Equation of state of dark energy in $f(R)$ gravity

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Motivation

- Many modified theories of gravity have been considered

- $f(R)$ gravity $\cdots$ one of the simplest generalizations of GR
  - There are some $f(R)$ models which are viable on both cosmological and local scales

- EoS of dark energy: $P_{\mathrm{DE}} = w\rho_{\mathrm{DE}}$
  - $w = -1$ in ΛCDM model
  - $w \neq -1$ in $f(R)$ theories

  → Important for distinguishing models.

- Observational constraint (Kowalski et al. (2008)) SN+BAO+CMB

$$|1 + w_{z<0.5}| < 0.1$$

- "Fifth force" must be small (Brax et al. (2008)) $\cdots$ local constraint

$$|(1 + w)\Omega_{\mathrm{DE}}| < 10^{-4}$$ ⟸ Extremely small!

→ We argue that this is incorrect

$f(R)$ gravity

- Action

$$S = \frac{M_{\rm Pl}^2}{2}\int d^4x\sqrt{-g}f(R) + \underline{S_{\rm m}(g_{\mu\nu},\Psi)}$$

matter field

- Einstein frame

$$g_{\mu\nu} \to \bar{g}_{\mu\nu} = f'(R)g_{\mu\nu}$$

$$f'(R) \equiv F(R) \equiv e^{-2\beta\phi/M_{\rm Pl}}, \quad \beta \equiv \frac{1}{\sqrt{6}}$$

$$S = \int d^4x\sqrt{-\bar{g}}\left(\frac{M_{\rm Pl}^2}{2}\bar{R} - \frac{1}{2}(\bar{\nabla}\phi)^2 - V(\phi)\right) + S_{\rm m}(e^{2\beta\phi/M_{\rm Pl}}\bar{g}_{\mu\nu},\Psi)$$

GR

scalar field

ϕ couples to matter

→ fifth force

$$\left(= \frac{\beta}{M_{\rm Pl}}\vec{\nabla}\phi\right)$$

$$V(\phi) = \frac{1}{16\pi G}\frac{RF - f}{F^2}$$

EoM in the Einstein frame

- Einstein field equation

$$\bar{G}_{\mu\nu} = 8\pi G \left(\bar{\nabla}_\mu \phi \bar{\nabla}_\nu \phi - \bar{g}_{\mu\nu} \left[\frac{1}{2} (\bar{\nabla}\phi)^2 + V(\phi) \right] + \bar{T}^{\mathrm{m}}_{\mu\nu} \right)$$

- Klein-Gordon equation

$$\bar{\Box}\phi = V'(\phi) + \frac{\beta}{M_{\mathrm{Pl}}} \tilde{\rho}_{\mathrm{m}} e^{\beta\phi/M_{\mathrm{Pl}}}$$

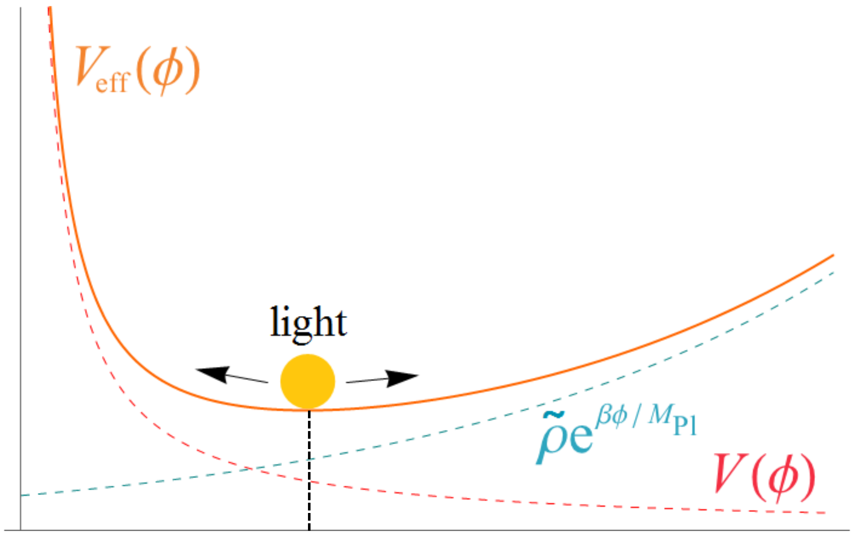
- The dynamics of ϕ are governed by an effective potential

$$\boxed{V_{\mathrm{eff}}(\phi) \equiv V(\phi) + \tilde{\rho}_{\mathrm{m}} e^{\beta\phi/M_{\mathrm{Pl}}}}$$

