DIFFUSION OF UHECR IN EXPANDING UNIVERSE

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(based on the works with R. Aloisio and A. Gazizov)

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PROPAGATION OF UHECR IN MAGNETIC FIELDS

MC simulation

Yoshiguchi et al., (K, Sato) 2003, propagation in random magnetic field , $(B_c,\ l_c)$,

Sigl et al., (2003, 2004), Blasi, De Marco (2004), Kachelriess and Semikoz (2005).

Diffusive analytic solution

Aloisio and VB (2004, 2005), Lemoine (2005), based on
Syrovatsky (1959) solution of diffusion equation.

SYROVATSKY (1959) SOLUTION OF DIFFUSION EQUATION

Equation for a single source:

$$\frac{\partial}{\partial t} n_p(E,\vec{r},t) - \mathrm{div}\left[D(E,\vec{r},t)\nabla n_p\right] - \frac{\partial}{\partial E}\left[b(E,\vec{r},t)n_p\right] = Q(E,\vec{r},t)\delta^3(\vec{r}-\vec{r}_g).$$

solution was obtained by exclusive method introducing the **Syrovatsky variables**

$$\lambda(E,E_g) = \int_E^{E_g} d\varepsilon \frac{D(\varepsilon)}{b(\varepsilon)}, \qquad \tau(E,E_g) = \int_E^{E_g} \frac{d\varepsilon}{b(\varepsilon)}.$$

This method is valid when $D(E)$, $b(E)$, Q(E) do not depend on time.

The Syrovatsky solution:

$$n_p(E,r) = \frac{1}{b(E)} \int_E^{\infty} dE_g Q(E_g) \frac{exp\left[-r^2/4\lambda(E,E_g)\right]}{\left[4\pi\lambda(E,E_g)\right]^{3/2}}.$$

SYROVATSKY SOLUTION AND PROPAGATION THEOREM

The Syrovatsky solution obeys the **propagation theorem** (Aloisio and VB 2004):

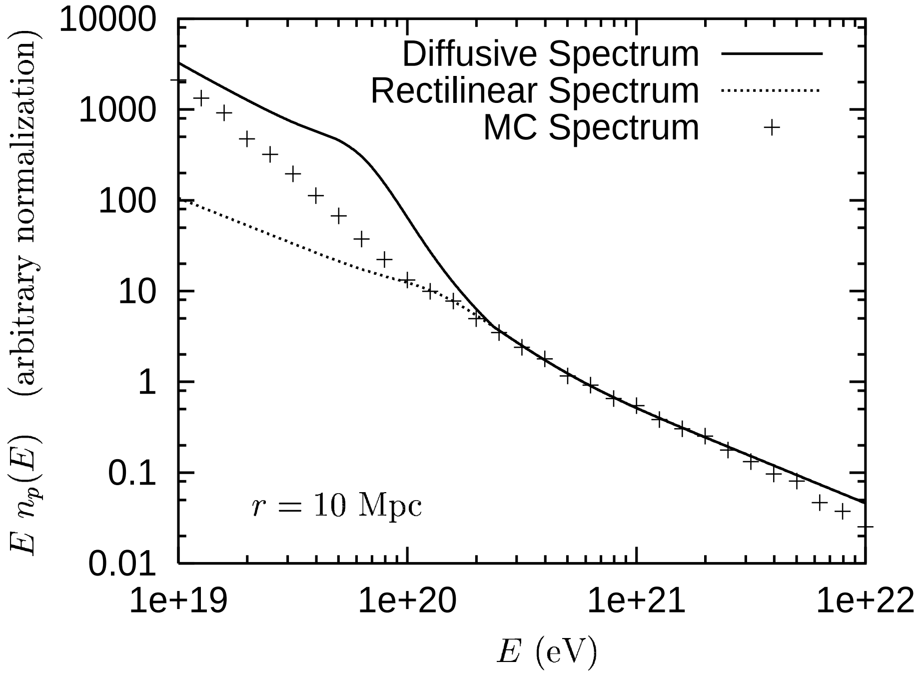
FOR UNIFORM DISTRIBUTION OF SOURCES WITH SEPARATION d MUCH LESS THAN CHARACTERISTIC LENGTHS OF PROPAGATION, SUCH AS $l_{\rm att}(E)$ and $l_{\rm diff}(E)$, **THE DIFFUSE SPECTRUM OF UHECR HAS AN UNIVERSAL (STANDARD) FORM INDEPENDENT OF MODE OF PROPAGATION .**

