Renormalization Group Flow and the Cosmological Constant Problem *

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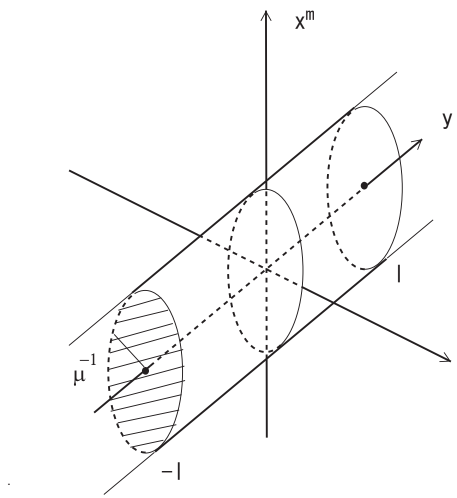
0. Introduction

5D Electromagnetism on the *flat* geometry
The extra space is *periodic* (periodicity $2l$) and Z_2-parity

Figure 1: IR-regularized geometry of 5D flat space (1).



$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + dy^2 \quad , \quad -\infty < x^\mu, y < \infty \quad , \quad y \to y + 2l,\ y \leftrightarrow -y \quad ,$$
$$(\eta_{\mu\nu}) = \mathsf{diag}(-1,1,1,1)\ , (X^M) = (X^\mu = x^\mu, X^5 = y) \equiv (x,y)\ ,$$
$$M, N = 0,1,2,3,5;\ \mu, \nu = 0,1,2,3. \quad (1)$$

The Casimir energy

$$E_{Cas}(\Lambda, l) = \frac{2\pi^2}{(2\pi)^4} \int_{1/l}^{\Lambda} d\tilde{p} \int_{1/\Lambda}^{l} dy\ \tilde{p}^3 W(\tilde{p}, y) F(\tilde{p}, y) \quad ,$$
$$F(\tilde{p}, y) \equiv F^-(\tilde{p}, y) + 4F^+(\tilde{p}, y) = \int_{\tilde{p}}^{\Lambda} d\tilde{k} \frac{-3\cosh \tilde{k}(2y - l) - 5\cosh \tilde{k}l}{2\sinh(\tilde{k}l)} \quad . \quad (2)$$

Λ the 4D-momentum cutoff; $W(\tilde{p}, y)$ the *weight function*

1) Un-weighted case: $W=1$

Un-restricted integral region :

$$E_{Cas}(\Lambda,l)=\frac{1}{8\pi^2}\left[-0.1249 l\Lambda^5-(1.41,0.706,0.353)\times 10^{-5}\ l\Lambda^5\ln(l\Lambda)\right] \quad ,$$

Randall-Schwartz integral region :

$$E_{Cas}^{RS}=\frac{1}{8\pi^2}[-0.0893\ \Lambda^4] \quad . \quad (3)$$

2) Weighted case

$$E_{Cas}^{W}=$$

$$
\begin{cases}
-2.50\frac{\Lambda}{l^3} + (-0.142, 1.09, 1.13) \times 10^{-4}\frac{\Lambda \ln(l\Lambda)}{l^3} & \text{for} \quad W_1 = (1/N_1)\mathrm{e}^{-(1/2)l^2\tilde{p}^2 - (1/2)y^2/l^2} \\
-6.03 \times 10^{-2}\frac{\Lambda}{l^3} & \text{for} \quad W_2 = (1/N_2)\mathrm{e}^{-\tilde{p}y} \\
-2.51\frac{\Lambda}{l^3} + (19.5, 11.6, 6.68) \times 10^{-4}\frac{\Lambda \ln(l\Lambda)}{l^3} & \text{for} \quad W_8 = (1/N_8)\mathrm{e}^{-(l^2/2)(\tilde{p}^2 + 1/y^2)}
\end{cases}
$$

(W_1: elliptic, W_2: hyperbolic, W_8: reciprocal).

