General Covariance in Gravity at a Lifshitz Point

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Review based on:

arXiv:0811.2217, arXiv:0812.4287, arXiv:0901.3775,
arXiv:0902.3657,
arXiv:0909.3841 (w/ Charles Melby-Thompson),
arXiv:1003.0009 (w/ Cenke Xu)

New results:

arXiv:1007.2410,
with Charles Melby-Thompson.

Work in progress:

w/ Charles Melby-Thompson, Kevin Grosvenor,
Patrick Zulkowski.

Central idea

Combine gravity with the concept of anisotropic scaling.

In a spacetime with coordinates $(t, \mathbf{x}) \equiv (t, x^i)$, $i = 1, \dots D$, consider

$$\mathbf{x} \to b\mathbf{x},$$
$$t \to b^z t.$$

Here z is the dynamical critical exponent.

In condensed matter (and now even in string theory!), many values of z are possible; integers (1, 2, . . .), fractions, . . .

Example: Lifts of static critical systems (Euclidean QFTs) to dynamical critical phenomena.

Goal: Construct similar models with propagating gravitons.

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(i) Phenomenology of gravity in our Universe, of $3 + 1$ dimensions; cosmology. How close can this resemble GR in IR?

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(i) Phenomenology of gravity in our Universe, of $3+1$ dimensions; cosmology. How close can this resemble GR in IR?

(ii) Gravity duals of field theories in AdS/CFT; in particular, candidates for duals of nonrelativistic field theories;

(iii) Gravity on worldvolumes of branes;

(iv) Mathematical applications (theory of the Ricci flow);

(iv) Emergent Gaussian IR fixed points in lattice systems of condensed matter.

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Main goal: Search for a UV fixed point in gravity.

Asymptotic safety: looking for relativistic, nontrivial fixed points. [Weinberg,. . .]

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Flow between UV and IR: from $z > 1$ to $z = 1$.

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The modified dynamics can mimic dark matter, [Mukohyama, . . .] mimic dark energy and suggest alternatives to inflation [Brandenberger et al, Kiritsis et al, Lüst et al, . . .].

Update on the status of Lifshitz gravity

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Example: Lifshitz scalar field theory

Many interesting features can be illustrated by:

$$S = \frac{1}{2}\int dt\, d^D\mathbf{x}\, \left\{\dot{\phi}^2 - (\Delta\phi)^2\right\}$$

A theory closely related to the better-known

$$W = \frac{1}{2}\int d^D\mathbf{x}\, \partial_i\phi\partial_i\phi$$

The critical dimension has shifted:

$$[\phi] = \frac{D-2}{2};$$

ϕ is dimensionless in $2+1$ dimensions.

[Lifshitz,1941]

Gravity at a Lifshitz point

Minimal starting point: fields $g_{ij}(t,\mathbf{x})$ (the spatial metric), action $S = S_K - S_V$, with the kinetic term

$$S_K = \frac{1}{\kappa^2}\int dt\, d^D\mathbf{x}\sqrt{g}\,\dot{g}_{ij}G^{ijk\ell}\dot{g}_{k\ell}$$

where $G^{ijk\ell} = g^{ik}g^{j\ell} - \lambda g^{ij}g^{k\ell}$ is the De Witt metric, and the "potential term"

$$S_V = \frac{1}{4\kappa^2}\int dt\, d^D\mathbf{x}\sqrt{g}\,V(R_{ijk\ell})$$

containing all terms of the appropriate dimension.
Special case, theory in "detailed balance": $V = (\delta W/\delta g_{ij})^2$.

Extending the symmetries

A good starting point, but this action is only invariant under time-independent spatial diffeomorphisms, $\tilde{x}^i = \tilde{x}^i(x^j)$, and describes dynamical propagating components g_{ij} of the spatial metric.

