COSMOLOGY WITH $f(R)$ GRAVITY

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Why modified gravity?

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Why modified gravity?

Why going beyond the pure Einstein gravity (GR) interacting with dust (baryons + CDM) and radiation? – Existence of dark energy (DE).

Two cases where DE shows itself:
1) inflation in the early Universe – primordial DE,
2) present accelerated expansion of the Universe – present DE.

Present view of the history of our Universe in one line:

$$? \longrightarrow DS \Longrightarrow FRWRD \Longrightarrow FRWMD \Longrightarrow \overline{DS} \longrightarrow ?$$

$f(R)$ gravity

The simplest model of modified gravity (= geometrical dark energy) considered as a phenomenological macroscopic theory in the fully non-linear regime and non-perturbative regime.

$$S = \frac{1}{16\pi G}\int f(R)\sqrt{-g}\, d^4x + S_m$$

$$f(R) = R + F(R), \quad R \equiv R_{\mu}^{\mu}\ .$$

Field equations

$$\frac{1}{8\pi G}\left(R_\mu^\nu - \frac{1}{2}\,\delta_\mu^\nu R\right) = -\left(T_{\mu\,(vis)}^\nu + T_{\mu\,(DM)}^\nu + T_{\mu\,(DE)}^\nu\right) \;,$$

where $G = G_0 = const$ is the Newton gravitational constant measured in laboratory and the effective energy-momentum tensor of DE is

$$8\pi G T_{\mu\,(DE)}^\nu = F'(R)\,R_\mu^\nu - \frac{1}{2}\,F(R)\delta_\mu^\nu + \left(\nabla_\mu \nabla^\nu - \delta_\mu^\nu \nabla_\gamma \nabla^\gamma\right) F'(R)\,.$$

Because of the need to describe DE, de Sitter solutions in the absence of matter are of special interest. They are given by the roots $R = R_{DS}$ of the algebraic equation

$$Rf'(R) = 2f(R)\;.$$

Degrees of freedom

I. In quantum language: particle content.
1. Graviton – spin 2, massless, transverse traceless.
2. Scalaron – spin 0, massive, mass - R-dependent:
$m_s^2(R) = \frac{1}{3f''(R)}$ in the WKB-regime.

II. Equivalently, in classical language: number of free functions of spatial coordinates at an initial Cauchy hypersurface.
Six, instead of four for GR – two additional functions describe massive scalar waves.

Thus, $f(R)$ gravity is a non-perturbative generalization of GR.
It is equivalent to scalar-tensor gravity with $\omega_{BD} = 0$ (if $f''(R) \neq 0$).

Why R-dependence only?

For almost all other geometric invariants – $R_{\mu\nu}R^{\mu\nu}$, $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$, $R_{,\mu}R^{,\mu}$ etc. (where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor) – ghosts appear if the theory is taken in full, in the non-perturbative regime.

The only known exception: $f(R,G)$ with $f_{RR}f_{GG} - f_{RG}^2 = 0$, where $G = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R\mu\nu R^{\mu\nu} + R^2$ is the Gauss-Bonnet invariant, does not possess ghosts but has other problems.

For $f_{RR}f_{GG} - f_{RG}^2 \neq 0$, a ghost was found very recently (A. de Felice, S. Tsujikawa, arViv:1006.4399).

Possible microscopic origin of $f(R)$

1. Quantum-gravitational loop corrections from all quantum field of matter including gravity itself.
Recent development: asymptotically safe gravity (existence of a fixed UV point).
Cannot produce "R-dominance" – corrections depending on $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ are of the same order as those depending on R.

2. Reduction from higher-dimensional models.
The same problem.

Viable ways.