The renormalization of the compactification size l.

$$
E^{W}_{Cas}/\Lambda l = -\frac{\alpha}{l^4}(1 - 4c\ln(l\Lambda)) = -\frac{\alpha}{l'^4} \quad , \tag{5}
$$

The quantity Λl is the normalization factor.

Casimir Energy of 4D Electromagnetism

Figure 2: Graph of Planck's radiation formula. $\mathcal{P}(\beta, k) = \frac{1}{(c\hbar)^3}\frac{1}{\pi^2}k^3/(\mathrm{e}^{\beta k} - 1) \quad (1 \leq \beta \leq 2,\ 0.01 \leq k \leq 10)$.

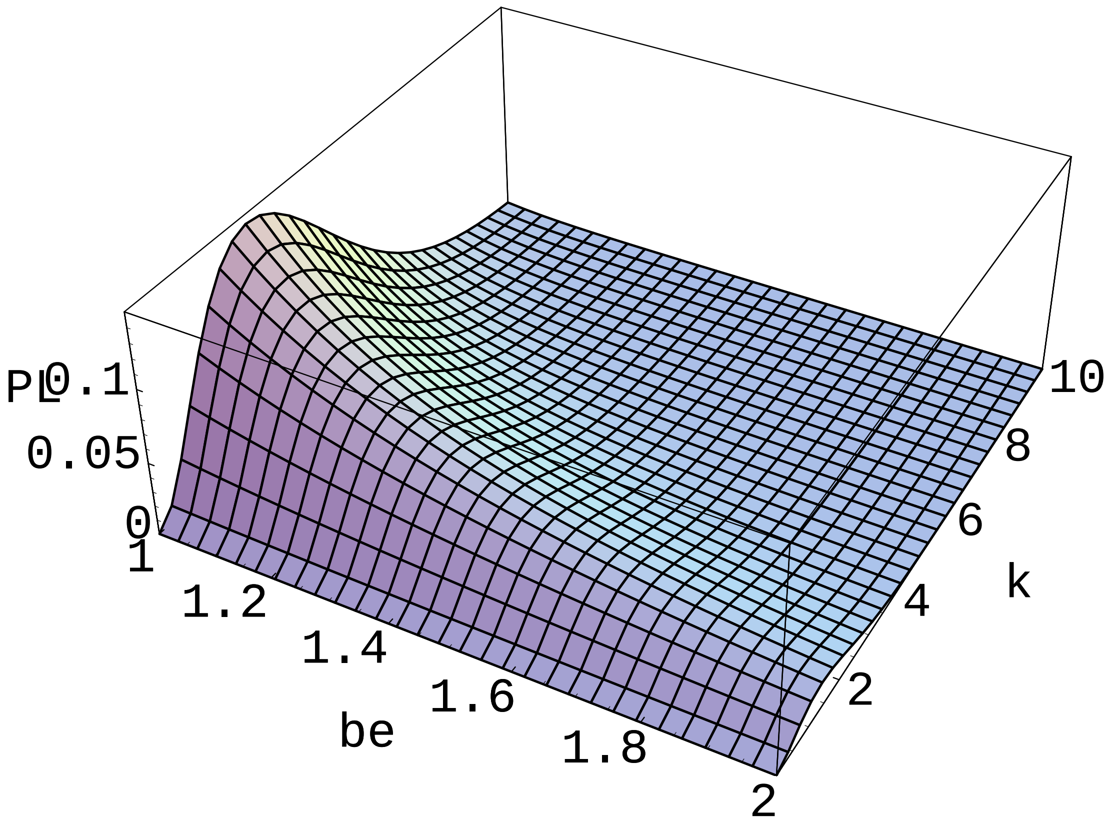
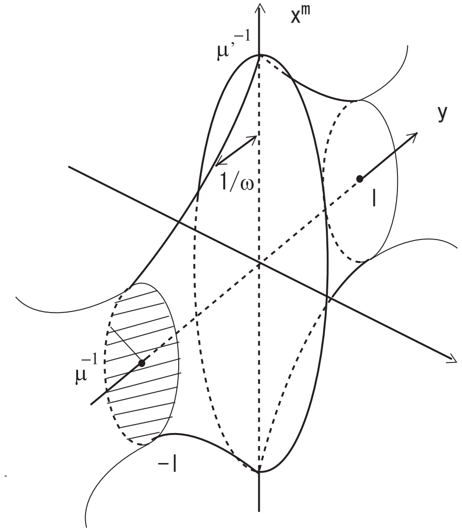