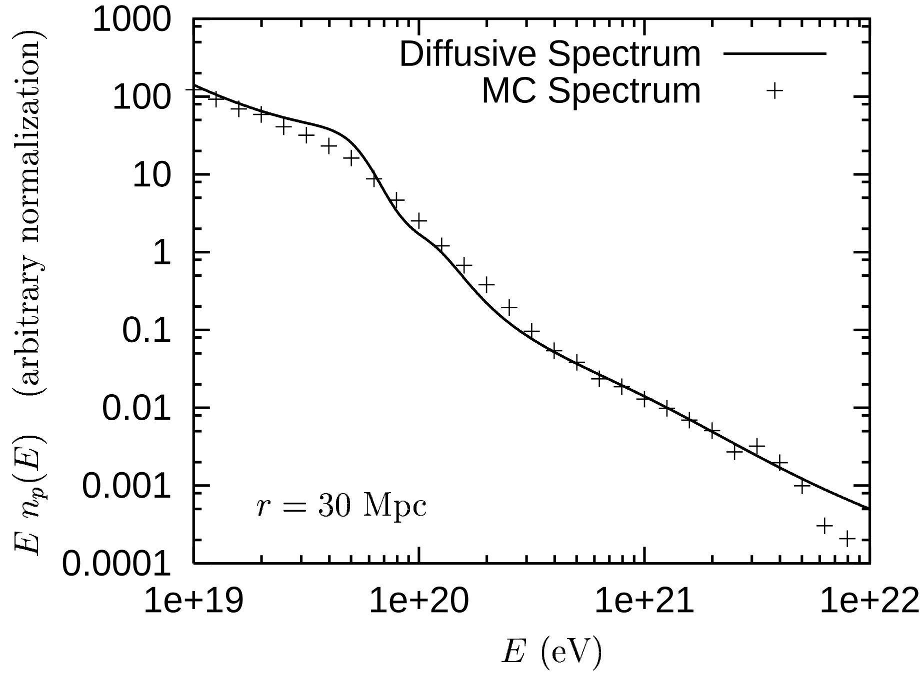
when $d \rightarrow 0$ solution for any mode of propagation tends to **universal spectrum**, which for homogeneous distribution of sources can be calculated from conservation of number of particles in the comoving volume $n_P(E)dE = \int dtq[E_g(t), t]dE_g$, where q is the production rate per unit comoving volume.

$$J_{\rm univ}(E) = \frac{c}{4\pi}\frac{\mathcal{L}_0(\gamma_g - 2)}{E_{\rm min}^2}\int_0^{z_{\rm max}} dz \left|\frac{dt}{dz}\right| (1+z)^m \left(\frac{E_g(E,z)}{E_{\rm min}}\right)^{-\gamma_g} \frac{dE_g}{dE},$$

where $\mathcal{L}_0$ is emissivity and m describes evolution.

COMPARISON WITH MC (K. Sato group)





CALCULATION OF THE DIFFUSE FLUX

We calculate diffuse spectrum for sources located in vertices of cubic lattice

$$J_p(E) = \frac{c}{4\pi}\frac{1}{b(E)}\sum_i \int_E^{E_{max}} dE_g Q(E_g) \frac{exp\left[-r_i^2/4\lambda(E,E_g)\right]}{(4\pi\lambda(E,E_g))^{3/2}}.$$

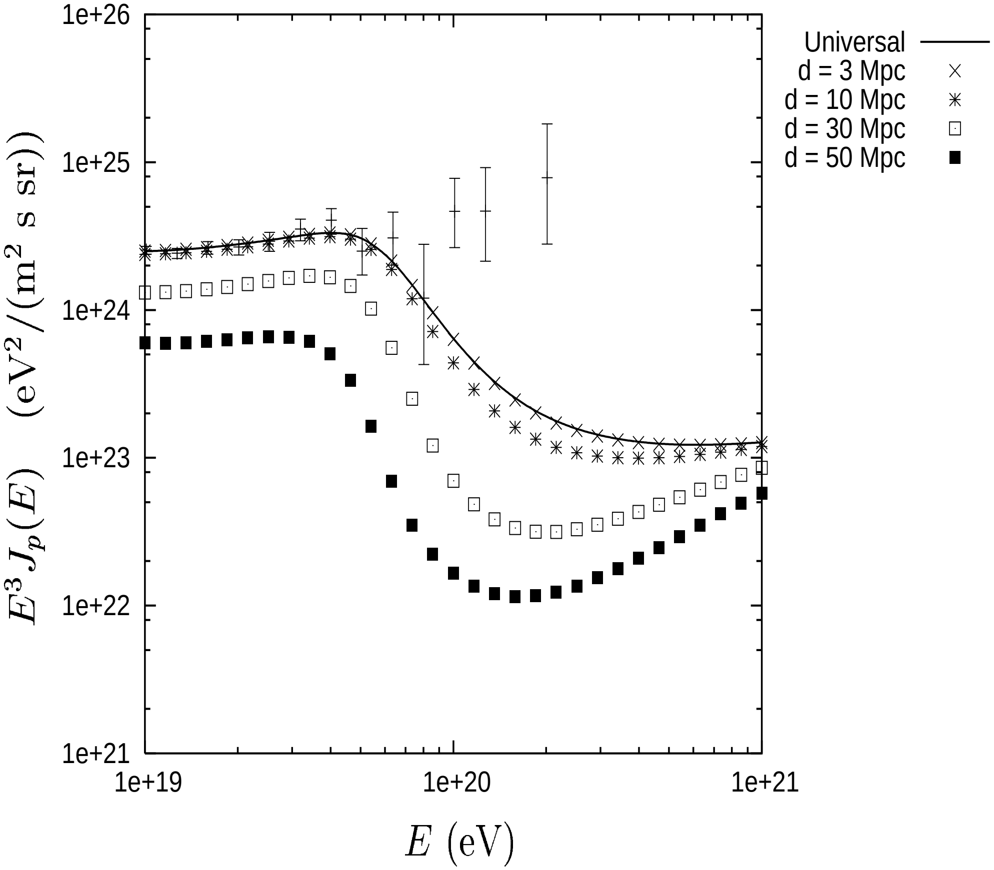
The diffusion coefficient $D(E)$ is needed for calculation of $\lambda(E, E_g)$.

