Second-order gauge-invariant cosmological perturbation theory:

--- Recent development and problems ---

Kouji Nakamura (NAOJ)

with Masa-Katsu Fujimoto (NAOJ)

References :

K.N. Prog. Theor. Phys., **110** (2003), 723. (arXiv:gr-qc/0303039).

K.N. Prog. Theor. Phys., **113** (2005), 413. (arXiv:gr-qc/0410024).

K.N. Phys. Rev. D **74** (2006), 101301R. (arXiv:gr-qc/0605107).

K.N. Prog. Theor. Phys., **117** (2007), 17. (arXiv:gr-qc/0605108).

K.N. Proc. of LT7 (2008) (arXiv:0711.0996[gr-qc]).

K.N. Phys. Rev. D **80** (2009), 124021. (arXiv:0804.3840[gr-qc]).

K.N. Prog. Theor. Phys. **121** (2009), 1321. (arXiv:0812.4865[gr-qc]).

K.N. Adv. in Astron. **2010** (2010), 576273. (arXiv:1001.2621[gr-qc]).

K.N. in preparation.

I. Introduction

- The second order perturbation theory in general relativity has very wide physical motivation.

  - Cosmological perturbation theory
    - Expansion law of inhomogeneous universe (ΛCDM v.s. inhomogeneous cosmology)
    - Non-Gaussianity in CMB (beyond WMAP)

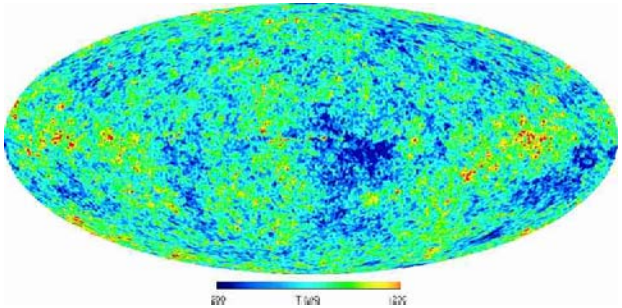
  - Black hole perturbations
    - Radiation reaction effects due to the gravitational wave emission.
    - Close limit approximation of black hole - black hole collision (Gleiser, et.al (1996))

  - Perturbation of a star (Neutron star)
    - Rotation – pulsation coupling (Kojima 1997)

There are many physical situations to which higher order perturbation theory should be applied.

The first order approximation of our universe from a homogeneous isotropic one is revealed by the recent observations of the CMB.

- It is suggested that the fluctuations are adiabatic and Gaussian at least in the first order approximation.
- One of the next research is to clarify the accuracy of this result.
  – Non-Gaussianity, non-adiabaticity …and so on.



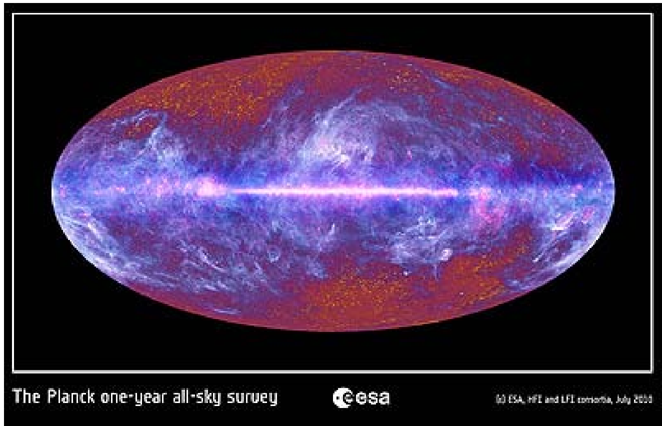
(Bennett et al., (2003).)

- To carry out this, it is necessary to discuss perturbation theories beyond linear order.
- The second-order perturbation theory is one of such perturbation theories.

c.f. Non-Gaussianity is a topical subject also in Observations.

– E. Komatsu, et al., APJ Supp. **180** (2009), 330; arXiv:1001.4538[astro-ph.CO].

– **The first full-sky map of Planck was press-released on 5 July 2010!!!** ----->



However, general relativistic perturbation theory requires very delicate treatments of “gauges”.

It is worthwhile to formulate the higher-order gauge-invariant perturbation theory from general point of view.