→ Depends on local matter densities

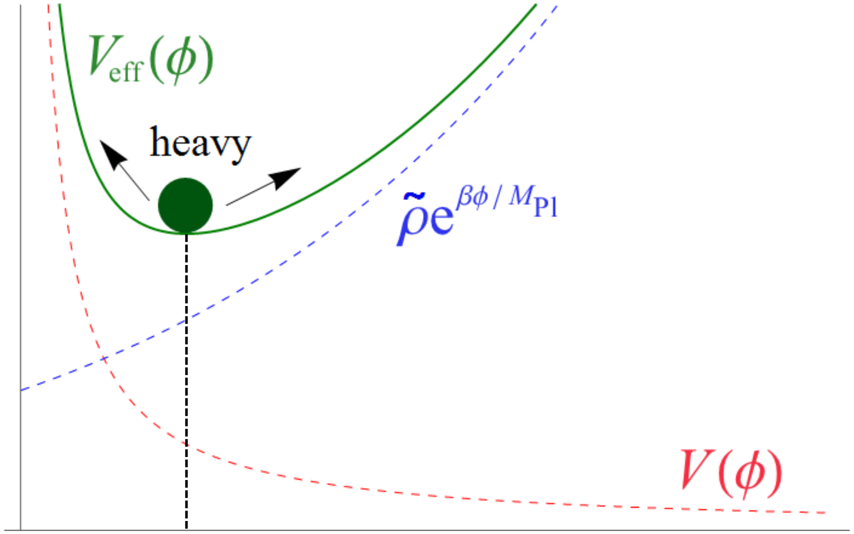
Chameleon mechanism

small ρ



large $\phi_{\rm min}$

large ρ

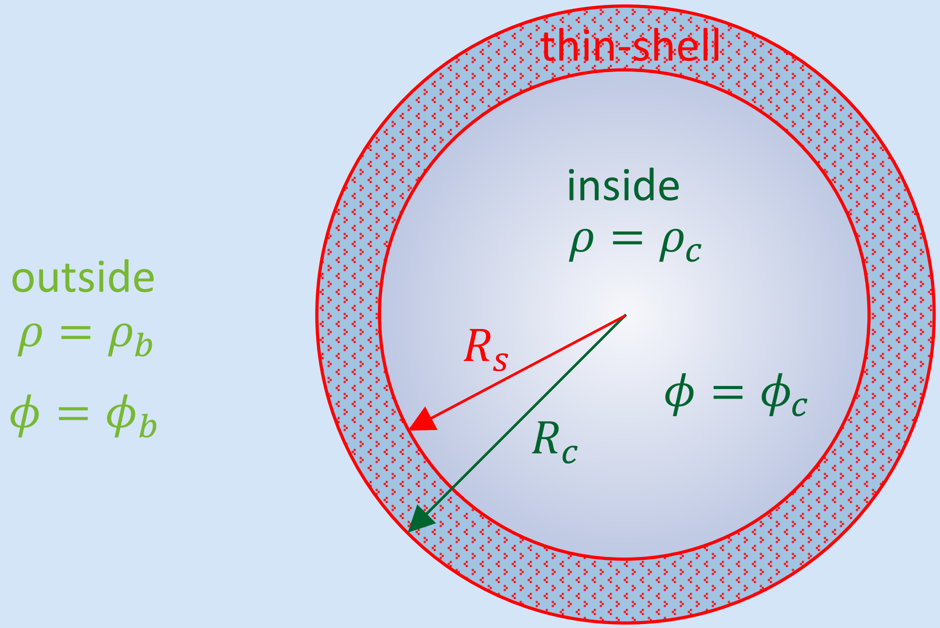


small $\phi_{\rm min}$

$$\boxed{\begin{gathered}\rho_1 < \rho_2 \\ \phi_1^{\rm min} > \phi_2^{\rm min} \\ m_1 < m_2\end{gathered}}$$

