Scalar perturbations induced by scalar perturbations

Keisuke Inomata
a postdoc in RESCEU
(moving to Chicago on Aug. 31)

Based on my ongoing work

Overview

We derive analytic solutions of the second-order scalar perturbations in a radiation-dominated (RD).

$$\Psi^{(2)}(\boldsymbol{k},\eta) = \int \frac{\mathrm{d}^3\tilde{k}}{(2\pi)^3} uv \underline{I_{\Psi,\mathrm{r}}(u,v,x)} \left(\frac{2}{3}\right)^2 \zeta^{(1)}(\tilde{\boldsymbol{k}})\zeta^{(1)}(\boldsymbol{k}-\tilde{\boldsymbol{k}})$$

$$(v \equiv \tilde{k}/k,\ \ u \equiv |\boldsymbol{k}-\tilde{\boldsymbol{k}}|/k,\ \ x \equiv k\eta)$$

We derive the analytic expression of this transfer function.

c.f. first-order perturbation

$$\Psi^{(1)}(\boldsymbol{k},\eta) = T_{\Psi,\mathrm{r}}(x)\frac{2}{3}\zeta^{(1)}(\boldsymbol{k}) \qquad \left(T_{\Psi,\mathrm{r}}(x) = \frac{3[\sin(x/\sqrt{3}) - (x/\sqrt{3})\cos(x/\sqrt{3})]}{(x/\sqrt{3})^3}\right)$$

We also discuss the power spectra of the second-order scalar perturbations.

Outline of this talk

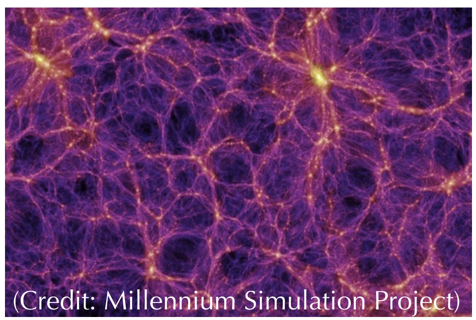
- Introduction
- Second-order perturbations in a RD era
- Summary

Scalar perturbation in Cosmology

Scalar perturbations induce the perturbations of energy density ($\delta\rho/\rho$).

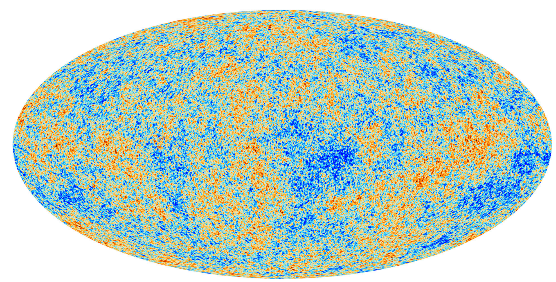
Scalar perturbations are origins of many things!

Examples



(Credit: Millennium Simulation Project)

Large Scale Structure



(Credit: ESA and the Planck Collaboration)

CMB perturbations

From the observations, we already know the amplitude of the scalar perturbations.

$$\mathcal{P}_\zeta = 2.1 \times 10^{-9} \quad \text{(Planck 2018)}$$

$$(\delta\rho/\rho \sim 10^{-5})$$

Scalar perturbations originate from vacuum fluctuations of an inflaton during the inflation era.

Theoretical motivation

Metric perturbations:

$$\mathrm{d}s^2 = a^2\left[-(1+2\Phi^{(1)}+\Phi^{(2)})\mathrm{d}\eta^2 + (1-2\Psi^{(1)}-\Psi^{(2)})\mathrm{d}x_i\mathrm{d}x^i\right]$$

We already know first-order perturbations very well.

$$\Psi^{(1)}(\boldsymbol{k},\eta) = \begin{cases} T_{\Psi,\mathrm{r}}(x)\frac{2}{3}\zeta^{(1)}(\boldsymbol{k}) & \text{(RD)} \\ \frac{3}{5}\zeta^{(1)}(\boldsymbol{k}) & \text{(MD)} \end{cases} \quad \left(T_{\Psi,\mathrm{r}}(x) = \frac{3[\sin(x/\sqrt{3}) - (x/\sqrt{3})\cos(x/\sqrt{3})]}{(x/\sqrt{3})^3}\right)$$

How about second-order perturbations?