- According to this motivation, we have been formulating the general relativistic second-order perturbation theory in a gauge-invariant manner.
  - **General framework**:
    - K.N. PTP**110** (2003), 723; ibid, **113** (2005), 413.
  - **Application to cosmological perturbation theory** :
    - Einstein equations (the first-order : scalar mode only):
      - K.N. PRD**74** (2006), 101301R; PTP**117** (2007), 17.
    - Equations of motion for matter fields:
      - K.N. PRD**80** (2009), 124021.
    - Consistency of the 2nd order Einstein equations including all modes (perfect fluid, scalar field):
      - K.N. PTP**121** (2009), 1321.
    - Summary of current status of this formulation:
      - K.N. Adv. in Astron. **2010** (2010), 576273.

In this poster, …

- I will give a brief explanation of our gauge-invariant formulation of the second-order cosmological perturbations through the single scalar field case (as a simple example of matter field).

II. The second kind gauge in GR.

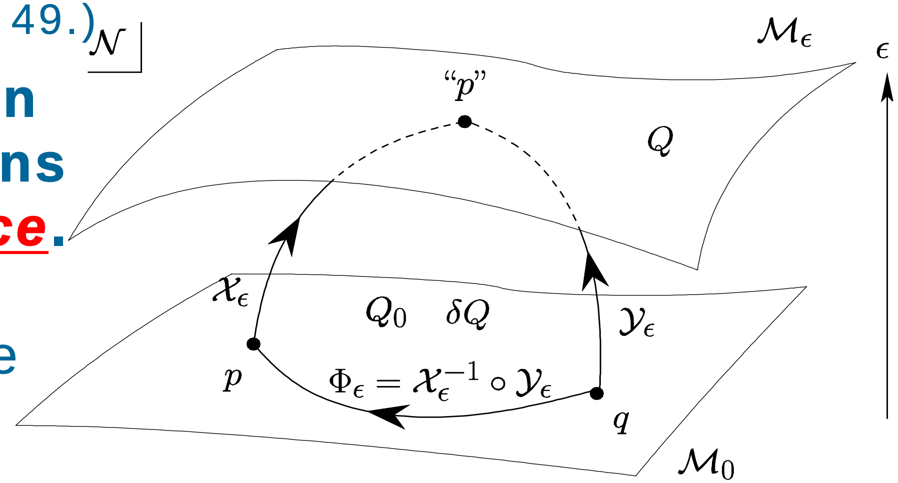
(Stewart and Walker, PRSL **A341** (1974), 49.)

■ "***Gauge degree of freedom***" **in general relativistic perturbations arises due to** ***general covariance*****.**

○ In any perturbation theories, we always treat two spacetimes :

- *Physical Spacetime (PS);*
- *Background Spacetime (BGS).*

Physical spacetime (PS)



Background spacetime (BGS)

○ In perturbation theories, we always write equations like

$$Q(\text{``}p\text{''}) = Q_0(p) + \delta Q(p)$$

Through this equation, we always identify the points on these two spacetimes and this identification is called "gauge choice" in perturbation theory.

Gauge transformation rules of each order

- **Expansion of gauge choices :**

We assume that each gauge choice is an exponential map.

$$Q_{\mathcal{X}} = \mathcal{X}_\epsilon^* Q = Q + \epsilon \pounds_u Q + \frac{1}{2}\epsilon^2 \pounds_u^2 Q + O(\epsilon^3)$$

$$Q_{\mathcal{Y}} = \mathcal{Y}_\epsilon^* Q = Q + \epsilon \pounds_v Q + \frac{1}{2}\epsilon^2 \pounds_v^2 Q + O(\epsilon^3)$$

$$\Phi_\epsilon^* Q = \left(\mathcal{X}_\epsilon^{-1} \circ \mathcal{Y}_\epsilon\right)^* Q = Q + \epsilon \pounds_{\xi_1} Q + \frac{1}{2}\epsilon^2 \left(\pounds_{\xi_2} + \pounds_{\xi_1}^2\right) Q + O(\epsilon^3)$$

(Sonego and Bruni, CMP, **193** (1998), 209.)

