# $\mathrm{rot}(\mathrm{rot}\mathbf{A})$の計算

ベクトル演算公式　$\mathrm{rot}(\mathrm{rot}\mathbf{A}) = \mathrm{grad}(\mathrm{div}\mathbf{A}) - \nabla^2\mathbf{A}$　を導く。

定義から、$\mathrm{rot}\mathbf{A}$ はベクトルで次のような成分をもつ。

$$\mathrm{rot}\mathbf{A} = \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

従って、これにもう一度 rot を演算すると次のようになる。

$$[\mathrm{rot}(\mathrm{rot}\mathbf{A})]_x = \frac{\partial}{\partial y}\left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) - \frac{\partial}{\partial z}\left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \quad ①$$

$$[\mathrm{rot}(\mathrm{rot}\mathbf{A})]_y = \frac{\partial}{\partial z}\left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) - \frac{\partial}{\partial x}\left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \quad ②$$

$$[\mathrm{rot}(\mathrm{rot}\mathbf{A})]_z = \frac{\partial}{\partial x}\left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) - \frac{\partial}{\partial y}\left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \quad ③$$

一方、右辺の第一項を成分で書くと次のようになる。

$$[\mathrm{grad}(\mathrm{div}\mathbf{A})]_x = \frac{\partial}{\partial x}\left( \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) \quad ④$$

$$[\mathrm{grad}(\mathrm{div}\mathbf{A})]_y = \frac{\partial}{\partial y}\left( \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) \quad ⑤$$

$$[\mathrm{grad}(\mathrm{div}\mathbf{A})]_z = \frac{\partial}{\partial z}\left( \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) \quad ⑥$$

また、右辺第二項は

$$-\nabla^2 A_x = -\left( \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2} \right) \quad ⑦$$

$$-\nabla^2 A_y = -\left( \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2} \right) \quad ⑧$$

$$-\nabla^2 A_z = -\left(\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2}\right) \qquad ⑨$$

これらから、それぞれ成分ごとに調べると

①=④＋⑦

②=⑤＋⑧

③=⑥＋⑨

が成り立つので、$\mathrm{rot}(\mathrm{rot}\mathbf{A}) = \mathrm{grad}(\mathrm{div}\mathbf{A}) - \nabla^2 \mathbf{A}$ が成り立っていることがわかる。