Perturbations of Cosmological and Black hole solutions in Bi-Gravity

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(In progress)

RESCEU Summer School 2015 @Kinugawa Green palace, 3rd August 2015

What is bi-gravity?

What is bi-gravity?

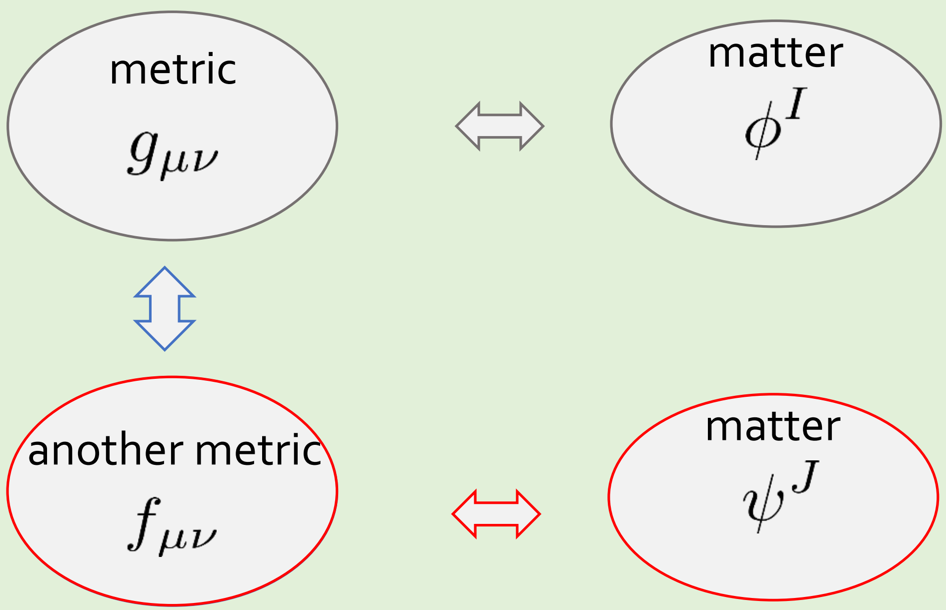
One of the theory of ***modified gravity***

$$S_{EH}[g] + S_{matter}[g, \phi^I]$$
$$+S_{int}[g, f] \quad \text{non-derivative interaction}$$
$$+S_{EH}[f] + S_{matter}[f, \psi^J]$$

General Relativity

Massive gravity — de Rham, Gabadadze, Tolley (2011)

bi-gravity — Hassan, Rosen (2011)



~~External field~~ → dynamical field

Cosmology in bi-gravity

- Known result about cosmology Comelli et al (2011) Comelli et al (2014) De Felice et al (2014)

  Starting from the simple bi-diagonal assumption

$$
\begin{aligned}
g_{\mu\nu}dx^\mu dx^\nu &= -N^2(t)dt^2 + a^2(t)\delta_{ij}dx^i dx^j \\
f_{\mu\nu}dx^\mu dx^\nu &= -n^2(t)dt^2 + b^2(t)\delta_{ij}dx^i dx^j
\end{aligned}
$$

  the self accelerated cosmological solutions can be obtained,

  but **these solutions suffer from ghost or gradient instability**.

- Does bi-gravity rejected from cosmology?

No!

There are many cosmological solutions in bi-gravity.
Other solution may express the real universe.

This is a motivation of our work!

Contents

Action and Equation of Motion

Action and Equation of Motion

- Action

$$S_{EH}[g] + S_{matter}[g, \phi^I] + S_{int}[g, f] + S_{EH}[f] + S_{matter}[f, \psi^J]$$

By ghost free requirement, S_{int} is limited following form:

$$S_{int}[g,f] = \frac{M_{PL}^2}{2}\int d^4x\sqrt{-g}$$
$$\times 2\Big[\beta_0 + \beta_1 \gamma^\mu{}_\mu + \frac{1}{2}\beta_2({\gamma^\mu{}_\mu}^2 - \gamma^\mu{}_\nu\gamma^\nu{}_\mu)$$
$$+\frac{1}{6}\beta_3({\gamma^\mu{}_\mu}^3 - 3\gamma^\mu{}_\mu\gamma^\nu{}_\rho\gamma^\rho{}_\nu + 2\gamma^\mu{}_\nu\gamma^\nu{}_\rho\gamma$$

de Rham, Gabadadze, Tolley (2011)

$$\begin{aligned}\beta_0 &= -\Lambda^g + m^2(6 + 4\alpha_3 + \alpha_3)\\ \beta_1 &= m^2(-3 - 3\alpha_3 - \alpha_4)\\ \beta_2 &= m^2(1 + 2\alpha_3 + \alpha_4)\\ \beta_3 &= m^2(-\alpha_3 - \alpha_4)\\ \beta_4 &= -\Lambda^f + m^2\alpha_4\end{aligned}$$

with $\gamma^\mu{}_\nu = \sqrt{g^{-1}f}^\mu{}_\nu$ $\quad (\gamma^\mu{}_\rho\gamma^\rho{}_\nu := g^{\mu\rho}f_{\rho\nu})$