-------> $\xi_1 = u - v, \quad \xi_2 = [u, v]$

- **Expansion of the variable :** $Q = Q_0 + \epsilon Q_1 + \frac{1}{2} Q_2 + O(\epsilon^3)$
- **Order by order gauge transformation rules :**

$${}_{\mathcal{Y}}Q_1 - {}_{\mathcal{X}}Q_1 = \pounds_{\xi_1} Q_0$$

$${}_{\mathcal{Y}}Q_2 - {}_{\mathcal{X}}Q_2 = 2\pounds_{\xi_1} {}_{\mathcal{X}}Q_1 + \left(\pounds_{\xi_2} + \pounds_{\xi_1}^2\right) Q_0$$

Through these understanding of gauges and the gauge-transformation rules, we develop second-order gauge-invariant perturbation theory.

III. Gauge invariant variables

- metric perturbation : metric on PS : $\bar{g}_{ab}$, metric on BGS : g_{ab}

  metric expansion : $\bar{g}_{ab} = g_{ab} + \epsilon h_{ab} + \frac{1}{2}\epsilon^2 l_{ab} + O(\epsilon^3)$

  Our general framework of the second-order gauge invariant perturbation theory **was** based on a single assumption.

- **linear order (assumption) :**

Suppose that the linear order perturbation h_{ab} is decomposed as

$$h_{ab} = \mathcal{H}_{ab} + \pounds_X g_{ab}$$

so that the variable $\mathcal{H}_{ab}$ and X^a are the gauge invariant and the gauge variant parts of h_{ab} , respectively.

These variables are transformed as

$$_{\mathcal{Y}}\mathcal{H}_{ab} - {}_{\mathcal{X}}\mathcal{H}_{ab} = 0 \qquad {}_{\mathcal{Y}}X^a - {}_{\mathcal{X}}X^a = \xi_1^a$$

under the gauge transformation $\Phi_\epsilon = \mathcal{X}_\epsilon^{-1} \circ \mathcal{Y}_\epsilon$.

- **Recently, this assumption has been justified on generic BGS.**

  In cosmological perturbations, we may choose $\mathcal{H}_{ab}$ as

  (longitudinal gauge, J. Bardeen (1980))

$$\mathcal{H}_{\eta\eta} = -2a^2 \overset{(1)}{\Phi}, \quad \mathcal{H}_{i\eta} = a^2 \overset{(1)}{\nu}_i, \quad \mathcal{H}_{ij} = -2a^2 \overset{(1)}{\Psi} + a^2 \overset{(1)}{\chi}_{ij},$$

Second order :

Since the above assumption for the linear-order metric perturbation h_{ab} has been justified, we can always decompose the second order metric perturbations l_{ab} as follows :

$$l_{ab} =: \mathcal{L}_{ab} + 2\pounds_X h_{ab} + \left(\pounds_Z - \pounds_X^2\right) g_{ab}$$

where $\mathcal{L}_{ab}$ is gauge invariant part and Z_a is gauge variant part.

Under the gauge transformation $\Phi_\epsilon = \mathcal{X}_\epsilon^{-1} \circ \mathcal{Y}_\epsilon$, the vector field Z_a is transformed as

$$_{\mathcal{Y}}Z^a - {}_{\mathcal{X}}Z^a = \xi_2^a + [\xi_1, X]^a$$

- In the case of cosmological perturbations, we may choose the components of $\mathcal{L}_{ab}$ as (Poisson gauge)

$$\mathcal{L}_{\eta\eta} = -2a^2 \overset{(2)}{\Phi}, \quad \mathcal{L}_{i\eta} = a^2 \overset{(2)}{\nu_i}, \quad \mathcal{L}_{ij} = -2a^2 \overset{(2)}{\Psi} + a^2 \overset{(1)}{\chi_{ij}},$$

Perturbations of an arbitrary matter field Q :

Using gauge variant part of the metric perturbation of each order, gauge invariant variables for an arbitrary fields Q other than metric are defined by

- First order perturbation of Q :

$$\mathcal{Q}_1 := Q_1 - \pounds_X Q_0$$

- Second order perturbation of Q :

$$\mathcal{Q}_2 := Q_2 - 2\pounds_X Q_1 - \left\{\pounds_Z - \pounds_X^2\right\} Q_0$$