Figure 3: IR-regularized geometry of 5D warped space (6).



$$ds^2 = \frac{1}{\omega^2 z^2}(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2) = \mathrm{e}^{-2\omega|y|}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2 \ , \ |z| = \frac{1}{\omega}\mathrm{e}^{\omega|y|} \quad . \quad (6)$$

Heat-Kernel Approach and Position/Momentum Propagator

$$G_p^{\mp}(z,z') = \mp\frac{\omega^3}{2}z^2{z'}^2\frac{\{\mathbf{I}_0(\frac{\tilde{p}}{\omega})\mathbf{K}_0(\tilde{p}z)\mp\mathbf{K}_0(\frac{\tilde{p}}{\omega})\mathbf{I}_0(\tilde{p}z)\}\{\mathbf{I}_0(\frac{\tilde{p}}{T})\mathbf{K}_0(\tilde{p}z')\mp\mathbf{K}_0(\frac{\tilde{p}}{T})\mathbf{I}_0(\tilde{p}z')\}}{\mathbf{I}_0(\frac{\tilde{p}}{T})\mathbf{K}_0(\frac{\tilde{p}}{\omega})-\mathbf{K}_0(\frac{\tilde{p}}{T})\mathbf{I}_0(\frac{\tilde{p}}{\omega})} \quad ,$$

$$\tilde{p}\equiv\sqrt{p^2} \quad , \quad p^2\geq 0 \text{ (space-like)} \quad .$$

Λ-regularized Casimir energy.

$$E_{Cas}^{\Lambda,\mp}(\omega,T) = \int \frac{d^4p}{(2\pi)^4}\bigg|_{\tilde{p}\leq\Lambda}\int_{1/\omega}^{1/T} dz\ F^{\mp}(\tilde{p},z) \quad ,$$

$$F^{\mp}(\tilde{p}, z) = \frac{2}{(\omega z)^3} \int_{\tilde{p}}^{\Lambda} \tilde{k} \; G_k^{\mp}(z, z) d\tilde{k} \equiv \int_{\tilde{p}}^{\Lambda} \mathcal{F}^{\mp}(\tilde{k}, z) d\tilde{k} \quad , \tag{8}$$

Figure 4: Space of (z,$\tilde{p}$) for the integration.