Covariantization of the theory:

(1) Introduce ADM-like variables N (lapse) and N_i (shift), known from the space-time decomposition of the spacetime metric;

(2) Replace $\dot{g}_{ij} \rightarrow K_{ij} = \frac{1}{N}\left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i\right)$,

$$\sqrt{g} \rightarrow N\sqrt{g}.$$

Gauge symmetries: Foliation-preserving diffeomorphisms $\mathrm{Diff}_{\mathcal{F}}(M)$,

$$\delta t = f(t),\ \delta x^i = \xi^i(t, x^j).$$

The transformation rules follow from a nonrelativistic contraction of spacetime diffeomorphisms; N and N_i are gauge fields of $\mathrm{Diff}_{\mathcal{F}}(M)$:

$$\delta N = \dot{f}(t)N + \ldots, \quad \delta N_i = \dot{\xi}_j + \ldots$$

In the minimal (="projectable") realization, N is a function of only t.

Symmetries reminiscent of the Causal Dynamical Triangulations (CDT) approach to quantum gravity on the lattice.

Simplest example: $z = 2$ gravity

The action is $S = S_K - S_V$, with

$$S_k = \frac{1}{\kappa^2} \int dt\, d^D\mathbf{x} \sqrt{g} N \, \left(K_{ij} K^{ij} - \lambda K^2 \right)$$

and

$$S_V = \int dt\, d^D\mathbf{x} \sqrt{g} N \, \left(\alpha R_{ij} R^{ij} + \beta R^2 + \ldots \right).$$

Shift in the critical dimension, as in the Lifshitz scalar:

$$[\kappa^2] = 2 - D.$$

The minimal theory with $N(t)$ has the usual number of transverse-traceless graviton polarizations, plus an extra scalar DoF, all with the dispersion relation $\omega^2 \sim k^4$.

Two special values of λ: 1 and 1/D.

Another example: $z = 3$ gravity

The action is again $S = S_K - S_V$, with

$$S_K = \frac{1}{\kappa^2} \int dt\, d^D\mathbf{x} \sqrt{g} N \, \left(K_{ij} K^{ij} - \lambda K^2 \right)$$

and

$$S_V = \int dt\, d^D\mathbf{x} \sqrt{g} N \, C_{ij} C^{ij}.$$

where $C^{ij} = \varepsilon^{ik\ell} \nabla_k (R^j_\ell - \frac{1}{4} R \delta^j_\ell)$ is the Cotton-York-ADM tensor. The shift of the critical dimension is

$$[\kappa^2] = 3 - D.$$

Anisotropic Weyl invariance eliminates the scalar graviton classically.

Theory with detailed balance

The role of the condition of detailed balance is twofold:

(1) A technical one: Reduces the number of independent couplings in the action.
In condensed matter, nongravitational examples of theories with detailed balance exhibit a simpler renormalization structure.

(2) Perhaps a more conceptual one: The condition of detailed balance arises in systems out of equilibrium, relating S to the equilibrium theory described by W.

Detailed balance can be softly broken, or eliminated altogether, in favor of the most general action of the effective field theory approach.

Entropic origin and detailed balance

Imposing detailed balance might be convenient for mathematical simplicity. However, a remarkable physics parallel exists: between gravity with detailed balance, and the Onsager-Machlup theory of non-equilibrium thermodynamics. [Onsager,Machlup 1953; Onsager 1931]

$$S = \int dt\, d^D\mathbf{x} \left(\dot{\Phi}_a M^{ab} \dot{\Phi}_b - \frac{\delta W}{\delta \Phi_a} M_{ab} \frac{\delta W}{\delta \Phi_b} \right).$$

This OM action describes the response of thermodynamic variables Φ_a to entropic forces $\delta W / \delta \Phi_a$; W itself is entropy!

Formally, gravity at a Lifshitz point with detailed balance has the same structure; mathematical formalism for understanding the possible entropic origin of gravity?

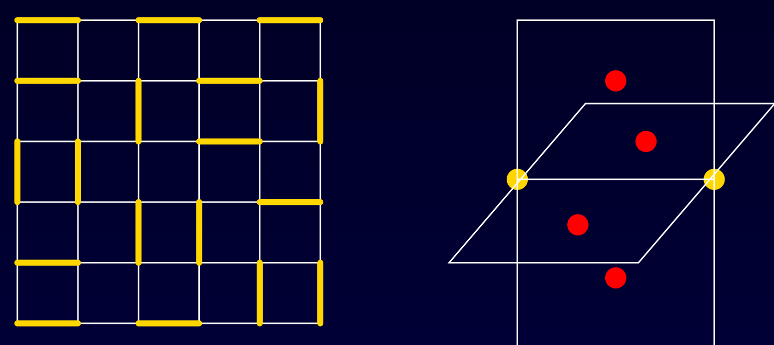
cf. the heuristic ideas of [Verlinde,Jacobson,Smoot et al,. . .]