Although some works derived the analytic solutions, they made some simplifications. (I will mention them later.)

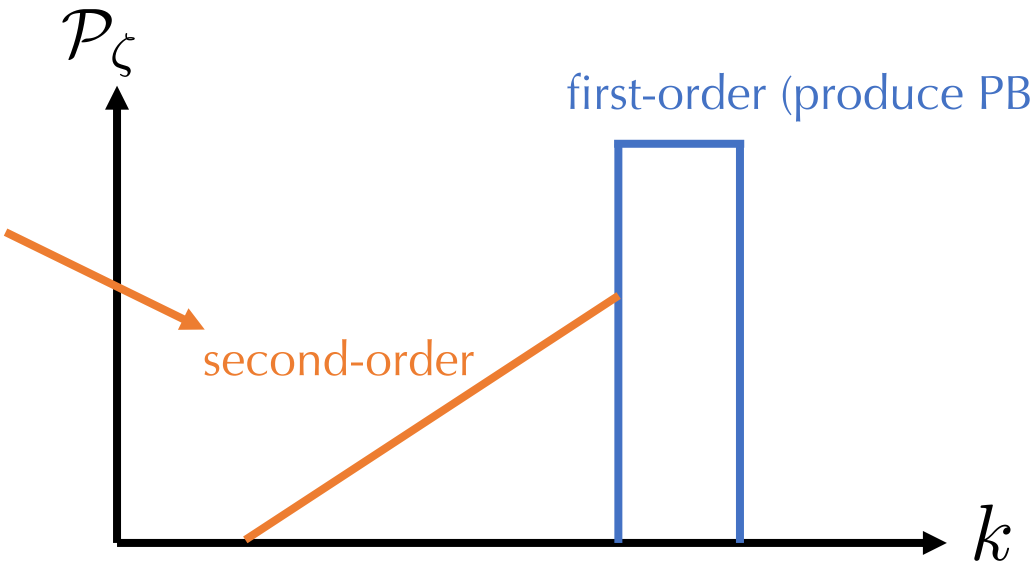
We derive the complete analytic expressions for a perfect fluid in a RD era.

Phenomenological motivation

Second-order perturbations can make modification of the large scale structure (Juszkiewicz 1981) and the CMB anisotropies (Hu, Scott, Silk, 1994).

Second-order perturbations can also be important in the context of PBHs.

This might be investigated by observation.



Second-order perturbations can be dominant on some scales.

Outline of this talk

- Introduction
- Second-order perturbations in a RD era
- Summary

Equations for second-order perturbations

Metric perturbations:

$$\mathrm{d}s^2 = a^2\left[-(1+2\Phi^{(1)}+\Phi^{(2)})\mathrm{d}\eta^2 + (1-2\Psi^{(1)}-\Psi^{(2)})\mathrm{d}x_i\mathrm{d}x^i\right]$$

Equation of motion (RD era):

$$\Psi^{(2)\prime\prime}(\boldsymbol{k},\eta) + \frac{4}{\eta}\Psi^{(2)\prime}(\boldsymbol{k},\eta) + \frac{k^2}{3}\Psi^{(2)}(\boldsymbol{k},\eta) = S_{\mathrm{r}}^{(2)}(\boldsymbol{k},\eta)$$

Source term:

$$\begin{aligned} S_{\mathrm{r}}^{(2)}(\boldsymbol{k},\eta) = \int \frac{\mathrm{d}^3\tilde{k}}{(2\pi)^3}\Bigg\{ & -\left(\frac{2}{3}\tilde{\boldsymbol{k}}\cdot(\boldsymbol{k}-\tilde{\boldsymbol{k}}) + \frac{4}{3}(\tilde{k}^2 + |\boldsymbol{k}-\tilde{\boldsymbol{k}}|^2)\right)\Phi(\tilde{\boldsymbol{k}})\Phi(\boldsymbol{k}-\tilde{\boldsymbol{k}}) + 2\Phi'(\tilde{\boldsymbol{k}})\Phi'(\boldsymbol{k}-\tilde{\boldsymbol{k}}) \\ & + \frac{3}{2k^2}\left(\frac{1}{k^2}(\boldsymbol{k}\cdot\tilde{\boldsymbol{k}})(\boldsymbol{k}\cdot(\boldsymbol{k}-\tilde{\boldsymbol{k}})) - \frac{1}{3}\tilde{\boldsymbol{k}}\cdot(\boldsymbol{k}-\tilde{\boldsymbol{k}})\right) \\ & \times \Bigg[\frac{1}{\eta}\left(6\Phi(\tilde{\boldsymbol{k}})\Phi(\boldsymbol{k}-\tilde{\boldsymbol{k}}) + 4\eta\Phi'(\tilde{\boldsymbol{k}})\Phi(\boldsymbol{k}-\tilde{\boldsymbol{k}}) + 2\eta^2\Phi'(\tilde{\boldsymbol{k}})\Phi'(\boldsymbol{k}-\tilde{\boldsymbol{k}})\right)' \\ & - \frac{k^2}{3}\left(6\Phi(\tilde{\boldsymbol{k}})\Phi(\boldsymbol{k}-\tilde{\boldsymbol{k}}) + 4\eta\Phi'(\tilde{\boldsymbol{k}})\Phi(\boldsymbol{k}-\tilde{\boldsymbol{k}}) + 2\eta^2\Phi'(\tilde{\boldsymbol{k}})\Phi'(\boldsymbol{k}-\tilde{\boldsymbol{k}})\right)\Bigg]\Bigg\} \end{aligned}$$

$$\Phi^{(1)}(\boldsymbol{k},\eta) = T_{\Phi,\mathrm{r}}(x)\frac{2}{3}\zeta^{(1)}(\boldsymbol{k})$$

Substituting the first-order transfer function, we solve the e.o.m.

How to solve the equation 1

The e.o.m can be rewritten as $(z \equiv a^2 \Psi^{(2)})$

$$z''(\boldsymbol{k},\eta) + \left(\frac{k^2}{3} - \frac{2}{\eta^2}\right) z(\boldsymbol{k},\eta) = a^2 S_{\rm r}^{(2)}(\boldsymbol{k},\eta)$$

The Green's function is given as the solution of the following equation:

$$G_{\rm r}'' + \left(\frac{k^2}{3} - \frac{2}{\eta^2}\right) G_{\rm r} = \delta(\eta - \bar{\eta})$$

The concrete expression is

$$kG_{\rm r}(k,\eta;\bar{\eta}) = -\Theta(\eta-\bar{\eta}) \frac{x\bar{x}}{\sqrt{3}} \left[j_1(x/\sqrt{3}) y_1(\bar{x}/\sqrt{3}) - j_1(\bar{x}/\sqrt{3}) y_1(x/\sqrt{3}) \right]$$

Then, we obtain

$$\Psi^{(2)}(\boldsymbol{k},\eta) = \Psi^{(2)}(\boldsymbol{k},0) T_{\Phi,{\rm r}}(x) + \int_0^\eta {\rm d}\bar{\eta} \left(\frac{a(\bar{\eta})}{a(\eta)}\right)^2 G_{\rm r}(k,\eta;\bar{\eta}) S_{\rm r}^{(2)}(\boldsymbol{k},\eta)$$

How to solve the equation 2

Then, we rewrite the equation as

$$\Psi^{(2)}(\boldsymbol{k},\eta) = \Psi^{(2)}(\boldsymbol{k},0)T_{\Phi,\mathrm{r}}(x) + \int_0^\eta \mathrm{d}\bar{\eta} \left(\frac{a(\bar{\eta})}{a(\eta)}\right)^2 G_\mathrm{r}(k,\eta;\bar{\eta}) S_\mathrm{r}^{(2)}(\boldsymbol{k},\eta)$$

$$= \Psi^{(2)}(\boldsymbol{k},0)T_{\Phi,\mathrm{r}}(x) + \int \frac{\mathrm{d}^3\tilde{k}}{(2\pi)^3} uv I_{\Psi,\mathrm{r,s}}(u,v,x) \left(\frac{2}{3}\right)^2 \zeta(\tilde{\boldsymbol{k}})\zeta(\boldsymbol{k}-\tilde{\boldsymbol{k}})$$

$$(v \equiv \tilde{k}/k,\ \ u \equiv |\boldsymbol{k}-\tilde{\boldsymbol{k}}|/k,\ \ x \equiv k\eta)$$