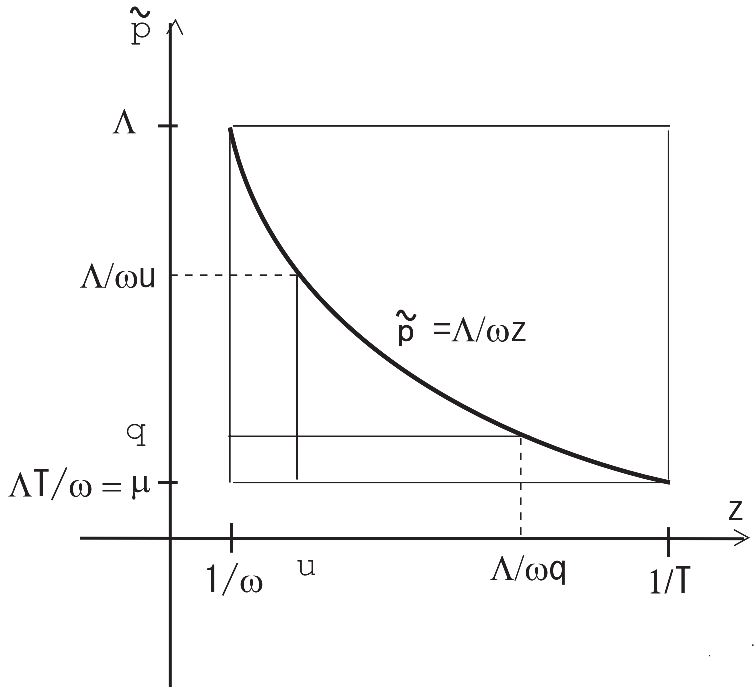
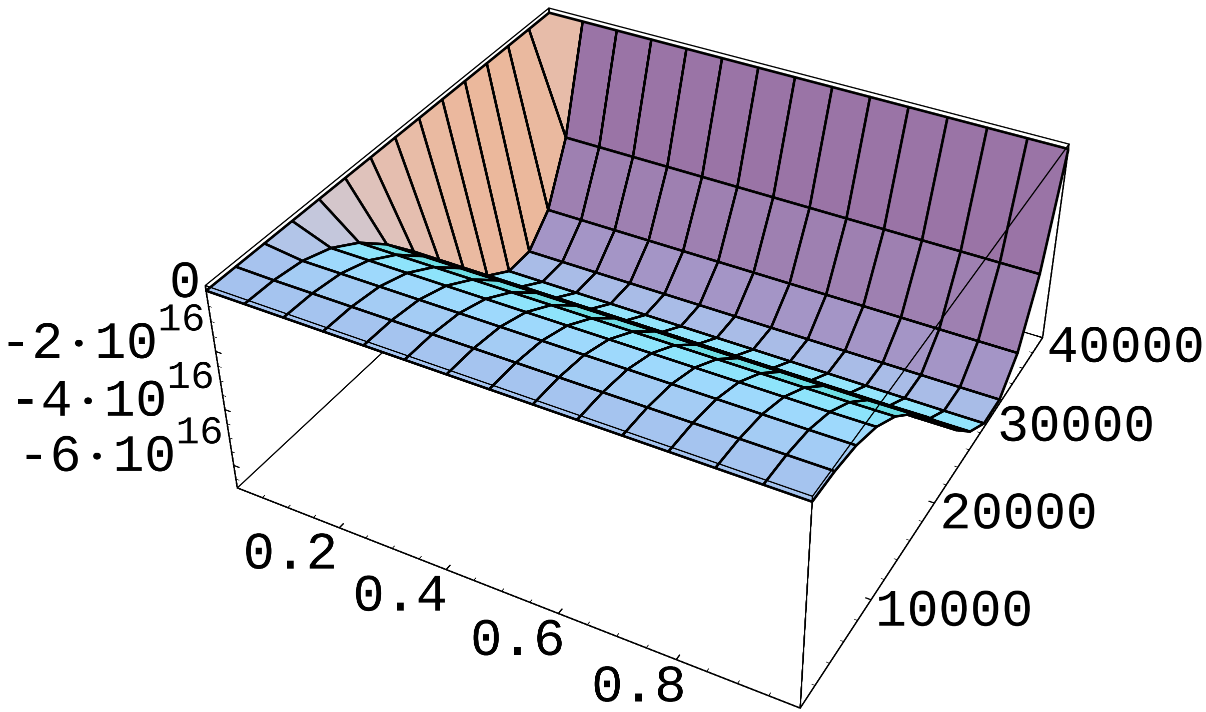


Figure 5: Behaviour of $(-1/2)\tilde{p}^3 F^-(\tilde{p}, z)$ (8). $T = 1, \omega = 10^4, \Lambda = 4 \cdot 10^4$. $1.0001/\omega \leq z < 0.9999/T$, $\Lambda T/\omega \leq \tilde{p} \leq \Lambda$.