Emergent gravity at a Lifshitz point

[Cenke Xu and P.H., arXiv:1003.0009]

These models with $z = 2$ or $z = 3$ gravitons can emerge as IR fixed points on the fcc lattice. Emergent gauge invariance stabilizes new algebraic bose liquid phases.

Recall the emergence of $U(1)$ "photons" in dimer models [Fradkin,Kivelson,Rokhsar,...]:



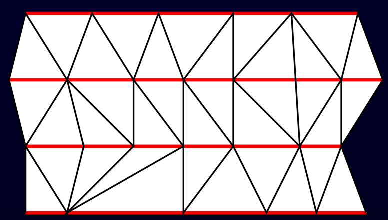
Lattice symmetries protect $z = 2$ or $z = 3$ in IR, forbid G_N. But: interacting Abelian gravity is possible!

Gravity on the lattice

Causal dynamical triangulations approach [Ambjørn,Jurkiewicz,Loll] to $3+1$ lattice gravity:

Naive sum over triangulations does not work (branched polymers, crumpled phases).

Modify the rules, include a preferred causal structure:



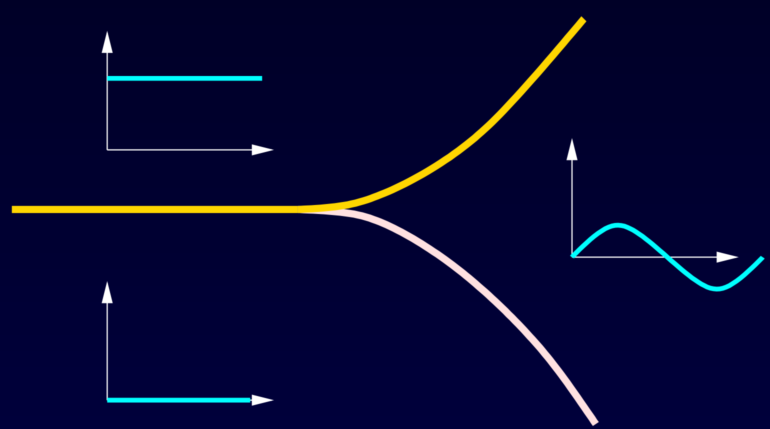
With this relevant change of the rules, a continuum limit appears to exist: The spectral dimension $d_s \approx 4$ in IR, and $d_s \approx 2$ in UV. Continuum gravity with anisotropic scaling: $d_s = 1 + D/z$. ([Benedetti,Henson,2009]: works in $2+1$ as well.)

Relevant deformations, RG flows, phases

The Lifshitz scalar can be deformed by relevant terms:

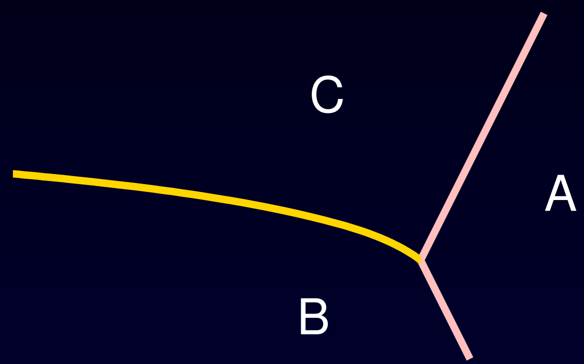
$$S = \frac{1}{2}\int dt\, d^D\mathbf{x}\,\left\{\dot{\phi}^2 - (\Delta\phi)^2 - \mu^2\partial_i\phi\partial_i\phi + m^4\phi^2 - \phi^4\right\}$$

The undeformed $z = 2$ theory describes a tricritical point, connecting three phases – disordered, ordered, spatially modulated ("striped") [A. Michelson, 1976]:



Phase structure in the CDT approach

Compare the phase diagram in the causal dynamical triangulations:
[Ambjørn et al, 1002.3298]



Note: $z = 2$ is sufficient to explain three phases.
Possibility of a nontrivial $z \approx 2$ fixed point in $3+1$ dimensions?