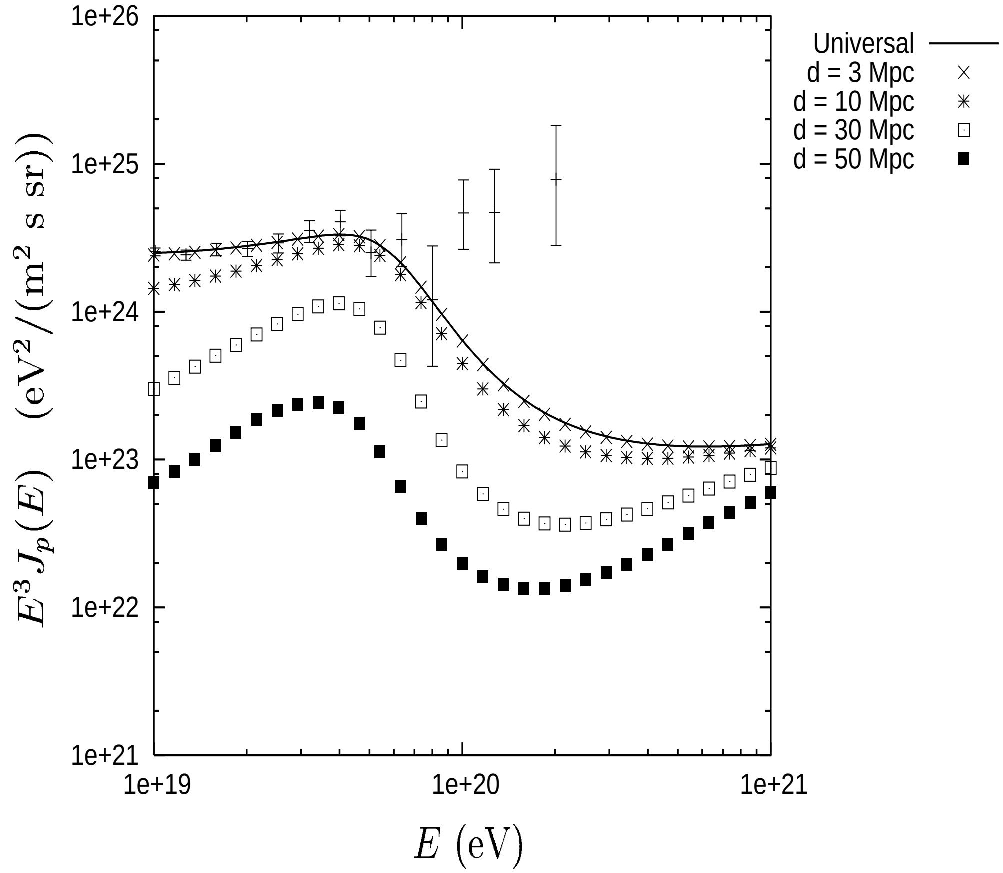
We assume magnetic turbulent plasma described as ensemble of MHD waves. Diffusion occurs due to resonant scattering on MHD waves. Magnetic turbulence has the basic (largest) scale l_c with magnetic field B_c.

It determines the critical energy E_c by relation $r_L(E_c) = l_c$.

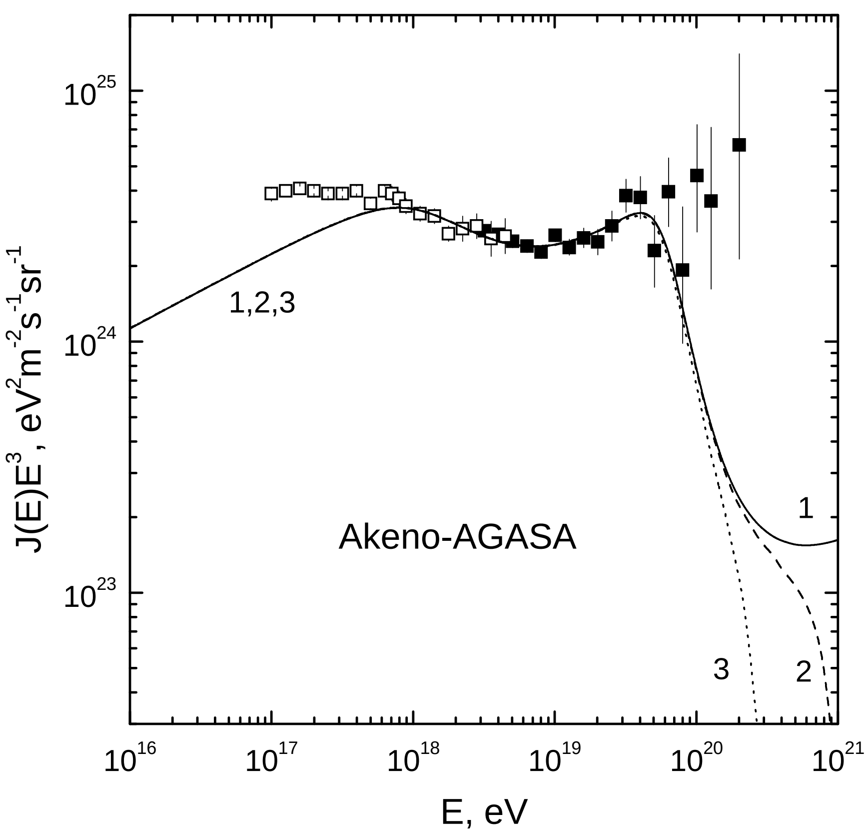
At $E \gg E_c$ $\;D(E) \approx c r_L^2/l_c \sim E^2$ for any spectrum of turbulence.
At $E \ll E_c$ $\;D(E)$ is determined by spectrum of turbulence,
e.g. $D(E) \sim E^{1/3}$ for the Kolmogorov spectrum.
Another option is the Bohm diffusion $D(E) = c r_L(E) \sim E$.