Weight Function and Casimir Energy Evaluation

$$E_{Cas}^{\mp\,W}(\omega, T) \equiv \int \frac{d^4p}{(2\pi)^4} \int_{1/\omega}^{1/T} dz\ W(\tilde{p}, z) F^{\mp}(\tilde{p}, z)$$

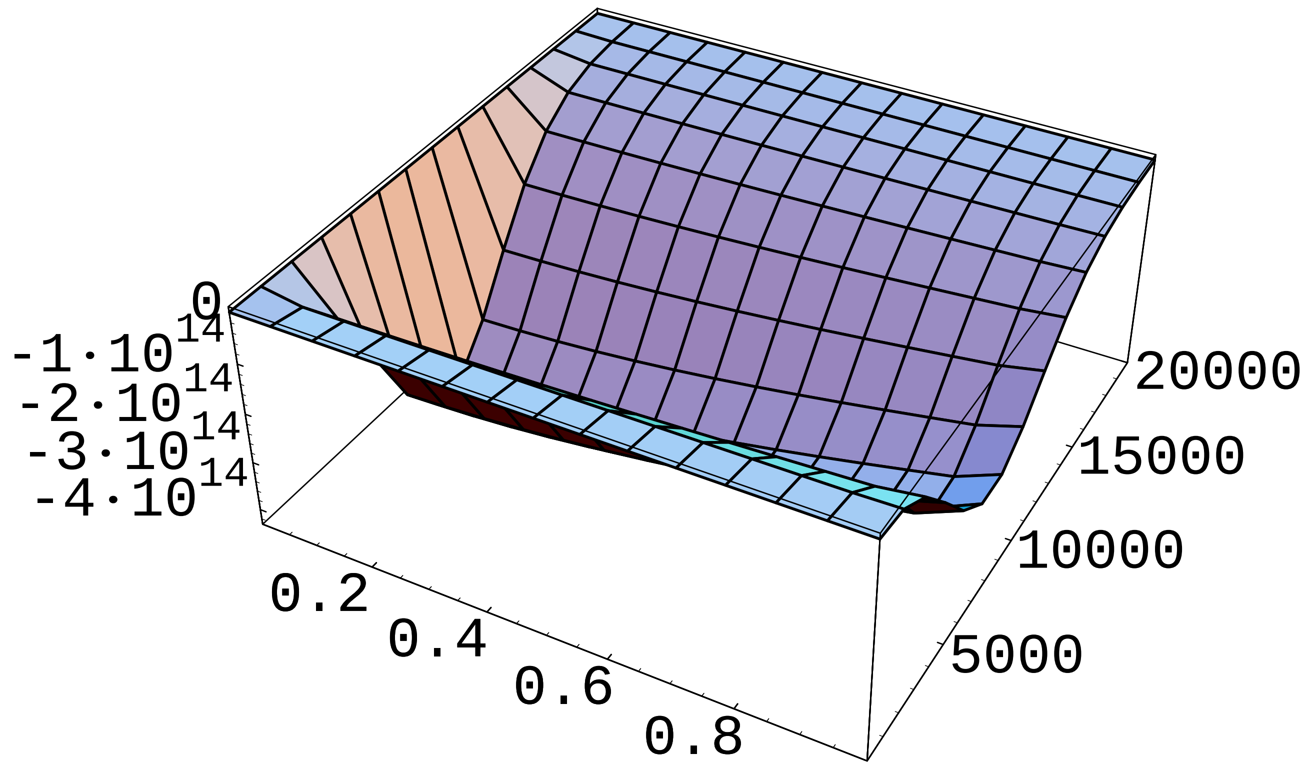
$$F^{\mp}(\tilde{p}, z) = s(z) \int_{p^2}^{\infty} \{G_k^{\mp}(z, z)\} dk^2 = \frac{2}{(\omega z)^3} \int_{\tilde{p}}^{\infty} \tilde{k}\ G_k^{\mp}(z, z) d\tilde{k}$$