These implies that each order perturbation of an arbitrary field is always decomposed as

$$Q_1 = \boxed{\mathcal{Q}_1} + \boxed{\pounds_X Q_0}$$

$$Q_2 = \boxed{\mathcal{Q}_2} + \boxed{2\pounds_X Q_1 + \left\{\pounds_Z - \pounds_X^2\right\} Q_0}$$

(green box) : gauge invariant part (yellow box) : gauge variant part

Perturbations of Einstein tensor and Energy momentum tensor

- First order :

$$ {}^{(1)}G_a{}^b = {}^{(1)}\mathcal{G}_a{}^b[\mathcal{H}] + \pounds_X G_a{}^b, $$

$$ \begin{aligned} {}^{(1)}T_a{}^b = & \nabla_a\varphi\nabla^b\varphi_1 - \nabla_a\varphi\mathcal{H}^{bc}\nabla_c\varphi + \nabla_a\varphi_1\nabla^b\varphi \\ & -\frac{1}{2}\delta_a{}^b\left(\nabla_c\varphi\nabla^c\varphi_1 - \nabla_c\varphi\mathcal{H}^{dc}\nabla_d\varphi + \nabla_c\varphi_1\nabla^c\varphi + 2\varphi_1\frac{\partial V}{\partial\varphi}\right) \\ & +\pounds_X T_a{}^b, \end{aligned} $$

- Second order :

$$ {}^{(2)}G_a{}^b = {}^{(1)}\mathcal{G}_a{}^b[\mathcal{L}] + {}^{(2)}\mathcal{G}_a{}^b[\mathcal{H},\mathcal{H}] + 2\pounds_X {}^{(1)}G_a{}^b + \left\{\pounds_Z - \pounds_X^2\right\}G_a{}^b, $$

$$ \begin{aligned} {}^{(2)}T_a{}^b = & \nabla_a\varphi\nabla^b\varphi_2 - 2\nabla_a\varphi\mathcal{H}^{bc}\nabla_c\varphi_1 + 2\nabla_a\varphi\mathcal{H}^{bd}\mathcal{H}_{dc}\nabla^c\varphi - \nabla_a\varphi g^{bd}\mathcal{L}_{dc}\nabla^c\varphi \\ & +2\nabla_a\varphi_1\nabla^b\varphi_1 - 2\nabla_a\varphi_1\mathcal{H}^{bc}\nabla_c\varphi + \nabla_a\varphi_2\nabla^b\varphi \\ & -\frac{1}{2}\delta_a{}^b\left(\nabla_c\varphi\nabla^c\varphi_2 - 2\nabla_c\varphi\mathcal{H}^{dc}\nabla_d\varphi_1 + 2\nabla^c\varphi\mathcal{H}^{de}\mathcal{H}_{ec}\nabla_d\varphi \right. \\ & \qquad -\nabla^c\varphi\mathcal{L}_{dc}\nabla^d\varphi + 2\nabla_c\varphi_1\nabla^c\varphi_1 - 2\nabla_c\varphi_1\mathcal{H}^{dc}\nabla_d\varphi \\ & \qquad \left. +\nabla_c\varphi_2\nabla^c\varphi + 2\varphi_2\frac{\partial V}{\partial\varphi} + 2\varphi_1^2\frac{\partial^2 V}{\partial\varphi^2}\right) \\ & +2\pounds_X {}^{(1)}T_a{}^b + \left(\pounds_Z - \pounds_X^2\right)T_a{}^b, \end{aligned} $$

(green box) : gauge invariant part

(yellow box) : gauge variant part

Perturbations of the Klein-Gordon equation

- Klein-Gordon equation ($\nabla_a T_b{}^a = 0$)

$$C_{(K)} := \nabla^a \nabla_a \varphi - \frac{\partial V}{\partial \varphi}(\varphi) = 0$$

- Perturbative expansion of the Klein-Gordon equation :

$$\bar{C}_{(K)} =: C_{(K)} + \epsilon \overset{(1)}{C}_{(K)} + \frac{1}{2}\epsilon^2 \overset{(2)}{C}_{(K)} + O(\epsilon^3)$$