The function I is given as

$$I_{\Psi,\mathrm{r,s}}(u,v,x) = \mathcal{J}(u,v,x) 3\sqrt{3}\frac{j_1(x/\sqrt{3})}{x} + \mathcal{Y}(u,v,x) 3\sqrt{3}\frac{y_1(x/\sqrt{3})}{x}$$

$$\mathcal{J}(u,v,x) = \mathcal{J}_0(u,v,x) + \mathcal{J}_\mathrm{h}(u,v,x),$$

$$\mathcal{J}_0(u,v,x) \equiv \int_0^x \mathrm{d}\bar{x} \sum_{n=1}^{8} \sin(\alpha_n \bar{x} + \phi_n) M^j_{n0},$$

$$\mathcal{J}_\mathrm{h}(u,v,x) \equiv \int_0^x \mathrm{d}\bar{x} \sum_{m=1}^{7}\sum_{n=1}^{8} \sin(\alpha_n \bar{x} + \phi_n) \frac{M^j_{nm}}{\bar{x}^m}$$

$$\mathcal{Y}(u,v,x) = \mathcal{Y}_0(u,v,x) + \mathcal{Y}_\mathrm{h}(u,v,x),$$

$$\mathcal{Y}_0(u,v,x) \equiv \int_0^x \mathrm{d}\bar{x} \sum_{n=1}^{8} \sin(\alpha_n \bar{x} + \phi_n) M^y_{n0},$$

$$\mathcal{Y}_\mathrm{h}(u,v,x) \equiv \int_0^x \mathrm{d}\bar{x} \sum_{m=1}^{7}\sum_{n=1}^{8} \sin(\alpha_n \bar{x} + \phi_n) \frac{M^y_{nm}}{\bar{x}^m}$$

Discussed in Bartolo, Matarrese , Riotto, 2006

$\mathcal{J}_0, \mathcal{Y}_0$ → Not converge in the late-time limit (x>>1)

Never been discussed ! $\mathcal{J}_\mathrm{h}, \mathcal{Y}_\mathrm{h}$ → Converge in the late-time limit (x>>1)

Concrete expressions

$$\mathcal{J}_0(u,v,x) = \frac{3(-1+3(u^2-v^2)^2-2(u^2+v^2))}{16u^2v^2} \times \left(\frac{\cos\left(\frac{-1+u-v}{\sqrt{3}}x\right)}{-1+u-v} - \frac{\cos\left(\frac{1+u-v}{\sqrt{3}}x\right)}{1+u-v} - \frac{\cos\left(\frac{-1+u+v}{\sqrt{3}}x\right)}{-1+u+v} + \frac{\cos\left(\frac{1+u+v}{\sqrt{3}}x\right)}{1+u+v} \right) - \frac{3(-1+3(u^2-v^2)^2-2(u^2+v^2))}{2uv((u+v)^2-1)((u-v)^2-1)}$$

$$\mathcal{Y}_0(u,v,x) = -\frac{3(-1+3(u^2-v^2)^2-2(u^2+v^2))}{16u^2v^2} \times \left(\frac{\sin\left(\frac{-1+u-v}{\sqrt{3}}x\right)}{-1+u-v} + \frac{\sin\left(\frac{1+u-v}{\sqrt{3}}x\right)}{1+u-v} - \frac{\sin\left(\frac{-1+u+v}{\sqrt{3}}x\right)}{-1+u+v} - \frac{\sin\left(\frac{1+u+v}{\sqrt{3}}x\right)}{1+u+v} \right)$$

$$\mathcal{J}_\mathrm{h}(u,v,x\to\infty) = \mathcal{J}_\mathrm{h,late}(u,v), \;\; \mathcal{Y}_\mathrm{h}(u,v,x\to\infty) = \mathcal{Y}_\mathrm{h,late}(u,v)$$

$$\mathcal{J}_\mathrm{h,late}(u,v) = \frac{1}{8u^3v^3}\left[-9(u^6+u^4-u^2)-9(v^6+v^4-v^2)+9+6u^2v^2(u^2-v^2)^2+5u^2v^2(u^2+v^2)+8u^2v^2\right] - \frac{3}{32u^4v^4}\left(3(u^4-v^4)^2-6(u^4+v^4)+3-8u^2v^2(1+u^2+v^2)\right)\log\left(\frac{1-(u-v)^2}{(u+v)^2-1}\right)$$

$$\mathcal{Y}_\mathrm{h,late}(u,v) = \frac{3\pi}{32u^4v^4}\left(3u^4(u^4-2)+3v^4(v^4-2)+3-8u^2v^2(1+u^2+v^2)-6u^4v^4\right)$$