RG flows in gravity: $z = 1$ in IR

Theories with $z > 1$ represent candidates for the UV description. Under relevant deformations, the theory will flow in the IR. Relevant terms in the potential:

$$\Delta S_V = \int dt\, d^D\mathbf{x}\sqrt{g}N\left(\ldots + \mu^2 R - 2\Lambda\right).$$

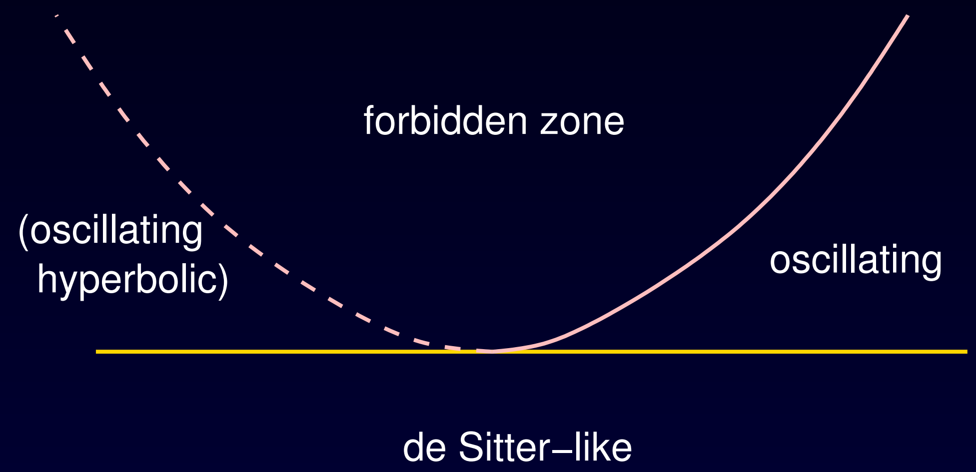
the dispersion relation changes in IR to $\omega^2 \sim k^2 + \ldots$
the IR speed of light is given by a combination of the couplings
μ^2 combines with $\kappa, \ldots$ to give an effective G_N.

Sign of k^2 in dispersion relation is opposite for the scalar and the tensor modes! Can we classify the phases of gravity? Can gravity be in a modulated phase?

Modulated phases of gravity

[in progress, w/ Patrick Zulkowski and Charles Melby-Thompson]

First, classify all spatially homogeneous and isotropic phases. Take $g_{ij} = a^2(t)\gamma_{ij}(k)$, with $k = 0, \pm 1$; set $N_i = 0$. The phase diagram for $k = 1$ (at fixed R^2 terms) looks like this:

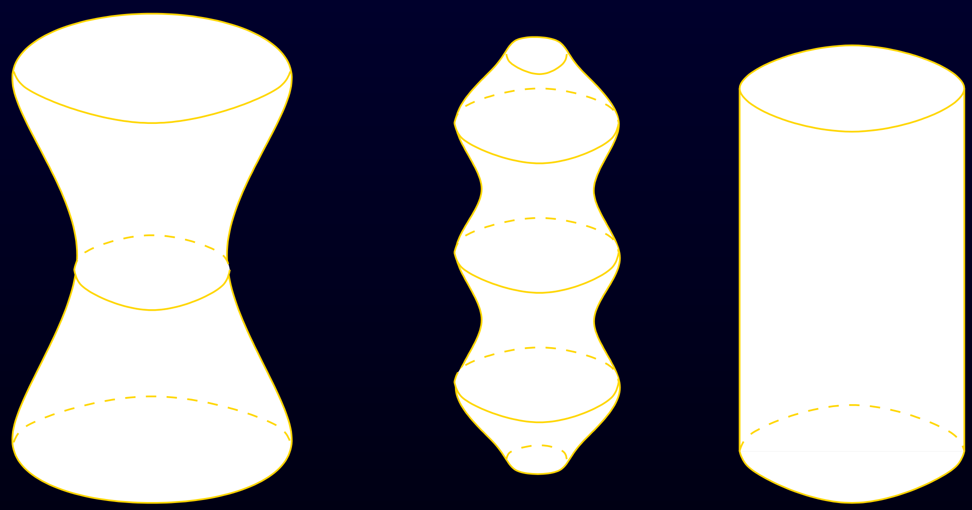


Governed by the Friedmann equation,

$$(\dot{g})^2 + R^2 + \mu^2 R - 2\Lambda = 0.$$

Spatially homogeneous isotropic phases of gravity

Examples of phases of gravity with $k = 1$: a de Sitter-like phase, an oscillating cosmology (= "temporally modulated" phase); the Einstein static universe appears at the phase transition line, where the theory satisfies detailed balance.



Cosmology: [Kiritsis et al, Brandenberger et al, Lüst et al, many others]

Comparison to GR in IR

The minimal, projectable theory in the IR:

$$S \sim \int dt\, d^D\mathbf{x}\, \sqrt{g}\, N \left\{ K_{ij}K^{ij} - \lambda K^2 + \ldots + \mu^2 (R - 2\Lambda) \right\}.$$

This looks accidentally as GR!

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Discrepancies:

(1) $\lambda = 1$ forced in GR;

(2) In GR, $N(t, \mathbf{x})$; here, $N(t)$;

(3) Gauge symmetries differ: $\mathrm{Diff}(M)$ vs. $\mathrm{Diff}(M, \mathcal{F})$.

Together, (2) and (3) imply the extra scalar graviton.

Projectable vs. nonprojectable

Simplest way to relax projectability: Declare N to be a function of everything, see what happens. This has worked in the ultralocal theory, leading to general covariance and the closure of the constraints.

Effective field theory says: Allow all terms in the action, compatible with the symmetries. New terms: built out of $\nabla_i N/N$. New constraints second-class, no additional gauge invariance.
(Literature: "healthy extension" of HL gravity)

Artificially disallowing such terms (an "unhealthy reduction"?): The constraint algebra in trouble, for $z > 1$. (Salvaging it leads to a reduction of DoF, effectively a topological theory.))

Ultralocal gravity

In retrospect, one example of a theory of gravity with anisotropic scaling has appeared in the literature already in the 1970's: the ultralocal theory of gravity [Isham;Teitelboim;Henneaux]

It results simply from eliminating all derivative terms from the potential, and setting

$$S_V = 2\Lambda.$$

Hamiltonian: as in GR, just

$$H = \int d^D\mathbf{x}\, \left(N\mathcal{H}_\perp + N_i\mathcal{H}^i\right).$$

Remarkably, this theory is "generally covariant" – it has the same number of gauge symmetries per spacetime point as GR. The symmetry algebra is not that of GR, instead it is deformed into spatial diffeomorphisms and a local $U(1)$ symmetry.

Nonrelativistic general covariance

Why do we want N to be the function of t and x^i? N is related to g_{00}, and that is where the Newton potential is.

Strategy: Keep the subleading, $\mathcal{O}(1/c^2)$ term in g_{00}:

$$g_{00} = -N(t)^2 + \frac{2NA(t,\mathbf{x})}{c^2} + \ldots,$$

and the subleading term α in the time reparametrizations as we take the $c \to \infty$ limit.

This α generates an extra $U(1)$ gauge symmetry,

$$\delta A_0 = \dot{\alpha} - N^i \partial_i \alpha, \quad \delta N_i = \partial_i \alpha, \quad \delta g_{ij} = 0.$$

Linearized theory

$U(1) \times \mathrm{Diff}(M, \mathcal{F})$ works beautifully, but only when $\lambda = 1$ (that's good!).