- We can show that each order perturbation of this Klein-Gordon equation is decomposed into the gauge-invariant and gauge-variant parts as

$$\overset{(1)}{C}_{(K)} =: \overset{(1)}{\mathcal{C}}_{(K)} + \pounds_X C_{(K)},$$

$$\overset{(2)}{C}_{(K)} =: \overset{(2)}{\mathcal{C}}_{(K)} + 2\pounds_X \overset{(1)}{C}_{(K)} + \left\{ \pounds_Z - \pounds_X^2 \right\} C_{(K)}$$

[green box] : gauge invariant part [yellow box] : gauge variant part

IV. Einstein eqs. and Klein-Gordon eq.

- We impose the Einstein equation of each order,

$$ {}^{(p)}G_a{}^b = 8\pi G {}^{(p)}T_a{}^b, \quad p = 0, 1, 2. $$

Then, the Einstein equation of each order is automatically given in terms of gauge invariant variables :

- **linear order :** $ {}^{(1)}\mathcal{G}_a{}^b[\mathcal{H}] = 8\pi G {}^{(1)}\mathcal{T}_a{}^b, $
- **second order :**

$$ {}^{(1)}\mathcal{G}_a{}^b[\mathcal{L}] + {}^{(2)}\mathcal{G}_a{}^b[\mathcal{H},\mathcal{H}] = 8\pi G {}^{(2)}\mathcal{T}_a{}^b. \quad . $$

- Further, each order perturbation of the Klein-Gordon equation is also automatically given in the gauge invariant form :
  - Background : $C_{(K)} = 0,$
  - Linear order : $\overset{(1)}{\mathcal{C}}_{(K)} = 0,$
  - Second order : $\overset{(2)}{\mathcal{C}}_{(K)} = 0,$

We do not have to care about gauge degree of freedom at least in the level where we concentrate only on the equations.

Second order Einstein equations (scalar)

- Master equation of the second-order Einstein equations :

$$\left\{\partial_\eta^2 + 2\left(\mathcal{H} - \frac{\partial_\eta^2\varphi}{\partial_\eta\varphi}\right)\partial_\eta - \Delta - 4K + 2\left(\partial_\eta\mathcal{H} - \frac{\partial_\eta^2\varphi}{\partial_\eta\varphi}\mathcal{H}\right)\right\} \overset{(2)}{\Phi}$$

$$+\Gamma_0 + \frac{1}{2}\Gamma_k{}^k - \Delta^{-1}D^jD^i\Gamma_{ij} - \left(\partial_\eta - \frac{\partial_\eta^2\varphi}{\partial_\eta\varphi}\right)\Delta^{-1}D^k\Gamma_k$$

$$-\frac{3}{2}\left[-\partial_\eta^2 + \left(2\frac{\partial_\eta^2\varphi}{\partial_\eta\varphi} - \mathcal{H}\right)\partial_\eta\right](\Delta + 3K)^{-1}\left(\Delta^{-1}D^jD^i\Gamma_{ij} - \frac{1}{3}\Gamma_k{}^k\right) = 0,$$

- Momentum constraint (scalar mode) :

$$2\partial_\eta \overset{(2)}{\Psi} + 2\mathcal{H} \overset{(2)}{\Phi} - 8\pi G\partial_\eta\varphi\varphi_2 - \Delta^{-1}D^k\Gamma_k = 0,$$

- potential perturbation :

$$\left(-\partial_\eta^2 - 5\mathcal{H}\partial_\eta + \frac{4}{3}\Delta + 4K\right)\overset{(2)}{\Psi} - \left(\mathcal{H}\partial_\eta + 2\partial_\eta\mathcal{H} + 4\mathcal{H}^2 + \frac{1}{3}\Delta\right)\overset{(2)}{\Phi}$$

$$-8\pi Ga^2\varphi_2\frac{\partial V}{\partial\varphi} - \Gamma_0 + \frac{1}{6}\Gamma_k{}^k = 0,$$

- traceless part of the spatial component of Einstein equation :

$$\overset{(2)}{\Psi} - \overset{(2)}{\Phi} = \frac{3}{2}(\Delta + 3K)^{-1}\left(\Delta^{-1}D^iD_j\Gamma_i{}^j - \frac{1}{3}\Gamma_k{}^k\right) .$$

Here, Γ_0, Γ_i, and Γ_{ij} are the collections of the quadratic terms of linear order perturbations.