Examples of $W(\tilde{p}, z)$: $\quad W(\tilde{p}, z) =$

$$\left\{ \begin{array}{ll} (N_1)^{-1} \mathrm{e}^{-(1/2)\tilde{p}^2/\omega^2 - (1/2)z^2T^2} \equiv W_1(\tilde{p}, z),\ N_1 = 1.711/8\pi^2 & \text{elliptic suppression} \\ (N_2)^{-1} \mathrm{e}^{-\tilde{p} z T/\omega} \equiv W_2(\tilde{p}, z),\ N_2 = 2\frac{\omega^3}{T^3}/8\pi^2 & \text{hyperbolic suppression1} \\ (N_8)^{-1} \mathrm{e}^{-1/2(\tilde{p}^2/\omega^2 + 1/z^2T^2)} \equiv W_8(\tilde{p}, z),\ N_8 = 0.4177/8\pi^2 & \text{reciprocal suppression1} \end{array} \right.$$

where $G_k^{\mp}(z,z)$ are defined in (7). N_i are normalization constants. We show the shape of the energy integrand $(-1/2)\tilde{p}^3 W_1(\tilde{p},z)F^{-}(\tilde{p},z)$ in Fig.6.

Figure 6: Behavior of $(-1/2)\tilde{p}^3 W_1(\tilde{p},z)F^-(\tilde{p},z)$(elliptic suppression). $\Lambda = 20000,\ \omega = 5000,\ T = 1\ .\ 1.0001/\omega \le z \le 0.9999/T,\ \mu = \Lambda T/\omega \le \tilde{p} \le \Lambda$.



We can check the divergence (scaling) behavior of $E^{\mp\ W}_{Cas}$ by $numerically$ evaluating the $(\tilde{p}, z)$-integral (9) for the rectangle region of Fig.4.

$$
-E^{W}_{Cas} =
$$

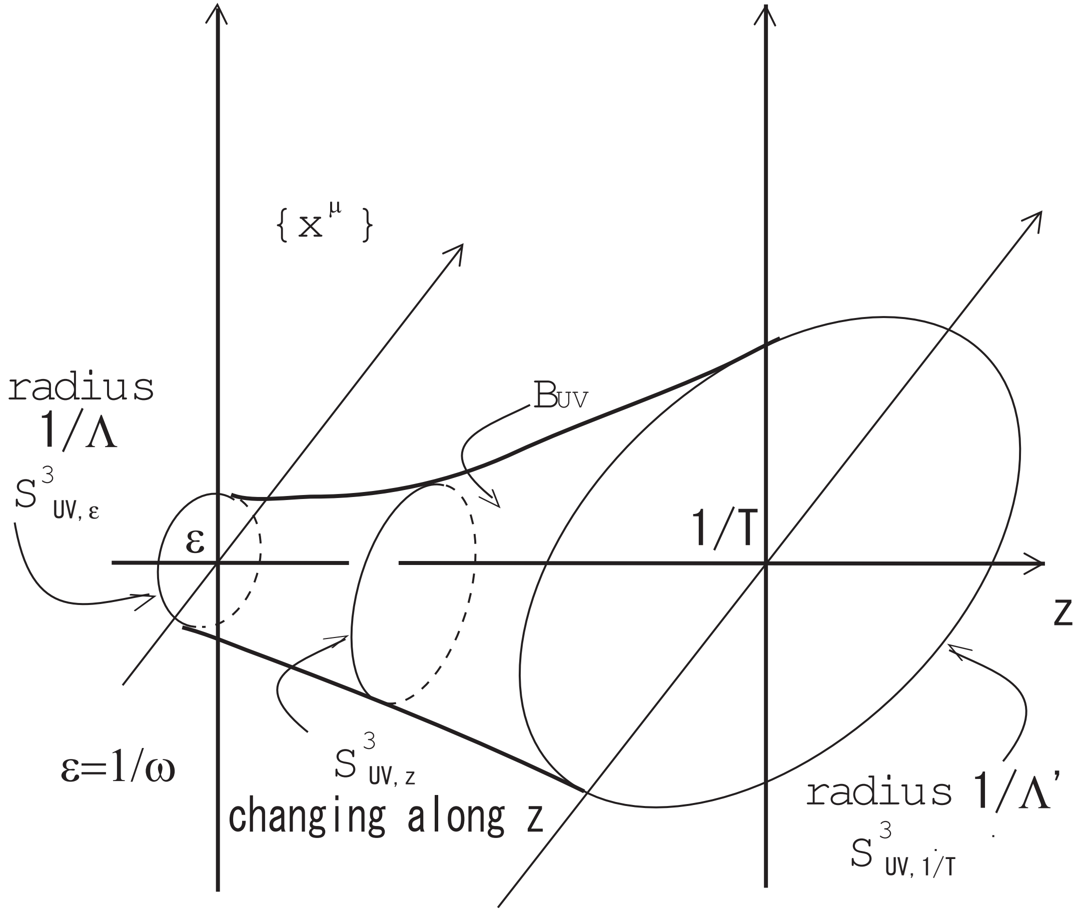
$$
\left\{ \begin{array}{ll}
\frac{\omega^4}{T}\Lambda\cdot 1.2\left\{1+0.11\ \ln\frac{\Lambda}{\omega}-0.10\ \ln\frac{\Lambda}{T}\right\} & \text{for}\quad W_1 \\
\frac{T^2}{\omega^2}\Lambda^4\cdot 0.062\left\{1+0.03\ \ln\frac{\Lambda}{\omega}-0.08\ \ln\frac{\Lambda}{T}\right\} & \text{for}\quad W_2 \\
\frac{\omega^4}{T}\Lambda\cdot 1.6\left\{1+0.09\ \ln\frac{\Lambda}{\omega}-0.10\ \ln\frac{\Lambda}{T}\right\} & \text{for}\quad W_8
\end{array} \right. \qquad (10)
$$