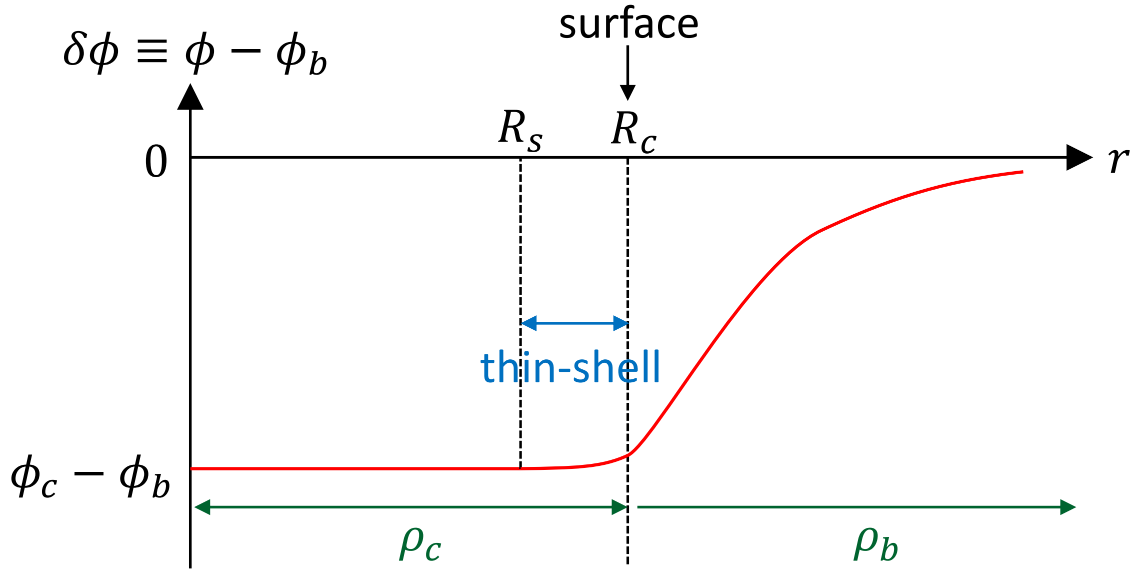
➢ Thin-shell solution

- Thin-shell solution ⋯ scalar field configuration around a uniform spherical object
- It played a crucial role in the previous work



Thin-shell solution

- General form of the thin-shell solution



- Thin-shell parameter

$$1 > \epsilon_{\mathrm{th}} = \frac{R_c - R_s}{R_c} \approx \frac{(\text{fifth force})}{(\text{Newtonian force})}$$

- It is a solution of the Poisson equation (static assumption):

$$\nabla^2\phi = V'_{\mathrm{eff}}(\phi)$$

Thin-shell solution

■ Functional form

$$\delta\phi = \begin{cases} \delta\phi_c & , r < R_s \\ \dfrac{\beta\rho_c}{3M_{\rm Pl}}\left(\dfrac{r^2}{2} + \dfrac{R_s^3}{r} - \dfrac{3}{2}R_s^2\right) + \delta\phi_c & , R_s < r < R_c \\ -\dfrac{\beta\rho_c}{3M_{\rm Pl}}\epsilon_{\rm th}\dfrac{R_c^3}{r}e^{-m_b(r-R_c)} & , r > R_c \end{cases}$$

$$\delta\phi \equiv \phi - \phi_b$$

$$\boxed{\epsilon_{\rm th} \equiv \frac{M_{\rm Pl}}{\beta}\frac{|\delta\phi_c|}{R_c^2\rho_c} \approx \frac{R_c - R_s}{R_c}}$$