CONVERSION OF DIFFUSIVE SPECTRUM TO UNIVERSAL SPECTRUM

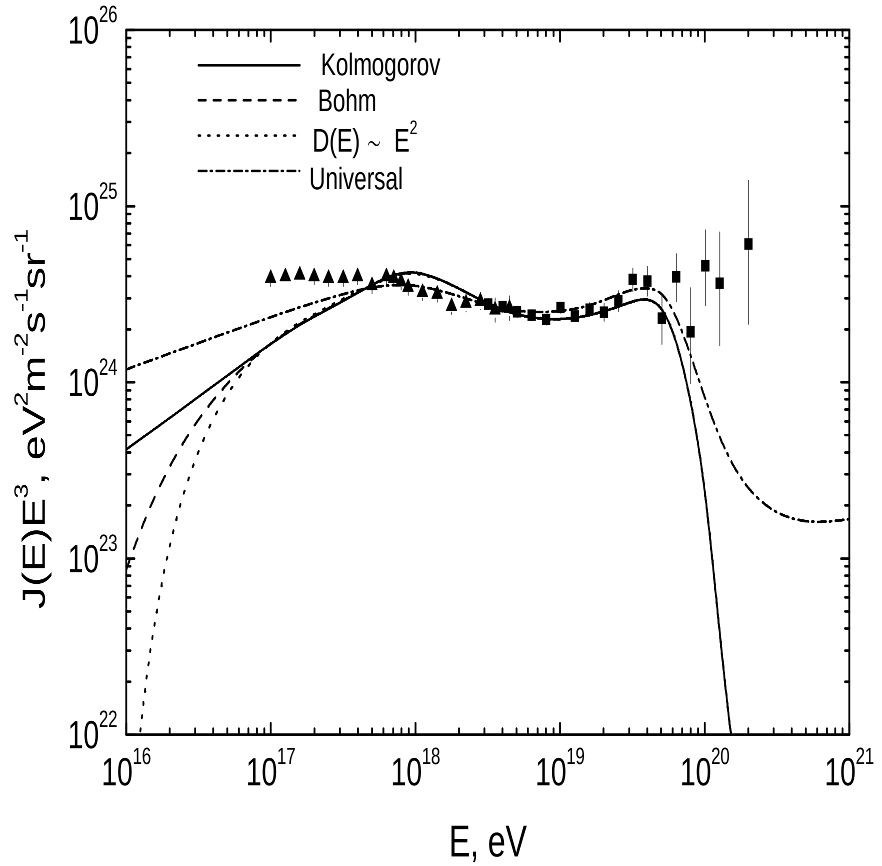




DIFFUSION at LOW-ENERGY END of UHECR



. rectilinear propagation



diffusive propagation

The low-energy ’diffusive cutoff’ at $E_b = 1 \times 10^{18}$ eV is universal and valid for all propagation modes. It is determined by fundamental energy $E_{\rm eq} = 2 \times 10^{18}$ eV, where pair-production and adiabatic energy losses become equal. The spectrum at $E < E_b$ depends on mode of propagation, e.g. rectilinear, Bohm or Kolmogorov diffusion. The low-energy ’cutoff’ provides transition from extragalactic to galactic CR.

DIFFUSION EQUATION IN EXPANDING UNIVERSE

Metric: $$ds^2 = c^2 dt^2 - a^2(t)d\vec{x}^{\,2} = -g_{\mu\nu}dx^{\mu\nu},$$

$$diag\ g_{\mu\nu} = (-1, a^2, a^2, a^2), \quad diag\ g^{\mu\nu} = (-1, 1/a^2, 1/a^2, 1/a^2),$$

Diffusive flux in the local frame:

$$j_k = -D\frac{\partial}{\partial x^k}n(\vec{x},t), \quad (k = 1,\ 2,\ 3).$$

Conservation of current j^μ :

$$\frac{\partial}{\partial x^\mu}\left(\sqrt{g}j^\mu\right) = 0.$$

Performing differentiation:

$$\frac{\partial}{\partial t}n(\vec{x},t) + 3H(t)n(\vec{x},t) - \frac{D}{a^2}\nabla_x^2 n(\vec{x},t) = 0,$$

Including energy losses and the source term:

$$\frac{\partial n}{\partial t} + 3H(t)n - \frac{D(E,t)}{a^2(t)}\nabla_x^2 n - \frac{\partial}{\partial E}\left[b(E,t)n\right] = \frac{Q(E,t)}{a^3(t)}\delta^3(\vec{x} - \vec{x}_g).$$

Analytic solution of the diffusion equation

Equation for the Fourier components $f_\omega(E,t)$:

$$\frac{\partial}{\partial t} f_\omega(E,t) - b(E,t)\frac{\partial}{\partial E} f_\omega(E,t) + \left[3H(t) - \frac{\partial b(E,t)}{\partial E} + \vec{\omega}^2 \frac{D(E,t)}{a^2(t)}\right] f_\omega(E,t) = \frac{Q(E,t)}{a^3(t)}.$$

The characteristic equation:

$$dE/dt = -b(E,t)$$

coincides with equation for energy evolution. Its solution is

$$\mathcal{E}' = E'(E,t,t').$$

The solution of equation for $f_\omega(E,t)$ with energies taken on characteristic:

$$f_\omega(E,t) = \int_{t_g}^{t} dt' \frac{Q(\mathcal{E}',t')}{a^3(t')} \exp\left\{ -\int_{t'}^{t} dt'' \left[3H(t'') - \frac{\partial b(\mathcal{E}'',t'')}{\partial \mathcal{E}''} + \vec{\omega}^2 \frac{D(\mathcal{E}'',t'')}{a^2(t'')}\right]\right\}$$