3. A non-minimally coupled scalar field with a large negative coupling ξ (for this choice of signs, $\xi_{conf} = \frac{1}{6}$):

$$L = \frac{R}{16\pi G} - \frac{\xi R\phi^2}{2} + \frac{1}{2}\phi_{,\mu}\phi^{,\mu} - V(\phi), \quad \xi < 0, \quad |\xi| \gg 1 \ .$$

Recent development: Higgs inflation (F. Bezrukov and M. Shaposhnikov, 2008). In the limit $|\xi| \gg 1$, the potential $V(\phi) = \frac{\lambda(\phi^2 - \phi_0^2)^2}{4}$ just produces $f(R) = \frac{1}{16\pi G}\left(R + \frac{R^2}{6M^2}\right)$ with $M^2 = \lambda/24\pi|\xi|^2 G$ (for this model, $|\xi| G\phi_0^2 \ll 1$).

4. Emergent gravity.
Less (not more!) symmetry at a more fundamental level.
Appearance of $f(R)$ gravity in the simplest case:
F. R. Klinkhamer and G. E. Volovik, JETP Lett. 88, 289 (2008) (arXiv:0807.3896).

Background FRW equations in $f(R)$ gravity

$$ds^2 = dt^2 - a^2(t)\left(dx^2 + dy^2 + dz^2\right)$$

$$H \equiv \frac{\dot{a}}{a}\,, \quad R = 6(\dot{H} + 2H^2)$$

The trace equation (4th order)

$$\frac{3}{a^3}\frac{d}{dt}\left(a^3\frac{df'(R)}{dt}\right) - Rf'(R) + 2f(R) = 8\pi G(\rho_m - 3p_m)$$

The 0-0 equation (3d order)

$$3H\frac{df'(R)}{dt} - 3(\dot{H} + H^2)f'(R) + \frac{f(R)}{2} = 8\pi G\rho_m$$

Conditions for viable $f(R)$ models

I. Conditions of classical and quantum stability:

$$f'(R) > 0\ , \quad f''(R) > 0\ .$$

Even the saturation of these inequalities should be avoided:
1. $f'(R_0) = 0$: a generic anisotropic space-like curvature singularity forms.
2. $f''(R_0) = 0$: a weak singularity forms, loss of predictability of the Cauchy evolution.

$$a(t) = a_0 + a_1(t - t_s) + a_2(t - t_s)^2 + a_3|t - t_s|^{5/2} + \dots\ .$$

The metric in C^2, but not C^3, continuous across this singularity, and there is no unambiguous relation between the coefficients a_3 for $t < t_s$ and $t > t_s$.

II. Conditions for the existence of the Newtonian limit:

$$|F| \ll R, \quad |F'| \ll 1, \quad RF'' \ll 1$$

for $R \gg R_{now}$ and up to some very large R.
The same conditions for smallness of deviations from GR.
III. Laboratory and Solar system tests.
No deviation from the Newton law up to 50 μ.
No deviation from the Einstein values of the post-Newtonian coefficients β and γ up to 10^{-4} in the Solar system.
IV. Existence of a future stable (or at least metastable) de Sitter asymptote:

$$f'(R_{DS})/f''(R_{DS}) \geq R_{DS} \; .$$

Required since observed properties of DE are close to that of a cosmological constant.

V. Cosmological tests:
among them the anomalous growth of matter perturbations for recent redshifts

$$\left(\frac{\delta\rho}{\rho}\right)_m \propto t^{\frac{\sqrt{33}-1}{6}}$$

at the matter-dominated stage for $k \gg m_s(R)a$, where
$$m_s^2(R) = 1/3F''(R)\ .$$
Results in apparent discrepancy between the linear σ_8 and the primordial slope n_s estimated from CMB data (assuming GR) and from galaxy/cluster data separately.

VI. $f(R)$ cosmology should not destroy previous successes of present and early Universe cosmology in the scope of GR, including the existence of the matter-dominated stage driven by non-relativistic matter preceded by the radiation-dominated stage with the correct BBN and, finally, inflation.