EoS of dark energy

- Einstein eqs. (background)

$$G_{\mathrm{eff},0} \equiv \frac{G}{F_0}$$

$$H^2 = \frac{8\pi G}{3}\left(\frac{\rho_{\mathrm{m}}}{F} + FV(\phi_b)\right) + \frac{2\beta}{M_{\mathrm{Pl}}} H\dot{\phi}_b \equiv \frac{8\pi G_{\mathrm{eff},0}}{3}(\rho_{\mathrm{m}} + \rho_{\mathrm{DE}})$$

$$\frac{2\ddot{a}}{a} + H^2 = -8\pi G V(\phi) + \frac{2}{3M_{\mathrm{Pl}}^2}\dot{\phi}_b^2 - \frac{2\beta}{M_{\mathrm{Pl}}}\left(\ddot{\phi} + 2H\dot{\phi}_b\right) \equiv -8\pi G_{\mathrm{eff},0} P_{\mathrm{DE}}$$

- (Effective) EoS of Dark Energy

$$\frac{P_{\mathrm{DE}}}{\rho_{\mathrm{DE}}} = w$$

$$\boxed{(1+w)\Omega_{\mathrm{DE}} = \frac{\rho_{\mathrm{DE}} + P_{\mathrm{DE}}}{\rho_{\mathrm{cr}}} = \frac{2\beta}{3M_{\mathrm{Pl}}}\left(\frac{\dot{\phi}_b}{H} - \frac{\ddot{\phi}_b}{H^2}\right) + \frac{2}{9M_{\mathrm{Pl}}^2}\frac{\dot{\phi}_b^2}{H^2} + \left(\frac{F_0}{F} - 1\right)\Omega_{\mathrm{m}}}$$

- Since $\dot{\phi}_b \sim H\Delta\phi$, we get

$\Delta\phi$: variation of ϕ_b in the last Hubble time

$$\boxed{|(1+w)\Omega_{\mathrm{DE}}| \sim O\left(\frac{\beta}{M_{\mathrm{Pl}}}\Delta\phi\right)}$$

EoS of dark energy

$$|(1+w_{\mathrm{DE}})\Omega_{\mathrm{DE}}| \sim O\left(\frac{\beta}{M_{\mathrm{Pl}}}\Delta\phi\right)$$

- $\Delta\phi$: variation of ϕ_b from time t to t_0

t: past time at which $z \gtrsim 1$

- Relation between the density and the minimum of the effective potential

$$\underbrace{\rho_c}_{\substack{\text{some celestial object} \\ \text{e.g. galaxy cluster}}} > \rho_b(t) > \rho_b(t_0) \quad \Longrightarrow \quad \phi_c < \phi_b(t) < \phi_b(t_0)$$

$$\frac{\beta}{M_{\mathrm{Pl}}}\Delta\phi < \frac{\beta}{M_{\mathrm{Pl}}}|\delta\phi_c(t_0)| \qquad \delta\phi_c(t_0) \equiv \phi_c - \phi_b(t_0)$$

- Consider an object with thin shell

$$\frac{\beta}{M_{\mathrm{Pl}}}|\delta\phi_c(t_0)| < \Phi_N \quad \Longleftrightarrow \epsilon_{\mathrm{th}} < 1$$

$$|(1+w_{\mathrm{DE}})\Omega_{\mathrm{DE}}| < \Phi_N$$

➡ Models with large $|1+w|$ cannot have a thin shell??

What is wrong?

- In a cosmological situation, the exterior solution of the Poisson equation does not satisfy the original Klein-Gordon equation

$$-\delta\ddot{\phi} - 3H\delta\dot{\phi} + \frac{\nabla^2}{a^2}\delta\phi - m_b^2\delta\phi = 0$$

since

$$\frac{\nabla^2}{a^2}\delta\phi \sim O\left(m_b^2\delta\phi\right)$$

$$\delta\ddot{\phi}, H\delta\dot{\phi} \sim O(H^2\delta\phi)$$

same order for models with $m_b \sim O(H)$

- Note that the interior solution need not be changed since $m_c \gg H$.

- For example, Starobinsky's model

$$f(R) = R + \lambda R_s\left[\left(1 + \left(\frac{R}{R_s}\right)^2\right)^{-n} - 1\right]$$

has $m_b \gtrsim H$ for small n, λ.

  - Actually, it is in such models that w deviates appreciably from -1.
  - For $n = 2$ and $\lambda = 1$, $m_b/H \approx 3.1$ and $w_0 \approx -0.94$

Our work

- Give a counterexample for the previous work

- Assume the background spacetime evolves as in wCDM model with $w \neq -1$, and solve the scalar field equation around a spherical object.