Key Point

- Bi-gravity has 5 free parameters $\beta_0, \beta_1, \beta_2, \beta_3, \beta_4 \Longleftrightarrow m, \alpha_3, \alpha_4, \Lambda^g, \Lambda^f$
- Interaction term is described by $\gamma^\mu{}_\nu = \sqrt{g^{-1}f}^\mu{}_\nu$

Action and Equation of Motion

- Action

$$S_{EH}[g] + S_{matter}[g, \phi^I] + S_{int}[g, f] + S_{EH}[f] + S_{matter}[f, \psi^J]$$

- Equation of motion

for $g_{\mu\nu}$: $$G^\mu{}_\nu + X[\gamma]^\mu{}_\nu = 8\pi G T^\mu{}_\nu$$

for $f_{\mu\nu}$: $$\mathcal{G}^\mu{}_\nu + \mathcal{X}[\gamma]^\mu{}_\nu = 8\pi \tilde{G} \mathcal{T}^\mu{}_\nu$$

Key Point

- Bi-gravity has 5 free parameters $\beta_0, \beta_1, \beta_2, \beta_3, \beta_4 \Longleftrightarrow m, \alpha_3, \alpha_4, \Lambda^g, \Lambda^f$
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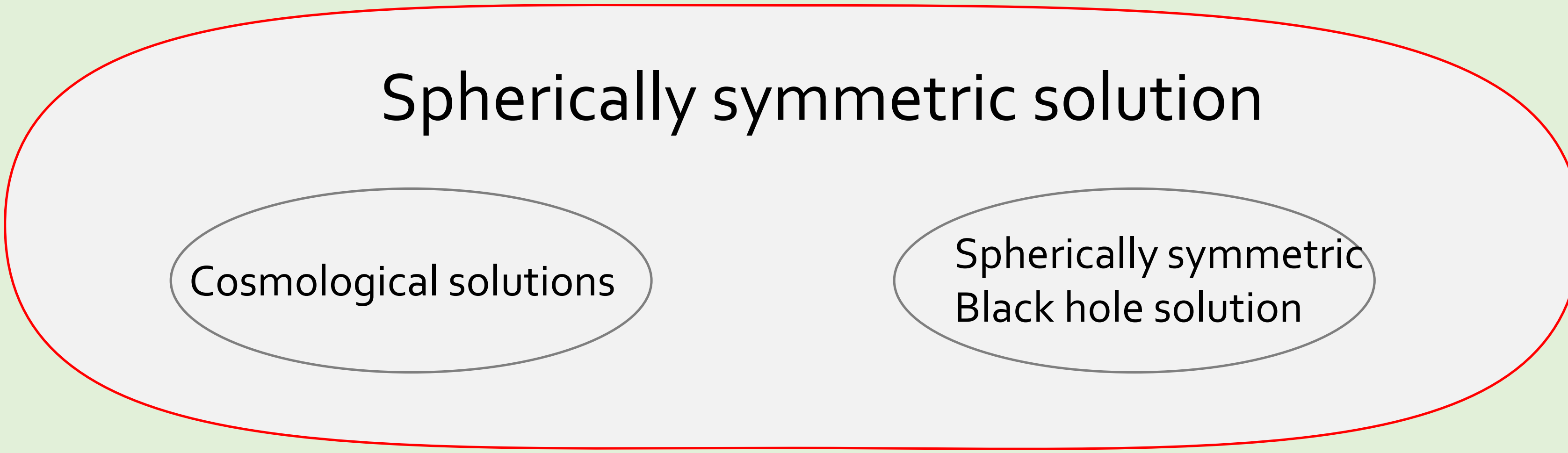
Back Ground Solutions

Metric Ansatz

- Ansatz of $g_{\mu\nu}$



$$g_{\mu\nu}dx^{\mu}dx^{\nu} \quad = \quad g_{tt}(t,r)dt^2 + 2g_{tr}(t,r)dtdr + g_{rr}(t,r)dr^2 + R^2(t,r)(d\theta^2 + \sin^2\theta d\phi)$$