Inflationary models in $f(R)$ gravity

1. The simplest one (Starobinsky, 1980):

$$f(R) = R + \frac{R^2}{6M^2}$$

with small one-loop quantum gravitational corrections producing the scalaron decay.
During inflation ($H \gg M$): $H = \frac{M^2}{6}(t_f - t), \quad |\dot{H}| \ll H^2$.

The only parameter M is fixed by observations – by the primordial amplitude of adiabatic (density) perturbations in the gravitationally clustered matter component:

$$M = 3.0 \times 10^{-6} M_{Pl} \, (50/N) \, ,$$

where $N \sim (50 - 55)$ is the number of e-folds between the first Hubble radius crossing during inflation of the present Hubble scale and the end of inflation, $M_{Pl} = \sqrt{G} \approx 10^{19}$ GeV.

Remains viable: $n_s = 1 - \frac{2}{N} \approx 0.96,\ r = \frac{12}{N^2} \approx 0.004$.

Observations: $n_s = 0.963 \pm 0.012;\ r < 0.24$ (95% CL).

2. Analogues of chaotic inflation: $F(R) \approx R^2 A(R)$ for $R \to \infty$ with $A(R)$ being a slowly varying function of R, namely

$$|A'(R)| \ll \frac{A(R)}{R}\ ,\ |A''(R)| \ll \frac{A(R)}{R^2}\ .$$

3. Analogues of new inflation, $R \approx R_1$:

$$F'(R_1) = \frac{2F(R_1)}{R_1}\ ,\ F''(R_1) \approx \frac{2F(R_1)}{R_1^2}\ .$$

Thus, all inflationary models in $f(R)$ gravity are close to the simplest one over some range of R.

Present DE models in $f(R)$ gravity

Much more difficult to construct. An example of the viable model satisfying the first 5 viability conditions (A. A. Starobinsky, JETP Lett. **86**, 157 (2007)):

$$f(R) = R + \lambda R_0 \left(\frac{1}{\left(1 + \frac{R^2}{R_0^2}\right)^n} - 1 \right)$$

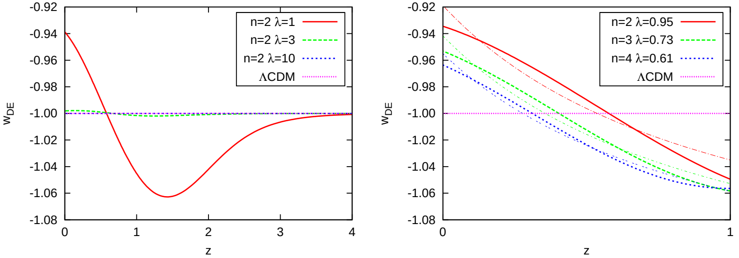
with $n \geq 2$. Similar models:

1. W. Hu and I. Sawicki, Phys. Rev. D **76**, 064004 (2007).
2. A. Appleby and R. Battye, Phys. Lett. B **654**, 7 (2007).

Phantom boundary crossing

H. Motohashi, A. A. Starobinsky, J. Yokoyama,
Progr. Theor. Phys. **123**, 887 (2010).

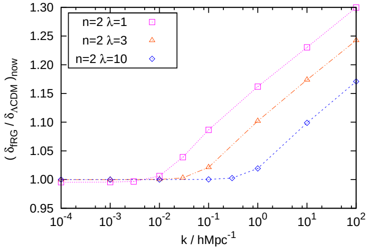
Generic feature: phantom behaviour for $z > 1$,
crossing of the phantom boundary $w_{DE} = -1$ for $z < 1$.

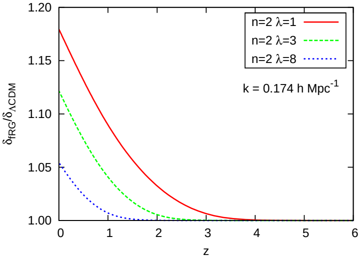


Anomalous growth of perturbations

Deeply in the sub-horizon regime:

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G_{eff}\,\rho\delta = 0\ , \quad G_{eff} = \frac{G}{f'}\,\frac{1 + 4\frac{k^2}{a^2}\frac{f''}{f'}}{1 + 3\frac{k^2}{a^2}\frac{f''}{f'}}\ .$$





Still not the end of the story!

