \lambda\varphi$ with $\lambda = 0.5$, $\gamma = 1$, $m_0 = 1$



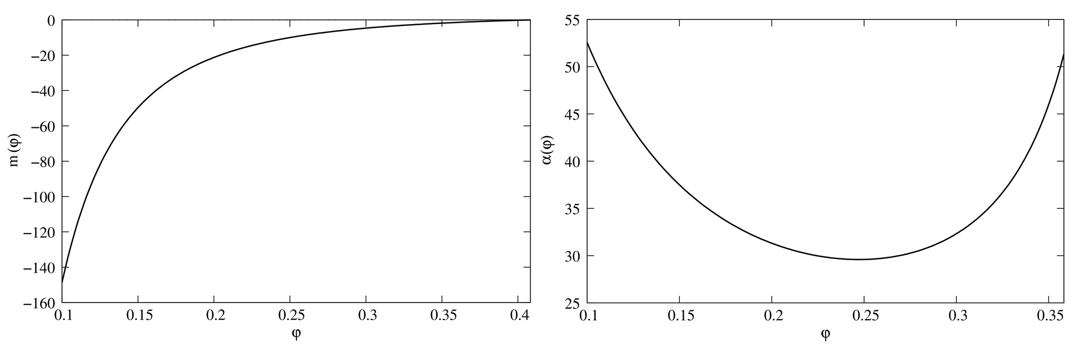
$m(\varphi) = m_0\left(3 - \frac{\lambda^2}{2}\right)e^{(-(1-\gamma/2)\lambda\varphi}$

$x = \lambda\varphi$ with $\lambda = -0.5$, $\gamma = 1$, $m_0 = 1$



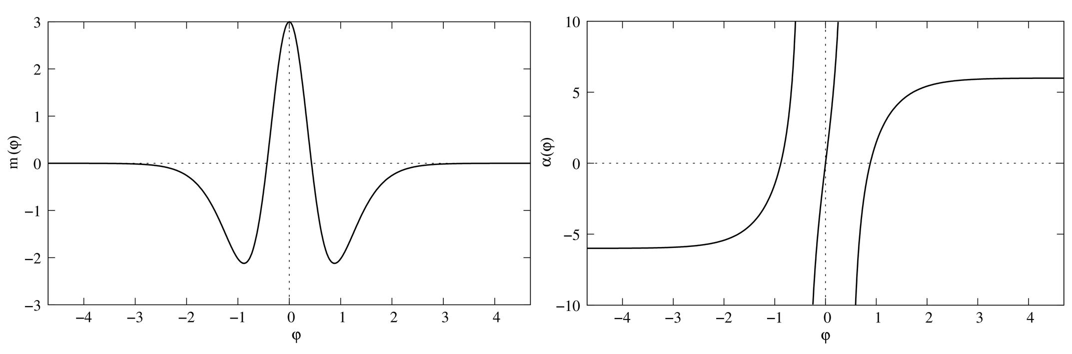
$m(\varphi) = m_0\left(3 - \frac{\lambda^2}{2}\right)e^{(-(1-\gamma/2)\lambda\varphi}$

$x = \lambda/\varphi$ with $\lambda = 1$, $\gamma = 1$, $m_0 = 1$



$m(\varphi) = m_0\left(3 - \frac{\lambda^2}{2\varphi^2}\right)\varphi^{-(1-\gamma/2)\lambda}$

$x = \lambda\tanh\varphi$ with $\lambda = 5.999$, $\gamma = 1$, $m_0 = 1$



$m(\varphi) = m_0\left(\frac{\lambda^2}{2}\tanh^2\varphi - 6\right)(\cosh\varphi)^{\gamma\lambda/2-2}$

5 Form-Invariance dualities

If we let H be transformed into $\bar{H}$ [9, 10, 5], we have a form-invariance duality when the following conditions are satisfied

$$\frac{\mathrm{d}\bar{H}}{\mathrm{d}H} = \frac{\bar{H}^2}{H^2}\frac{3\bar\gamma + (1-\bar\gamma/2)\bar{x}^2}{3\gamma + (1-\gamma/2)x^2}. \tag{8}$$

and

$$\frac{d\bar\varphi}{\bar{x}\bar{H}} = \frac{d\varphi}{xH}, \tag{9}$$

which translates the fact that the form invariance transformation preserves the time variable (5). We have also have the redundant condition

$$\frac{d\bar{H}}{dH} = \left(\frac{\mathrm{d}\bar\varphi}{\mathrm{d}\varphi}\right)^2 \frac{3\bar\gamma/\bar{x}^2 + (1-\bar\gamma/2)}{3\bar\gamma/x^2 + (1-\gamma/2)}. \tag{10}$$

We recover the general relativistic equations taking $\bar\gamma = 0 = \gamma$.