Introducing the analogue of the **Syrovatsky variable**

$$\lambda(E,t') = \int_{t'}^{t} dt'' \frac{D(\mathcal{E}'',t'')}{a^2(t'')},$$

we obtain for spherically symmetric case

$$\mathbf{n(x_g, E) = \int_0^{z_g} dz \left|\frac{dt}{dz}\right| Q[E_g(E,z),z] \; \frac{\exp[-x_g^2/4\lambda(E,z)]}{[4\pi\lambda(E,z)]^{3/2}} \; \frac{dE_g}{dE}},$$

where

$$\frac{dE_g}{dE} = (1+z)\exp\left[\int_0^z dz' \left|\frac{dt'}{dz'}\right| \frac{\partial b_{int}(\mathcal{E}',z')}{\partial \mathcal{E}'}\right],$$

$$-dt/dz = 1/\left[H_0(1+z)\sqrt{\Omega_m(1+z)^3+\Lambda}\right],$$

to be compared with the Syrovatsky solution:

$$\mathbf{n_S(E, x_g) = \frac{1}{b(E)} \int_E^{\infty} dE_g Q(E_g) \frac{\exp\left[-x_g^2/4\lambda(E,E_g)\right]}{\left[4\pi\lambda(E,E_g)\right]^{3/2}}}.$$

SUPERLUMINAL PROBLEM IN DIFFUSION EQUATION

Simple case:

Solution of energy-independent stationary diffusion equation:

$$n(r) = \frac{Q_0}{4\pi D r}, \qquad \frac{\partial n(r)}{\partial r} = -\frac{Q_0}{4\pi D r^2}$$

$$j = -D\frac{\partial n}{\partial r} = nu, \qquad u = -\frac{D}{n}\frac{\partial n}{\partial r} = \frac{D}{r} = c\frac{l_d}{r}$$

at $r < l_d$, $u > c$.

Similar example:

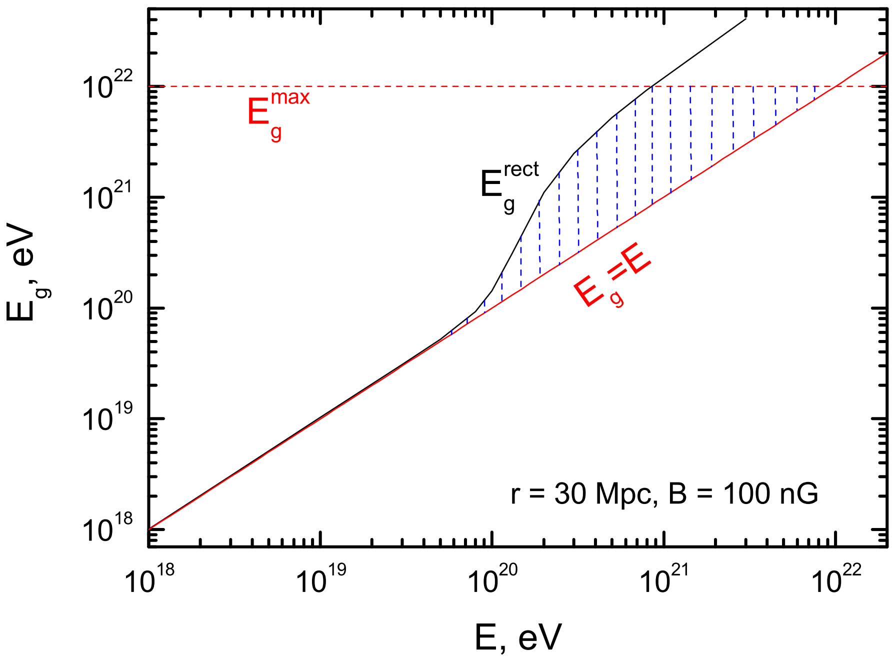
$r^2 \sim Dt, \quad u \sim \frac{r}{t} \sim \frac{D}{r} = c\,\frac{l_d}{r}$

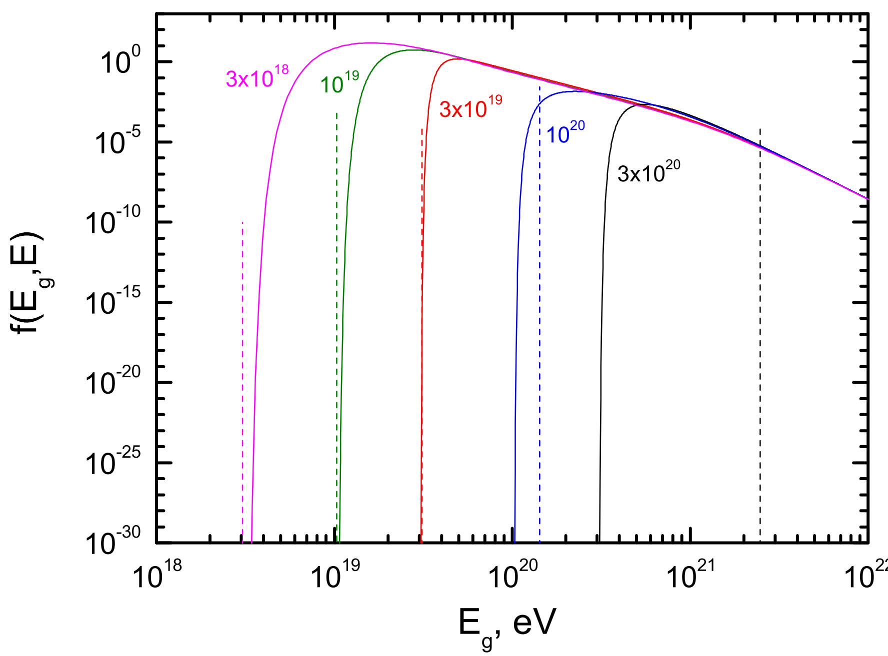
at $r < l_d$, $u > c$

SUPERLUMINAL PROBLEM DUE TO ENERGY LOSSES

$$n_p(E,r) = \frac{1}{b(E)} \int_E^\infty dE_g Q(E_g) \frac{exp\left[-r^2/4\lambda(E,E_g)\right]}{\left[4\pi\lambda(E,E_g)\right]^{3/2}}.$$

$$E \to E_g^{\mathrm{rect}}(E,r)$$





PRACTICAL RECIPES

Relativistic equation with diffusion as low-energy asymptotic:
is not found during last 100 years.