Structure of corrections to GR

$$R = R^{(0)} + \delta R_{ind} + \delta R_{osc} \ ,$$

$$R^{(0)} = 8\pi G T_m \propto a^{-3} \ ,$$

$$\delta R_{ind} = \left(RF'(R) - 2F(R) - 3\nabla_\mu \nabla^\mu F'(R)\right)_{R=R^{(0)}} \ ,$$

$$R \gg R_0, \quad \delta R_{ind} \approx \mathrm{const} = -F(\infty) = 4\Lambda(\infty) \ .$$

No Dolgov-Kawasaki instability.

$$MD: \qquad \delta R_{osc} \propto t^{-(3n+4)} \sin\left(c_1 t^{-(2n+1)} + c_2\right) ,$$

$$RD: \quad \delta R_{osc} \propto t^{-3(3n+4)/4} \sin\left(c_3 t^{-(3n+1)/2} + c_4\right) .$$

$\delta a/a$ is small but $\delta R_{osc}/R^{(0)}$ diverges for $t \to 0$.

δR_{osc} should be very small just from the beginning – a problem for those $f(R)$ models which do not let R become negative due to crossing of the $f''(R) = 0$ point.

The "scalaron overproduction" problem.

Three new problems

In the early Universe:

- Unlimited growth of $m_s(R)$ for $t \to 0$: when $m_s(R)$ exceeds M_{Pl}, quantum-gravitational loop corrections invalidate the use of an effective quasi-classical $f(R)$ gravity.
- Unlimited growth of the amplitude of δR oscillations for $t \to 0$ (the "scalaron overproduction" problem).
- "Big Boost" singularity before the Big Bang:

$$a(t) = a_0 + a_1(t - t_0) + a_2|t - t_0|^k + \ldots,\ 1 < k = \frac{2n+1}{n+1} < 2\,,$$

if $|F(R) - F(\infty)| \propto R^{-2n}$ for $R \to \infty$, so $f''(\infty) = 0$.

Curing all three problems

S. A. Appleby, R. A. Battye and A. A. Starobinsky,
JCAP **1006**, 005 (2010).

Add $\frac{R^2}{6M^2}$ to $f(R)$ with M not less than the scale of inflation. Then the first and and third problems go away. The second problem still remains, but (any) inflation can solve it.
However, in all known inflationary models R may be negative during reheating after inflation (e.g. when $V(\phi) = 0$).
Necessity of an extension of $f(R)$ to $R < 0$ keeping $f''(R) > 0$.
As a result, a non-zero g-factor $(0 < g < 1/2)$ arises:

$$g = \frac{f'(R) - f'(-R)}{2f'(R)} , \quad R_0 \ll R \ll M^2 .$$

An example satisfying all 6 viability conditions: the g-extended R^2-corrected AB model

$$f(R) = (1-g)R + g\epsilon \log\left[\frac{\cosh\left(R/\epsilon - b\right)}{\cosh b}\right] + \frac{R^2}{6M^2}\ .$$

$m_s \approx M = const$ for $\rho_m \gg 10^{-27}$ g/cm^3 –
no "chameleon" behaviour in laboratory and Solar system experiments.