Second order Klein-Gordon equation

- The second-order perturbation of the Klein-Gordon equation in the cosmological situation is given by

$$-a^2 \overset{(2)}{\mathcal{C}}_{(K)} = \partial_\eta^2\varphi_2 + 2\mathcal{H}\partial_\eta\varphi_2 - \Delta\varphi_2 + a^2\varphi_2\frac{\partial^2 V}{\partial\bar{\varphi}^2}(\varphi) - \partial_\eta\varphi\partial_\eta\left(\overset{(2)}{\Phi} + 3\overset{(2)}{\Psi}\right) + 2a^2\overset{(2)}{\Phi}\frac{\partial V}{\partial\bar{\varphi}}(\varphi) - \Xi_{(K)} = 0,$$

where

$$\Xi_{(K)} := 8\partial_\eta\overset{(1)}{\Phi}\partial_\eta\varphi_1 + 8\overset{(1)}{\Phi}\Delta\varphi_1 + 8\overset{(1)}{\Phi}\partial_\eta\overset{(1)}{\Phi}\partial_\eta\varphi - 4a^2\overset{(1)}{\Phi}\varphi_1\frac{\partial^2 V}{\partial\bar{\varphi}^2}(\varphi) - a^2(\varphi_1)^2\frac{\partial^3 V}{\partial\bar{\varphi}^3}(\varphi) - 2\overset{(1)}{\chi}{}^{ij}D_jD_i\varphi_1 + \overset{(1)}{\chi}{}^{ij}\partial_\eta\overset{(1)}{\chi}_{ij}\partial_\eta\varphi.$$

This second-order Klein-Gordon equation is not independent of the second-order Einstein equations.

We use this fact to check the consistency of all equations.

V. Consistency of 2nd-order equations

- Since momentum constraint for the vector mode is an initial value constraint, it should be consistent with the evolution equation of the vector mode. This consistency of equations leads an identity :

$$\partial_\eta \Gamma_k + 2\mathcal{H}\Gamma_k - D^l \Gamma_{lk} = 0,$$

Further, through this identity, we can confirm that the all Einstein equations for the second-order scalar mode are consistent with each other.

- The consistency between the Klein-Gordon equation and the Einstein equation of the second order leads an identity :

$$2\left(\partial_\eta + \mathcal{H}\right)\Gamma_0 - D^k \Gamma_k + \mathcal{H}\Gamma_k{}^k + 8\pi G \partial_\eta \varphi \Xi_{(K)} = 0,$$

- Actually, we have confirmed that these identities are guaranteed by the background Einstein equations and the first-order perturbations of the Einstein equation.

→ Our derived set of equations for the second-order perturbations are self-consistent.

VI. Summary

Based on the general framework of the general relativistic 2nd-order perturbations in [**K.N., PTP 110 (2003), 723; ibid, 113 (2005), 413.**], **we have derived the all components of the 2nd-order Einstein equations and equations of motion for matter fields in gauge-invariant, and self-consistent manner.**

[K.N., arXiv:0804.3840[gr-qc]; PTP **121** (2009), 1321.]

Current Status of our formulation (Problems)

- **Second-order cosmological perturbations (in progress)**

Second-order perturbation of the Einstein equation :

$$ {}^{(1)}\mathcal{G}_a{}^b[\mathcal{L}] + {}^{(2)}\mathcal{G}_a{}^b[\mathcal{H},\mathcal{H}] = 8\pi G\, {}^{(2)}\mathcal{T}_a{}^b. $$

In the case of the cosmological perturbations, these terms are **almost** completely derived.

The next task is to clarify the nature of the second-order perturbations of this energy momentum tensor.

- Incomplete parts are in the treatment of zero-mode!!!
  - At this moment, zero-modes are not included in our formulation, which should be included for completion.

    (Now I am trying!!!!!)
- Classical behaviors of the second-order perturbations.
  - This is a preliminary step to clarify the quantum behaviors of perturbations in inflationary universes.
- Comparison with the long-wavelength approximations.

- Multi-fluid or multi-field system
- Einstein Boltzmann system (treatments of photon and neutrino) [cf. N. Bartolo, et. al., (2006-); C. Pitrou, et. al., (2008-); L. Senatore, et. al. (2008-).]

  ---> **Non-linear effects in CMB physics.**