New coupling required:

$$\int dt\, d^D\mathbf{x}\, \sqrt{g}\, AR.$$

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Extension to nonlinear theory: Obstructed!

$$\delta_\alpha S \sim \int dt\, d^D\mathbf{x}\, \sqrt{g}\, \alpha \left(R^{ij} - \frac{1}{2} R g^{ij} \right) (\dot{g}_{ij} - 2\nabla_{(i} N_{j)}).$$

Three easy ways out

The obstruction

$$\delta_\alpha S \sim \int dt\, d^D\mathbf{x}\, \sqrt{g}\, \alpha \left(R^{ij} - \frac{1}{2} R g^{ij} \right) K_{ij}$$

goes away in three simple cases:

(1) in $2+1$ spacetime dimensions (but no spatial curvature);

(2) in Abelian gravity (an interacting theory);

(3) if we also add subleading A_{ij} fields in $1/c$ expansion of g_{ijj}; gives a topological theory.

General covariance at a Lifshitz point

At $\lambda = 1$, the obstruction exists for $U(1)_\Sigma$ even before gauging.
Strategy: First repair the global $U(1)_\Sigma$, then gauge it.

Introduce an auxiliary scalar, the Newton prepotential: ν

$$\delta\nu = \alpha.$$

Repairing the global $U(1)_\Sigma$:

$$\Delta S \sim \int dt\, d^D\mathbf{x}\,\sqrt{g}\,\nu\left(R^{ij} - \frac{1}{2}Rg^{ij}\right)K_{ij}$$

$$+\int dt\, d^D\mathbf{x}\,\sqrt{g}\,\nu\left(R^{ij} - \frac{1}{2}Rg^{ij}\right)\nabla_i\nabla_j\nu.$$

Gauging the global $U(1)_\Sigma$

Now introduce A, add new terms

$$\int dt\, d^D\mathbf{x}\, \sqrt{g}\, A(R - 2\Omega).$$

(Ω is a new relevant coupling, compatible with the repaired $U(1)_\Sigma$.)

Spectrum: Just the transverse-traceless (=tensor) graviton polarizations; the scalar graviton is a gauge artifact of $U(1)$.

Detailed analysis of Hamiltonian constraints confirms this count of DoF.

Preview of IR regime: Compact objects

Static compact object solutions? Schwarzschild geometry solves the equations of motion of the infrared limit of our theory with $\Omega = \Lambda = 0$.

Proof: For static solutions, $K_{ij} = 0$ and the rest of EoM is equivalent to EoM of a reduced action,

$$\int d^D\mathbf{x}\sqrt{g}\,(N - A)(R - 2\Omega).$$

The same is true for GR, if we identify $\mathcal{N} = N - A$ as the GR lapse function, and set $\Omega = \Lambda$. This gives a map between static solutions of GR and the IR limit of our theory (and $\nu = 0$).

Consequence: the β and γ coefficients of PPN take the GR values!

Preview of IR regime: Lorentz symmetry

Perhaps the most difficult challenge: How to explain the high degree of Lorentz invariance seen in Nature.
In particular, what makes all species see the same speed of light? (These might be strong coupling issues.)

Lorentz invariance as a global symmetry: Consider boosts

$$\delta t = b_i x^i, \qquad \delta x^i = b_i t.$$

In the minimal (=projectable) theory, this is not a symmetry: the background defines a preferred frame.

In the theory with nonrelativistic general covariance, the boost is a symmetry of the flat spacetime! It decomposes into a $U(1)$ transformation with $\alpha = b_i x^i$, and a $\mathrm{Diff}(M, \mathcal{F})$.
Preferred frame effects are only associated with ν.

Preview of IR regime: Cosmology?

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Observational cosmology prefers the homogeneous, isotropic and hence time-dependent foliation of the FRW Ansatz. However, the variation of A gives

$$R - 2\Omega = 0,$$

and time-dependent foliations by maximally symmetric slices will not be solutions.

Three possible ways out: (1) add matter;
(2) put cosmology in an unconventional gauge, with $N_i \neq 0$;
(3) when $\Omega = 0$, de Sitter in inflationary, spatially-flat coordinates is a solution!

Conclusions

The map of the new continent of gravity with anisotropic scaling is getting more precise.

Quantum gravity with nonrelativistic general covariance:

• exhibits an improved short-distance behavior associated with anisotropic scaling and $z > 1$,

• closely resembles general relativity at long distances,

• but the role of the Newton prepotential is still rather mysterious . . .