Initial condition

$$\Psi^{(2)}(\boldsymbol{k},\eta) = \Psi^{(2)}(\boldsymbol{k},0)T_{\Phi,\mathrm{r}}(x) + \int \frac{\mathrm{d}^3\tilde{k}}{(2\pi)^3} uv I_{\Psi,\mathrm{r,s}}(u,v,x) \left(\frac{2}{3}\right)^2 \zeta(\tilde{\boldsymbol{k}})\zeta(\boldsymbol{k}-\tilde{\boldsymbol{k}})$$
$$= \int \frac{\mathrm{d}^3\tilde{k}}{(2\pi)^3} uv (I_{\Psi,\mathrm{r,i}}(u,v,x) + I_{\Psi,\mathrm{r,s}}(u,v,x)) \left(\frac{2}{3}\right)^2 \zeta(\tilde{\boldsymbol{k}})\zeta(\boldsymbol{k}-\tilde{\boldsymbol{k}})$$

The initial condition of $\Psi^{(2)}$ is determined by $\zeta^{(2)}$.

Here, as a fiducial example, we impose

$$\zeta^{(2)} = 2a_{\mathrm{NL}}(\zeta^{(1)})^2$$

Then, we obtain

$$I_{\Psi,\mathrm{r,i}}(u,v,x) \equiv \mathcal{J}_{\mathrm{i}}(u,v)T_{\Phi,\mathrm{r}}(x)$$
$$\mathcal{J}_{\mathrm{i}}(u,v) \equiv \frac{2(u^2+v^2) - 3(u^2-v^2)^2 + 5 - 12a_{\mathrm{NL}}}{4uv}$$

Finally, we obtain the analytic solution of $\Psi^{(2)}$

$$\Psi^{(2)}(\boldsymbol{k},\eta) = \int \frac{\mathrm{d}^3\tilde{k}}{(2\pi)^3} uv I_{\Psi,\mathrm{r}}(u,v,x) \left(\frac{2}{3}\right)^2 \zeta^{(1)}(\tilde{\boldsymbol{k}})\zeta^{(1)}(\boldsymbol{k}-\tilde{\boldsymbol{k}})$$