e.g. the telegraph equation

$$\tau_d \frac{\partial^2}{\partial t^2} n + \frac{\partial}{\partial t} n - D\nabla^2 n = Q,$$

$c_d = (D\tau_d)^{1/2}, \quad \tau_d \to 0$ gives diffusion equation.

Standard practice: to avoid regions of superluminal regimes.

In UHECR diffusion we used:

- at $r < l_d$: rectilinear propagation,
- at $r > l_d$: diffusive propagation,
- with interpolation between,
- exclusion regions with large superluminal contribution.

JÜTTNER APPROACH

E. J. Jüttner (Ann. Phys. (Leipzig) 1911) found relativization of the non-relativistic Maxwell distribution.

Dunkel, Talkner, Hänggi (2007) observed the identity of the Maxwell distribution,

$$P_M(v) = \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mv^2}{2kT}\right).$$

with the Green function of energy and time independent 3D diffusion equation,

$$P_{\rm diff}(r,t) = \frac{1}{(4\pi Dt)^{3/2}} \exp\left(-\frac{r^2}{4Dt}\right),$$

obtained after substitution $v \to r$ and $kT/m \to 2Dt$.

Aloisio, V.B., Gazizov (2008) generalized the Jüttner function for energy and time dependent diffusion solution in expanding universe.

PROPAGATOR FORMALISM

We introduce phenomenologically the **propagator** $P(E,r,t)$ as

$$n(E,r) = \int_0^\infty dt\; Q[E_g(E,t),t]\; P(E,t,r)\; \frac{dE_g}{dE}(E,t),$$

where Q is a source generation function, and $P(E,r,t)$ **can be thought of as the Green function of unknown relativistic propagation equation.**

$P(E,r,t)$ must satisfy the following conditions:

- absence of superluminal signal: $P(E,r,t) = 0$ at $r > ct$,
- normalized probability to find a particle $\int dV\, P(E,r,t) = 1$,
- rectilinear propagation at large energies:
$$P(E,t,r) = \frac{1}{4\pi c^3 t^2}\delta(t - \frac{r}{c}),$$
- diffusive propagation at low energies:
$$P(E,r,t) = \frac{1}{[4\pi\lambda(E,t)]^{3/2}} \exp\left(-\frac{r^2}{4\lambda(E,t)}\right)$$

JÜTTNER PROPAGATOR

In terms of $v = r$ and $kT/m = 2Dt$ the Jüttner propagator is given by

$$P_J(E,t,r) = \frac{\theta(ct-r)}{(ct)^3 Z(c^2t/2D)\left[1 - r^2/(c^2t^2)\right]^2} \exp\left[-\frac{c^2t/2D}{[1-r^2/(ct)^2]^{1/2}}\right],$$

where $Z(y) = 4\pi K_1(y)/y$ and $K_1(y)$ is the modified Bessel function.

This Jüttner propagator corresponds to the simplest diffusion equation,

To obtain the propagator for energy-dependent propagation with Syrovatsky solution, one should change the variables as

$$\begin{aligned} \frac{c^2t}{2D} = \frac{c^2t^2}{2Dt} \quad &\to \quad \frac{c^2t^2}{2\int D(E,t)dt} = \frac{c^2t^2}{2\lambda(E,t)} \equiv \alpha(E,t). \\ t \quad &\to \quad \xi(t) = r/ct \end{aligned}$$

This gives the **modified Jüttner propagator**.

MODIFIED JÜTTNER PROPAGATOR

$$P_{mJ}(E,t,r) = \frac{\theta(1-\xi)}{4\pi(ct)^3}\frac{1}{(1-\xi^2)^2}\frac{\alpha(E,\xi)}{K_1[\alpha(E,\xi)]}\exp\left[-\frac{\alpha(E,\xi)}{\sqrt{1-\xi^2}}\right],$$

$$n(E,r) = \frac{1}{4\pi cr^2}\int\limits_{\xi_{\min}}^{1}\xi d\xi \frac{Q[E_g(E,\xi)]}{(1-\xi^2)^2}\frac{\alpha(E,\xi)}{K_1[\alpha(E,\xi)]}\exp\left[-\frac{\alpha(E,\xi)}{\sqrt{1-\xi^2}}\right]\frac{dE_g}{dE}.$$

High-energy regime $\alpha \ll 1, \quad \xi \to 1$ **(rectilinear propagation)**:

$$n(E,r) = \frac{1}{4\pi cr^2}Q\left[E_g\left(\frac{r}{c}\right)\right]\frac{dE_g}{dE}.$$

Low-energy regime $\alpha \gg 1, \quad \xi \ll 1$ **(Syrovatsky solution)**:

$$P(E,r,t) = \frac{\theta(ct-r)}{[4\pi\lambda(E,t)]^{3/2}}\exp\left[-\frac{r^2}{4\lambda(E,t)}\right].$$