We establish a form-invariance transformation between any pair of scalar field solutions by selecting the corresponding generating functions $x(\varphi)$ and $\bar{x}(\bar\varphi)$, deriving the corresponding $H(\varphi)$ and $\bar{H}(\bar\varphi)$ functions in accordance to equation (8), and plugging them into equation (9) from which we derive the relation $\bar\varphi = \bar\varphi(\varphi)$. Subsequently, we obtain $\bar{H} = \bar{H}(H)$ by using the conditions (10).

5.1 The case $\bar{H} = cH + \beta$

We consider the simplest case which corresponds to the transformation $\bar{H} = cH + \beta$, where both c and β are constants. To begin with we will set $\beta = 0$. In this case we have the expansion of the universe in the two related theories satisfy $\bar{a} = a^c$ and a particular case of interest is the case where $c = -1$ which we will address below. First we see from eq. (8) that $H \to cH$ implies

$$(1-\bar\gamma/2)\bar{x}^2 = 3(\gamma/c - \bar\gamma) + (1-\gamma/2)x^2/c\,. \tag{11}$$

From Eq. (9) we then derive

$$\frac{d\bar\varphi}{d\varphi} = c^{-1}\frac{\bar{x}}{x}. \tag{12}$$

If $\bar\gamma = \gamma/c$ we see that Eq.(11) simplifies and we obtain $\bar{x} = \sigma x$ where $\sigma = \pm\sqrt{(2-\gamma)/(2c-\gamma)}$ is a constant. It follows that $\bar\phi = c\sigma\phi + \phi_0$.
A particularly interesting case, is that when $c = -1$ which yields an inverse relation for the scale factors, $\bar{a} = a^{-1}$ and plays an important role in pre-big-bang scenarios [11]. We see this case is fulfilled when γ (say $\bar\gamma$) is negative. This means that the perfect fluid of one of the solutions related by this duality must have a phantom equation of state.
The particular ST solutions in this duallity belong to the same theory in this case since the corresponding generating functions x and $\bar{x}$ satisfy a linear relation. For instance they can thus be two Brans-Dicke solutions with different values of σ, that is hence of α.

6 Discussion

We have introduced a procedure that permits to derive exact scalar tensor Friedmann solutions with a perfect fluid satisfying the barotropic equation of state $p = (\gamma - 1)\rho$.
The method devised here is much simpler than all the methods put forward earlier in the literature.
In addition it permits a rigorous discussion of the asymptotic behaviours of the scalar tensor cosmologies. This reveals that in addition to the attractor mechanism towards GR disclosed by Damour and Nordtvedt [1], we also find that there is an attractor mechanism towards Brans-Dicke theory.
Finally, our method further allows us to analyse form-invariance dualities between different solutions either belonging to the same or to diverse scalar-tensor theories.

References

[1] T. Damour & K. Nordvedt, *Phys. Rev. Lett.* **70**, 2217 (1993); *Phys. Rev. D***48**, 3436 (1993).

[2] C. M. Will, Theory and Experiment in Gravitation (Cambridge University Press, Cambridge, England, 1993).

[3] P G Bergmann, *Int. J. Theor. Phys.* **1**, 25 (1968); R V Wagoner, *Phys. Rev.* **D1**, 3209 (1970); K Nordtvedt, *Astrophys. J.* **161**, 1059 (1970).

[4] C. Brans & R.H. Dicke, *Phys. Rev.* **124**, 925 (1961).

[5] T. Charters and J. P. Mimoso, JCAP **1008** (2010) 022.

[6] A. Nunes and J. P. Mimoso, Phys. Lett. B **488**, 423 (2000).

[7] J. P. Mimoso and D. Wands, Phys. Rev. D **51**, (1995) 477.

[8] J. P. Mimoso and T. Charters, Exact scalar field cosmologies, in preparation (2010).

[9] L. P. Chimento, J. Math. Phys. **38**, 2565 (1997).

[10] L. P. Chimento, Phys. Rev. D **65**, 063517 (2002).

[11] M. Gasperini and G. Veneziano, Astropart. Phys. **1**, 317 (1993).