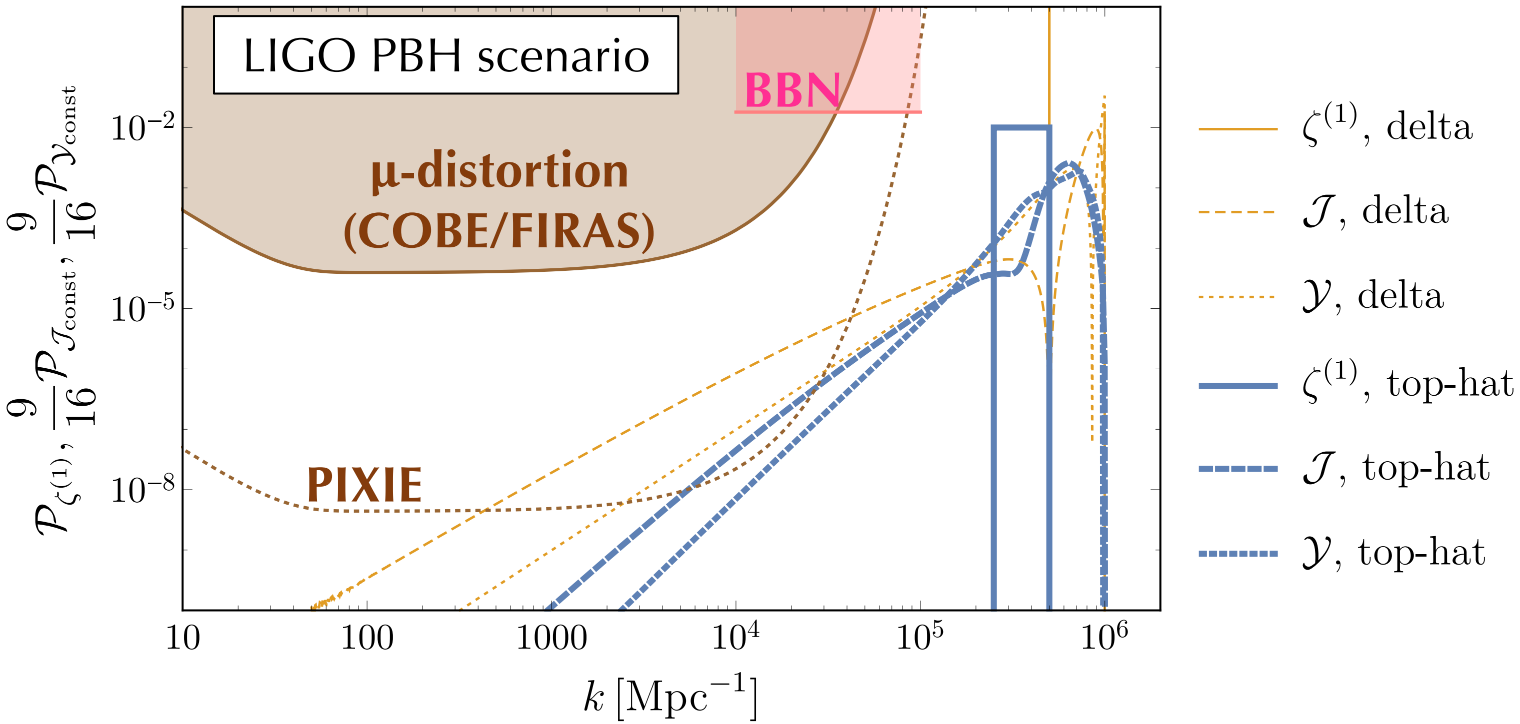
$(I_{\Psi,\mathrm{r}} \equiv I_{\Psi,\mathrm{r,i}} + I_{\Psi,\mathrm{r,s}})$

Power spectra

For simplicity, we focus on the terms that converge in the late-time limit.

$$\mathcal{P}_{\mathcal{J}_{\rm const}}(k) \equiv \int_0^\infty \mathrm{d}v \int_{|v-1|}^{v+1} \mathrm{d}u \; \left(\mathcal{J}_{\rm h,late}(u,v) + \mathcal{J}_{\rm i}(u,v)\right)^2 \left(\frac{2}{3}\right)^4 \mathcal{P}_{\zeta^{(1)}}(kv)\mathcal{P}_{\zeta^{(1)}}(ku)$$

$$\mathcal{P}_{\mathcal{Y}_{\rm const}}(k) \equiv \int_0^\infty \mathrm{d}v \int_{|v-1|}^{v+1} \mathrm{d}u \, \mathcal{Y}^2_{\rm h,late}(u,v) \left(\frac{2}{3}\right)^4 \mathcal{P}_{\zeta^{(1)}}(kv)\mathcal{P}_{\zeta^{(1)}}(ku).$$



Note: The spectra can be modified if we take into account the neglected terms.

Outline of this talk

- Introduction
- Second-order perturbations in a RD era
- Summary

Summary

We have derived analytic solutions of the second-order scalar perturbations in a radiation-dominated (RD).

$$\Psi^{(2)}(\boldsymbol{k},\eta) = \int \frac{\mathrm{d}^3\tilde{k}}{(2\pi)^3} uv \underline{I_{\Psi,\mathrm{r}}(u,v,x)} \left(\frac{2}{3}\right)^2 \zeta^{(1)}(\tilde{\boldsymbol{k}})\zeta^{(1)}(\boldsymbol{k}-\tilde{\boldsymbol{k}})$$

$$(v \equiv \tilde{k}/k,\ u \equiv |\boldsymbol{k}-\tilde{\boldsymbol{k}}|/k,\ x \equiv k\eta)$$

We have derived the analytic expression of this transfer function.

Future work:

1. Effects of the diffusion damping
2. Effects of the transition between a MD era and a RD era
3. Cross-correlation between the first-order and the third-order perturbations.