JÜTTNER PROPAGATOR FOR EXPANDING UNIVERSE

We use as variables $\xi(t)$ and $\alpha(E,t)$, where

$$\xi(t) = \frac{x_g}{\zeta(t)}, \quad \frac{c^2 t}{D} = \frac{c^2 t^2}{Dt} \to \frac{\zeta^2(t)}{2\lambda(E,t)} \equiv \alpha(E,t),$$

$$\zeta(t) = \int_t^{t_0} \frac{cdt}{a(t)} = \frac{c}{H_0}\int_0^{z_g} \frac{dz}{\sqrt{\Omega_m(1+z)^3+\Omega_\Lambda}}.$$

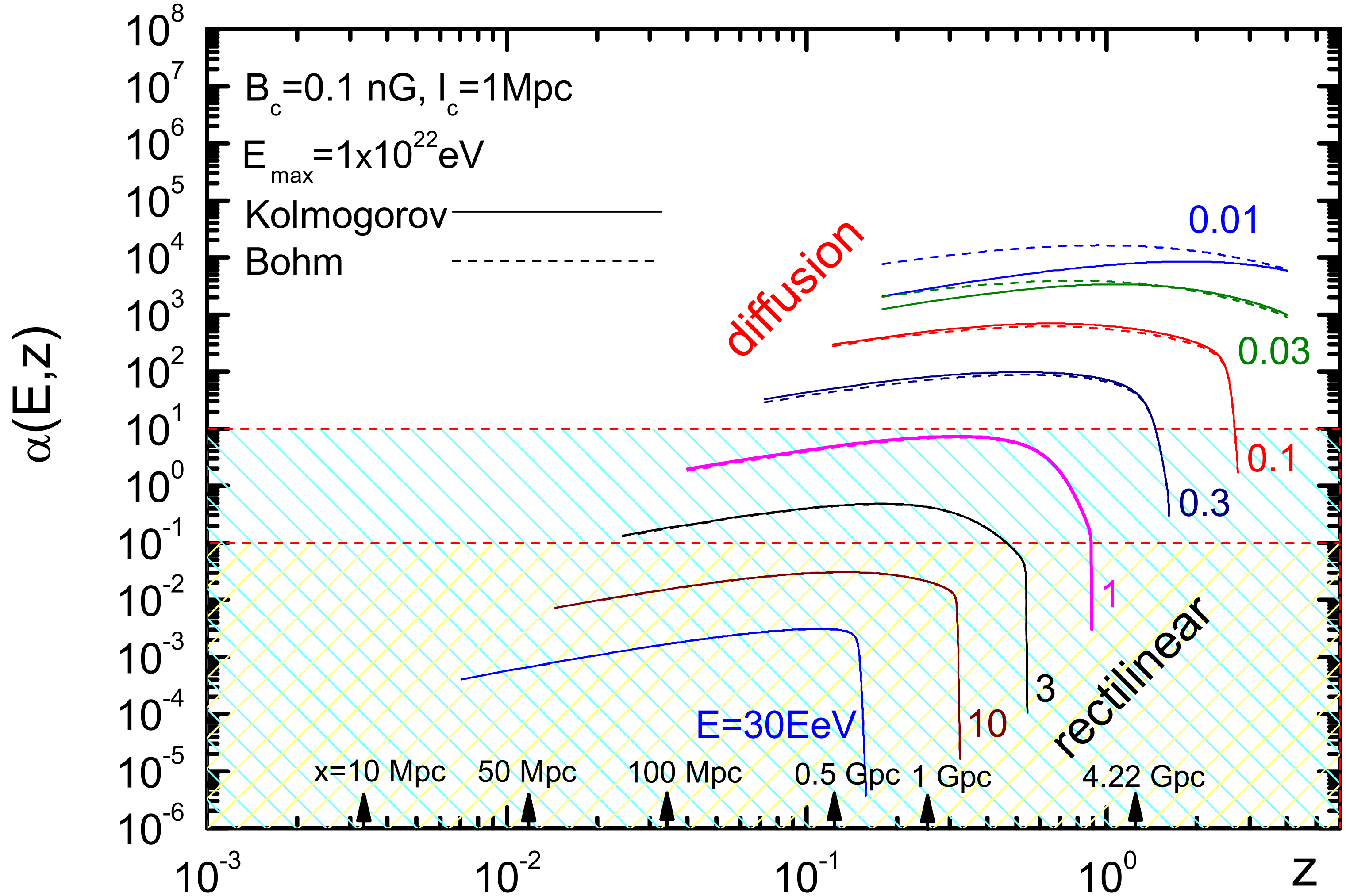
is comoving length of particle trajectory.

$$P_{eJ}(E,t,x_s) = \theta(1-\xi)\frac{\xi^3}{x_s^3(1-\xi^2)^2}\frac{\alpha}{4\pi K_1(\alpha)}\exp\left(-\frac{\alpha}{\sqrt{1-\xi^2}}\right).$$