- Ansatz of $f_{\mu\nu}$

$$f_{\mu\nu}dx^{\mu}dx^{\nu} \quad = \quad f_{tt}(t,r)dt^2 + 2f_{tr}(t,r)dtdr + f_{rr}(t,r)dr^2 + A^2(t,r)R^2(t,r)(d\theta^2 + \sin^2\theta d\phi)$$

- The bi-diagonal solution corresponds to $g_{tr} = f_{tr} = 0$

Cosmological Constant Solution

The prediction of GR with c.c. coincide with observation at least back ground level.
Then, we look for the case that the effect of bi-gravity behave a cosmological constant.

impose $X^{\mu}{}_{\nu} = \Lambda^{g}_{eff} \delta^{\mu}{}_{\nu}$

remainder
EoM: $G^{\mu}_{\nu} + X^{\mu}_{\nu} = 8\pi G\, T^{\mu}_{\nu}$
$m, \alpha_3, \alpha_4, \Lambda^g, \Lambda^f$: parameters
$A^2(t,r) := f_{\theta\theta}/g_{\theta\theta}$
$\gamma^{\mu}_{\nu} = \sqrt{g^{-1}f}^{\mu}_{\nu}$

$$\begin{aligned} X^{t}{}_{r} &= -m^2\gamma^{t}{}_{r}(3-2A+(A-3)(A-1)\alpha_3+(A-1)^2\alpha_4)=0 \\ X^{r}{}_{t} &= -m^2\gamma^{r}{}_{t}(3-2A+(A-3)(A-1)\alpha_3+(A-1)^2\alpha_4)=0 \end{aligned}$$

$$3-2A+(A-3)(A-1)\alpha_3+(A-1)^2\alpha_4=0$$

(A is constant)

or bi-diagonal case $\gamma^{t}{}_{r} = \gamma^{r}{}_{t} = 0$

$$X^{t}{}_{t} - X^{\theta}{}_{\theta} = m^2 \frac{A^2 - A(\gamma^{t}{}_{t}+\gamma^{r}{}_{r}) + \gamma^{t}{}_{t}\gamma^{r}{}_{r} - \gamma^{t}{}_{r}\gamma^{r}{}_{t}}{1-A}(A-2+(A-1)\alpha_3) = 0$$

$$A-2+(A-1)\alpha_3=0$$

$$\begin{aligned} A &= \frac{2+\alpha_3}{1+\alpha_3} \\ \alpha_4 &= 1+\alpha_3+\alpha_3^2 \end{aligned}$$

$$X^{\mu}{}_{\nu} = (m^2(A-1)+\Lambda^g)\delta^{\mu}{}_{\nu}$$

$$\mathcal{X}^{\mu}{}_{\nu} = \frac{G}{\tilde{G}}\left(m^2\frac{1-A}{A}+\Lambda^f\right)\delta^{\mu}_{\nu}$$

Summary of the Background Solution

Choice of the parameter of theory

$$\alpha_4 = 1 + \alpha_3 + \alpha_3^2$$

Metric

$$g_{\mu\nu} = g_{tt}(t,r)dt^2 + 2g_{tr}(t,r)dtdr + g_{rr}(t,r)dr^2 + R^2(t,r)(d\theta^2 + \sin^2\theta d\phi)$$
$$f_{\mu\nu} = f_{tt}(t,r)dt^2 + 2f_{tr}(t,r)dtdr + f_{rr}(t,r)dr^2 + \underline{A^2R^2(t,r)}(d\theta^2 + \sin^2\theta d\phi)$$

$$A = \frac{2+\alpha_3}{1+\alpha_3}$$

Equation of Motion

$$G^\mu{}_\nu + \Lambda^g_{eff}\delta^\mu{}_\nu = 8\pi G T^\mu{}_\nu \qquad (= X^\mu{}_\nu)$$

$$\mathcal{G}^\mu{}_\nu + \Lambda^f_{eff}\delta^\mu{}_\nu = 8\pi\tilde{G}\mathcal{T}^\mu{}_\nu$$

$$\Lambda^g_{eff} = m^2(A-1) + \Lambda^g$$

$$\Lambda^f_{eff} = \frac{G}{\tilde{G}}(m^2(1-A)/A + \Lambda^f)$$

Any spherically symmetric solution of GR can be a solution of bi-gravity !!