They give, after normalizing the factor Λ/T, only the log-divergence.

$$
E^{W}_{Cas}/\Lambda T^{-1} = -\alpha\omega^4\left(1-4c\ln(\Lambda/\omega)-4c'\ln(\Lambda/T)\right) \quad , \qquad (11)
$$

This means the 5D Casimir energy is $finitely$ obtained by the ordinary renormalization of the warp factor ω. In the above result of the warped case, the IR parameter l in the flat result (5) is replaced by the inverse of the warp factor ω.

Figure 7: UV regularization surface in 5D coordinate space.



Meaning of Weight Function and Quantum Fluctuation of Coordinates and Momenta

We *propose* to *replace* the 5D space integral with the weight W, by the following path-integral. We newly define the Casimir energy in the higher-dimensional theory as follows.

$$\mathcal{E}_{Cas}(\omega, T, \Lambda) \equiv \int_{1/\Lambda}^{1/\mu} d\rho \int_{\tilde{p}(1/\omega)=\tilde{p}(1/T)=1/\rho} \prod_{a,z} \mathcal{D}p^a(z) \left\{ \int_{1/\omega}^{1/T} F(\tilde{p}(z'), z') dz' \right\}$$

$$\times \exp\left[-\frac{1}{2\alpha'} \int_{1/\omega}^{1/T} \frac{1}{\omega^4 z^4} \frac{1}{\tilde{p}^3} \sqrt{\frac{\tilde{p}'^2}{\tilde{p}^4} + 1}\; dz \right]$$

$$= \int_{1/\Lambda}^{1/\mu} d\rho \int_{r(1/\omega)=r(1/T)=\rho} \prod_{a,z} \mathcal{D}x^a(z) \left\{ \int_{1/\omega}^{1/T} F(\frac{1}{r(z')}, z')dz' \right\}$$

$$\times \exp\left[-\frac{1}{2\alpha'} \int_{1/\omega}^{1/T} \frac{1}{\omega^4 z^4} \sqrt{r'^2 + 1}\, r^3 dz \right] \quad , (12)$$

where $\mu = \Lambda T/\omega$ and the limit $\Lambda T^{-1} \to \infty$ is taken. The string (surface) tension parameter $1/2\alpha'$ is introduced. (Note: Dimension of α' is [Length]^4.) The square-bracket $([\cdots])$-parts of (12) are $-\frac{1}{2\alpha'}$Area $= -\frac{1}{2\alpha'}\int \sqrt{\text{det} g_{ab}} d^4x$ (See (**??**)) where g_{ab} is the induced metric on the 4D surface. $F(\tilde{p}, z)$ is defined in (9) or (8) and shows the *field-quantization* of the bulk scalar (EM) fields.