Combined models of primordial and present DE

If $M \approx 3 \times 10^{-6} M_{Pl}$, the scalaron can play the role of an inflaton, too. Then the inflationary predictions are formally the same as for the pure $R + R^2/6M^2$ inflationary model which does not describe the present DE:

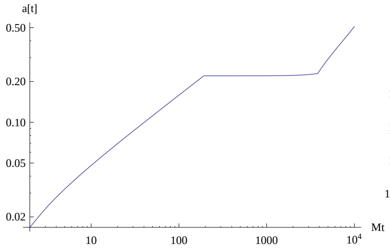
$$n_s = 1 - \frac{2}{N}\ , \quad r = \frac{12}{N^2}.$$

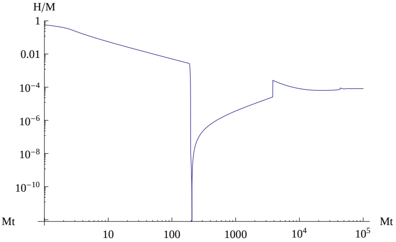
However, N is different, $N \sim 70$ for the unified model (versus $N \sim (50 - 55)$ for the purely inflationary one) because the stage if reheating after inflation becomes completely different: it consists of unequal periods with $a \approx const$ and $a \propto t^{1/2}$. Duration of the periods in terms of $\ln t$:
$-\ln(1-2g)$ and $-2\ln(1-2g)$ respectively.
So, $a(t) \propto t^{1/3}$ on average for a long time after the end of inflation, in contrast to

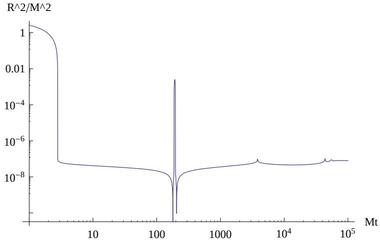
$$a(t) \propto t^{2/3}\left(1 + \frac{2}{3Mt}\sin M(t-t_1)\right)$$

for the pure inflationary $f(R) = R + R^2/6M^2$ model.

Observable prediction which is, however, degenerate with other inflationary models in $f(R)$ gravity.

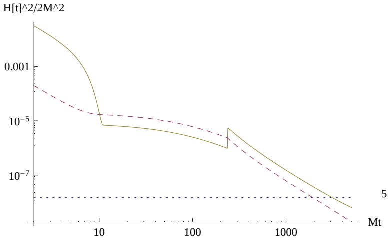

a[t]
0.50
0.20
0.10
0.05
0.02
10
100
1000
10^4
Mt


H/M
1
0.01
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10
100
1000
10^4
10^5
Mt


R^2/M^2
1
0.01
10^{-4}
10^{-6}
10^{-8}
10
100
1000
10^4
10^5
Mt

Reheating – due to gravitational particle creation which occurs mainly at the end of inflation. Less efficient than in the pure inflationary $f(R) = R + R^2/6M^2$ model,

$$t = t_{\rm reh} \sim M^{-4} M_{\rm Pl}^3 \sim 10^{-18}\ {\rm s}\ .$$



Conclusions about $f(R)$ cosmological models

- The simplest pioneer inflationary $f(R)$ model remains viable. Critical test: measurement of the low value for the tensor-to-scalar ratio of primordial metric perturbations $r \sim 0.4\%$.
- Much more problems with models of present DE. Still a narrow class among all $f(R)$ model s of present DE remains viable: it is possible to construct predictive models satisfying all existing cosmological, Solar system and laboratory data, and distinguishable from ΛCDM.
- To achieve this, previously constructed viable $f(R)$ DE models should be extended to large R with the $\sim R^2$ asymptotic behaviour and to negative R keeping $f'(R) > 0, \; f''(R) > 0$ at least up to the scale of inflation.

- Combined description of primordial DE producing inflation and present DE in the scope of $f(R)$ gravity is possible for the specific choice of M: $M \approx 3 \times 10^{-6} M_{Pl}$.
- Combined inflationary – DE $f(R)$ models have a significantly different reheating stage after inflation as compared to pure inflationary $f(R)$ models, with strongly non-linear oscillations of the scale factor $a(t)$.
- The most critical test for all $f(R)$ models of present dark energy: anomalous growth of density perturbations in the matter component at recent redshifts $z \sim 1 - 3$ (see also the talk by H. Motohashi about the possibility to raise the sum of neutrino masses to ~ 0.5 eV).