Perturbation

Linear Perturbation

- To investigate the stability of the solution, we consider the linear perturbation

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$$
$$f_{\mu\nu} = \bar{f}_{\mu\nu} + \delta f_{\mu\nu}$$

remainder

$$\bar{g}_{\mu\nu} = g_{tt}(t,r)dt^2 + 2g_{tr}(t,r)dtdr + g_{rr}(t,r)dr^2 + R^2(t,r)(d\theta^2 + \sin^2\theta d\phi)$$
$$\bar{f}_{\mu\nu} = f_{tt}(t,r)dt^2 + 2f_{tr}(t,r)dtdr + f_{rr}(t,r)dr^2 + A^2R^2(t,r)(d\theta^2 + \sin^2\theta d\phi)$$

↓

- Equation of motion can be obtained as

$$\delta G^\mu{}_\nu + \delta X^\mu{}_\nu = 8\pi G \delta T^\mu{}_\nu$$

Einstein equation

$$\delta X^\mu{}_\nu = m^2 \frac{A^2 - A(\gamma^t{}_t + \gamma^r{}_r) + \gamma^t{}_t\gamma^r{}_r - \gamma^t{}_r\gamma^r{}_t}{(1-A)^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \delta\gamma^3{}_3 & -\delta\gamma^2{}_3 \\ 0 & 0 & -\delta\gamma^3{}_2 & \delta\gamma^2{}_2 \end{pmatrix}$$

$$\delta\gamma^I{}_J = \sqrt{g^{-1}f}^I{}_J - \sqrt{\bar{g}^{-1}\bar{f}}^I{}_J = \frac{1}{2A}\bar{g}^{IK}(-A^2\delta g_{KJ} + \delta f_{KJ}) \qquad I, J = \theta, \phi$$

Solution of Conservation Equation

- Key observation:
  $X^{\mu}{}_{\nu}$ must satisfy the conservation equation, $\nabla_{\mu} X^{\mu}{}_{\nu} = 0$, if matter is conserved.
  ($\nabla_{\mu} X^{\mu}{}_{\nu} = 8\pi G \nabla_{\mu} T^{\mu}_{\nu} - \nabla_{\mu} G^{\mu}_{\nu}$)

$$\bar{\nabla}_{\mu} \delta X^{\mu}{}_{\nu} = 0$$

$$\delta X^{\mu}{}_{\nu} = m^2 \frac{A^2 - A(\gamma^t{}_t + \gamma^r{}_r) + \gamma^t{}_t \gamma^r{}_r - \gamma^t{}_r \gamma^r{}_t}{(1-A)^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \delta\gamma^3{}_3 & -\delta\gamma^2{}_3 \\ 0 & 0 & -\delta\gamma^3{}_2 & \delta\gamma^2{}_2 \end{pmatrix}$$

$$\delta\gamma^2{}_2 + \delta\gamma^3{}_3 = 0$$

$$\partial_\theta (\sin\theta \delta\gamma^2{}_3) = -\partial_\phi (\sin\theta \delta\gamma^2{}_2)$$

$$\frac{1}{\sin\theta} \partial_\theta (\sin\theta \partial_\theta (\sin^2\theta \delta\gamma^2{}_2)) + \frac{1}{\sin^2\theta} \partial_\phi \partial_\phi (\sin^2\theta \delta\gamma^2{}_2) = 0$$

Laplace equation on sphere, then solution is $\sin^2\theta \delta\gamma^2{}_2 = f(t,r)$

⇒ If $\delta\gamma^2{}_2$ is non-singular at $\theta = 0, \pi$, the solution is $\delta\gamma^2{}_2 = 0$

⇒ From remaining equation, $\delta\gamma^3{}_3 = \delta\gamma^2{}_3 = 0$

Solution of Conservation Equation

- Key observation:
  $X^\mu{}_\nu$ must satisfy the conservation equation, $\nabla_\mu X^\mu{}_\nu = 0$, if matter is conserved.

  $\nabla_\mu X^\mu{}_\nu = 8\pi G \nabla_\mu T^\mu_\nu - \nabla_\mu G^\mu_\nu$

$$\bar{\nabla}_\mu \delta X^\mu{}_\nu = 0$$

$$\delta X^\mu{}_\nu = m^2 \frac{A^2 - A(\gamma^t{}_t + \gamma^r{}_r) + \gamma^t{}_t\gamma^r{}_r - \gamma^t{}_r\gamma^r{}_t}{(1-A)^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \delta\gamma^3{}_3 & -\delta\gamma^2{}_3 \\ 0 & 0 & -\delta\gamma^3{}_2 & \delta\gamma^2{}_2 \end{pmatrix}$$

$$\delta\gamma^I{}_J = \frac{1}{2A}\bar{g}^{IK}(-A^2\delta g_{KJ} + \delta f_{KJ})$$