The proposed definition, (12), clearly shows the 4D space-coordinates x^a or the 4D momentum-coordinates p^a are quantized (quantum-statistically, not field-theoretically) with the Euclidean time z and the "area Hamiltonian" $A =$

$\int \sqrt{\det g_{ab}}\ d^4x$. Note that $F(\tilde{p}, z)$ or $F(1/r, z)$ appears, in (12), as the energy density operator in the quantum statistical system of $\{p^a(z)\}$ or $\{x^a(z)\}$.

Discussion and Conclusion

$$
\begin{aligned}
E^W_{Cas}/\Lambda T^{-1} = -\alpha\omega^4\,(1 - 4c\ln(\Lambda/\omega) - 4c'\ln(\Lambda/T)) &= -\alpha\omega'^4 \quad , \\
\omega' = \omega\sqrt[4]{1 - 4c\ln(\Lambda/\omega) - 4c'\ln(\Lambda/T)} & \quad . \qquad (13)
\end{aligned}
$$

we find the renormalization group function for the warp factor ω as

$$
\begin{aligned}
|c| \ll 1 \quad , \quad |c'| \ll 1 \quad , \quad \omega' = \omega(1 - c\ln(\Lambda/\omega) - c'\ln(\Lambda/T)) & \quad , \\
\beta(\beta\text{-function}) \equiv \frac{\partial}{\partial(\ln\Lambda)}\ln\frac{\omega'}{\omega} = -c - c' & \quad . \qquad (14)
\end{aligned}
$$

We should notice that, in the flat geometry case, the IR parameter (extra-space size) l is renormalized . In the present warped case, however, the corresponding parameter T is not renormalized, but the warp parameter ω is renormalized. Depending on the sign of $c+c'$, the 5D bulk curvature ω flows as follows. When $c+c'>0$, the bulk curvature ω decreases (increases) as the the measurement energy scale Λ increases (decreases). When $c+c'<0$, the flow goes in the opposite way.

$$\frac{1}{G_N}\lambda_{obs}\sim\frac{1}{G_N{R_{cos}}^2}\sim m_\nu^4\sim(10^{-3}eV)^4 \quad , \tag{15}$$

where R_{cos} is the cosmological size (Hubble length), m_ν is the neutrino mass.

$$\frac{1}{G_N}\lambda_{th}\sim\frac{1}{{G_N}^2}={M_{pl}}^4\sim(10^{28}eV)^4 \quad . \tag{16}$$

The famous huge discrepancy factor: $\lambda_{th}/\lambda_{obs} \sim 10^{124}$. If we apply the present approach, we have the warp factor ω, and the result (13) strongly suggests the following choice:

$$\text{INPUT 1} \quad \Lambda = M_{pl} \quad ,$$

$$\text{INPUT 2(Newton's law exp.)} \; \omega \sim \frac{1}{\sqrt[4]{G_N {R_{cos}}^2}} = \sqrt{\frac{M_{pl}}{R_{cos}}} \sim m_\nu \sim 10^{-3}\text{eV}$$

$$\text{FACT} \quad S \sim \int d^4x\sqrt{-g}\frac{1}{G_N}\lambda_{obs} \sim R_{COS}^4\omega^4$$

$$\text{Result(13)requires} \quad \mathrm{e}^{-S} \leftrightarrow \mathrm{e}^{-E_{Cas}/T^4} = \exp\{-T^{-4}\Lambda T^{-1}\omega^4\}$$

$$\Longrightarrow \; T^5 = \frac{M_{pl}}{R_{cos}^4} \quad \text{OUTPUT} \quad . \quad (17)$$

We do not yet succeed in obtaining the right sign, but succeed in obtaining the finiteness and its gross absolute value of the cosmological constant. Now we understand that the smallness of the cosmological constant comes from the renormalization flow for the non asymptotic-free case ($c + c' < 0$ in (14)).