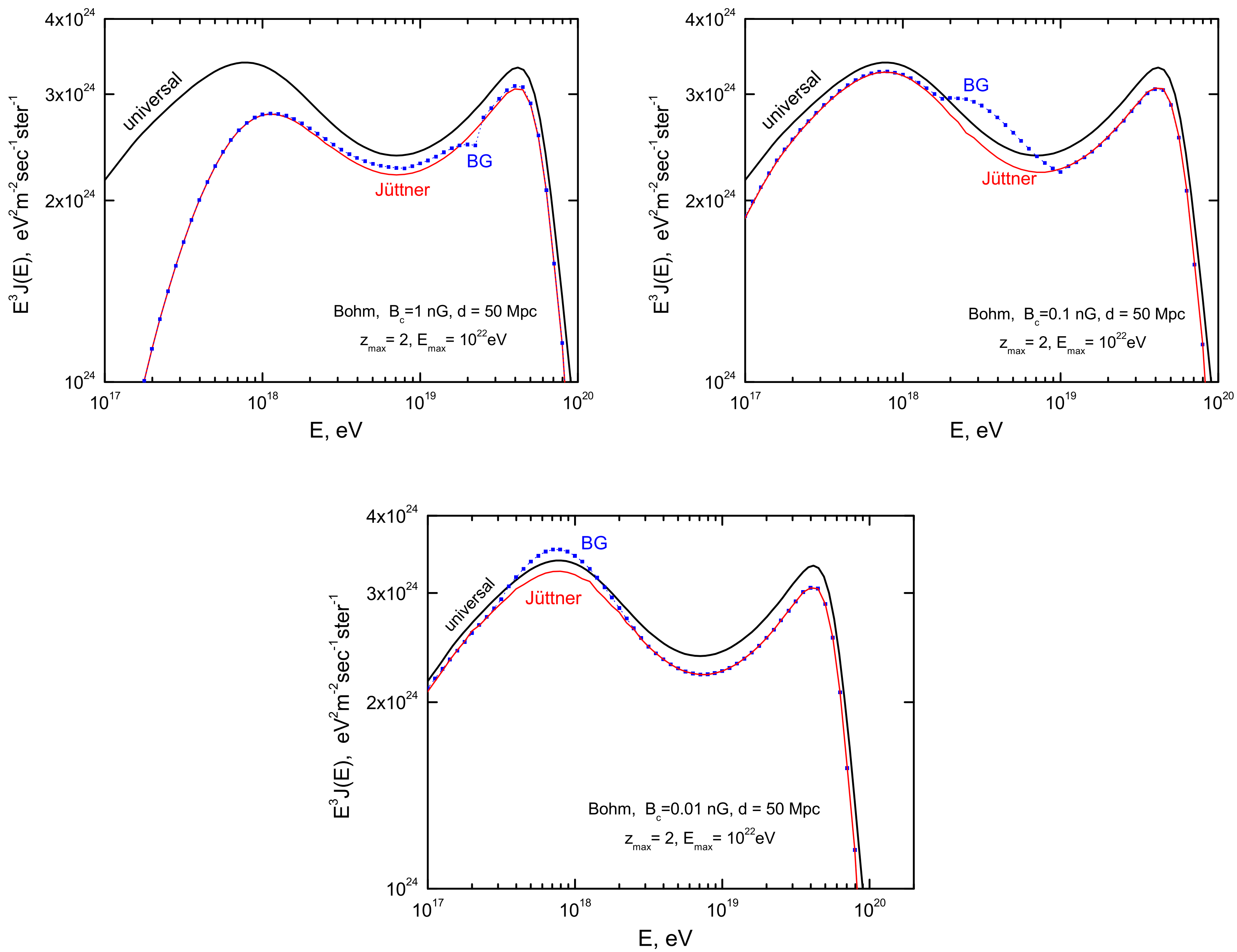
$$n(E,x_s) = \frac{1}{4\pi c x_s^2}\int\limits_{\xi_{\min}}^{1} \frac{\xi d\xi}{1+z(\xi)}\frac{Q[E_g(E,\xi)]}{(1-\xi^2)^2}\frac{\alpha}{K_1(\alpha)}\exp\left(-\frac{\alpha}{\sqrt{1-\xi^2}}\right)\frac{dE_g}{dE}.$$

$P_{eJ}(E,t,x_s)$ has correct asymptotics.

PROPAGATION IN TERMS OF $\alpha(E,z)$



DIFFUSE SPECTRA IN EXPANDING UNIVERSE



CONCLUSIONS

- **We obtained the analytic solution of diffusion equation for ultra-relativistic ($E \approx p$) particles (electrons, protons, nuclei). The solution is valid for expanding universe and for diffusion coefficient D and energy loss b with arbitrary dependence on E and t.**

- **Transition between diffusive propagation and rectilinear propagation at high energies is described by the modified Jüttner function.**

- **Comparison of diffusive (Syrovatsky) spectra from a source show the good agreement with numerical simulations by Yoshiguchi et al 2003.**

- **The method of diffusion equation is important for low-energy end of UHECR $1 \times 10^{17} \lesssim E < 1 \times 10^{19}$ eV , where numerical simulations need unrealistically long computation time.**

- **At $E < 1 \times 10^{18}$ eV spectrum of extragalactic protons has the diffusion cutoff, which provides transition from extragalactic to galactic cosmic rays at the second knee at $E_{2\text{kn}} \sim (0.4 - 0.8) \times 10^{18}$ eV, as measured in different experiments.**

THREE TESTS OF THE SOLUTION

1. The solution coincides with the Syrovatsky solution when

$$D(E,t) = D(E), \;\; b(E,t) = b(E), \;\; a(t) = 1 \; .$$

2. In case of homogeneous distribution of sources, the solution gives the universal spectrum as must be according to propagation theorem.

3. Solution for rectilinear-propagation equation

$$\frac{\partial n}{\partial t} + \frac{c\vec{e}}{a(t)}\frac{\partial n}{\partial \vec{x}} - b(E,t)\frac{\partial n}{\partial E} + 3H(t)n - n\frac{\partial b}{\partial E} = \frac{Q(E,t)}{a^3(t)}\delta^3(\vec{x}-\vec{x}_g),$$

obtained by the same formal method gives the correct (known) solution

$$n(t_0,E) = \frac{Q(E_g,t_g)}{4\pi c x_g^2(1+z_g)}\frac{dE_g}{dE}$$