⇩

$$\delta\gamma^I{}_J = 0 \iff \delta X^\mu{}_\nu = 0 \iff \delta f_{IJ} = A^2 \delta g_{IJ} \qquad I,J = \theta,\phi$$

- The equation of motion reduces to

Perturbed Einstein equation for $\delta g_{\mu\nu}$: $\delta G^\mu{}_\nu = 8\pi G \delta T^\mu{}_\nu$

3 Constraints between perturbations $\delta g_{IJ}, \delta f_{IJ}$: $\delta f_{IJ} = A^2 \delta g_{IJ}$

(Perturbed Einstein equation for $\delta f_{\mu\nu}$: $\delta \mathcal{G}^\mu{}_\nu = 8\pi \tilde{G} \delta \mathcal{T}^\mu{}_\nu$)

Dynamics of metric perturbations is completely same as GR !!

There is no unphysical instability peculiar to bi-gravity.

Quadratic Perturbations

The same result is true for quadratic perturbations

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} + g^{(2)}_{\mu\nu}$$
$$f_{\mu\nu} = \bar{f}_{\mu\nu} + \delta f_{\mu\nu} + f^{(2)}_{\mu\nu}$$

With the solutions of linear order $\delta f_{IJ} = A^2 \delta g_{IJ}$

$$\delta X^{(2)\mu}{}_{\nu} \propto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \delta X^{(2)2}{}_2 & \delta X^{(2)2}{}_3 \\ 0 & 0 & \delta X^{(2)3}{}_2 & \delta X^{(2)3}_3 \end{pmatrix}$$

$$\nabla_\mu X^\mu{}_\nu = 0 \quad \Longrightarrow \quad \delta X^{(2)\mu}{}_\nu = 0$$

The dynamics of quadratic perturbations is also same as GR !!

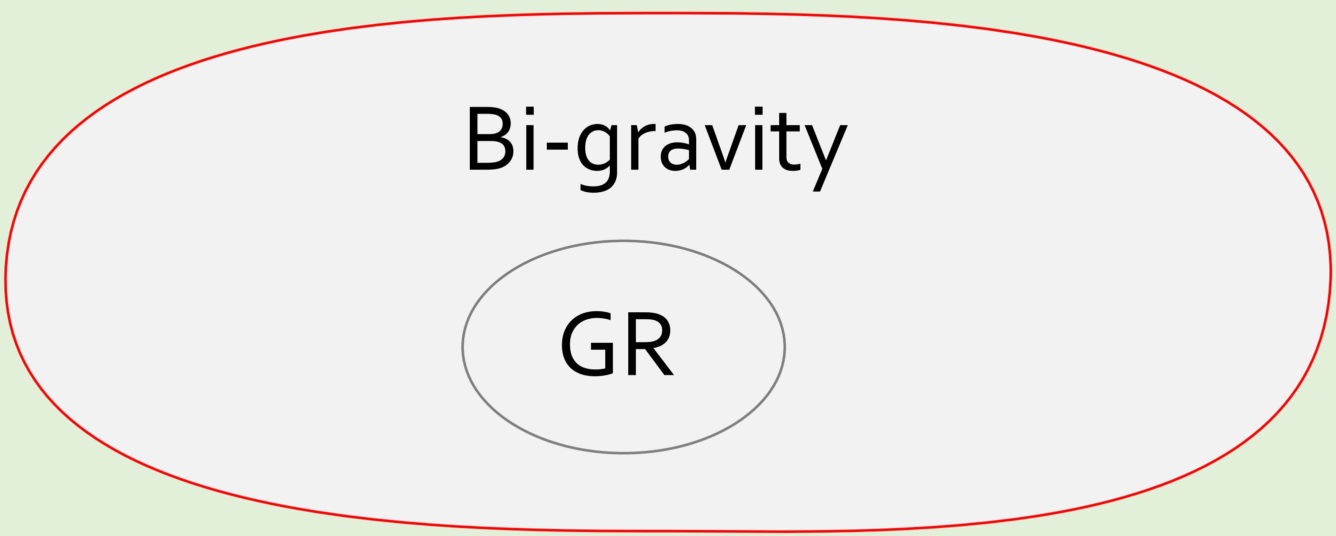
Summary

We showed that

1. Any spherically symmetric solution of GR can be a solution of bi-gravity.

2. Linear and quadratic perturbation of this solution is obeyed by Einstein equation.
   There is no instability peculiar to bi-gravity.

Bi-gravity can reproduce the prediction of GR,



about spherically symmetric space time.