- 2 steps:
  - Construct the solution in $w = -1$ (de Sitter) case.
  - Construct the solution in $w \neq -1$ case perturbatively, up to first order in $\epsilon \equiv 1 + w$.

- Conformal time η is used as time variable in order that $\epsilon \to 0$ limit is well-defined.

$$\eta \equiv \int^t \frac{dt'}{a(t')}$$

$$a(t) \propto \begin{cases} t^{\frac{2}{3\epsilon}} & ,\epsilon > 0 \\ \left(t_{\mathrm{rip}} - t\right)^{\frac{2}{3\epsilon}} & ,\epsilon < 0 \end{cases} \xrightarrow{t \to \eta} a(\eta) \propto (-\eta)^{-1-\frac{3}{2}\epsilon}$$

$$\Big\downarrow \; \epsilon \to 0$$

$$a_{\mathrm{dS}}(t) \propto e^{Ht} \xrightarrow{t \to \eta} a_{\mathrm{dS}}(\eta) \propto (-\eta)^{-1}$$

➢ $w = -1$ case

$$' \equiv \frac{\partial}{\partial \eta}$$

$$\mathcal{H} \equiv \frac{a'}{a} = aH$$

- Klein-Gordon equation outside the object

$$-\delta\phi''_{\mathrm{out}} - 2\mathcal{H}\delta\phi'_{\mathrm{out}} + \left(\nabla^2 - m_b^2 a^2\right)\delta\phi_{\mathrm{out}} = 0$$

- Find a solution which is smoothly connected to the interior solution

$$\delta\phi_{\mathrm{in}} = \begin{cases} \delta\phi_c & , ar < R_s \\ \dfrac{\beta\rho_c}{3M_{\mathrm{Pl}}}\left(\dfrac{(ar)^2}{2} + \dfrac{R_s^3}{ar} - \dfrac{3}{2}R_s^2\right) + \delta\phi_c & , R_s < ar < R_c \end{cases}$$

$r \to ar$ in the original Thin-shell solution

- If we make an ansatz

$$\delta\phi(\eta, r) = \varphi(Har),$$

using some one variable function $\varphi(u)$, the KG equation becomes an ODE:

$$\frac{d^2\varphi_{\mathrm{out}}(u)}{du^2} + \frac{4u^2 - 2}{u(u^2-1)}\frac{d\varphi_{\mathrm{out}}(u)}{du} + \left(\frac{m_b}{H}\right)^2 \frac{\varphi_{\mathrm{out}}(u)}{u^2 - 1} = 0$$

$$u \equiv Har$$

with the following boundary conditions:

- $\varphi_{\mathrm{out}} = \varphi_{\mathrm{in}}$ and $\partial_u\varphi_{\mathrm{out}} = \partial_u\varphi_{\mathrm{in}}$ at $u = HR_c$
- $\varphi_{\mathrm{out}} \to 0$ as $u \to \infty$

➢ $w = -1$ case

- The exterior solution is obtained as

$$\boxed{\delta\phi_{\text{out}} = -\frac{\beta\rho_c R_c^2}{M_{\text{Pl}}}\epsilon_{\text{th}} H R_c g_\alpha(Har)}$$

$$\boxed{\epsilon_{\text{th}} \equiv \frac{M_{\text{Pl}}}{\beta}\frac{|\delta\phi_c|}{R_c^2\rho_c} \approx \frac{R_c - R_s}{R_c}}$$

where

$$\boxed{\alpha \equiv \sqrt{\left(\frac{m_b}{H}\right)^2 - \frac{9}{4}}}$$

$$g_\alpha(u) \equiv \varphi_\alpha^{(2)}(u) - 2\frac{\Gamma\left(\frac{3+2i\alpha}{4}\right)\Gamma\left(\frac{3-2i\alpha}{4}\right)}{\Gamma\left(\frac{1+2i\alpha}{4}\right)\Gamma\left(\frac{1-2i\alpha}{4}\right)}\varphi_\alpha^{(1)}(u)$$

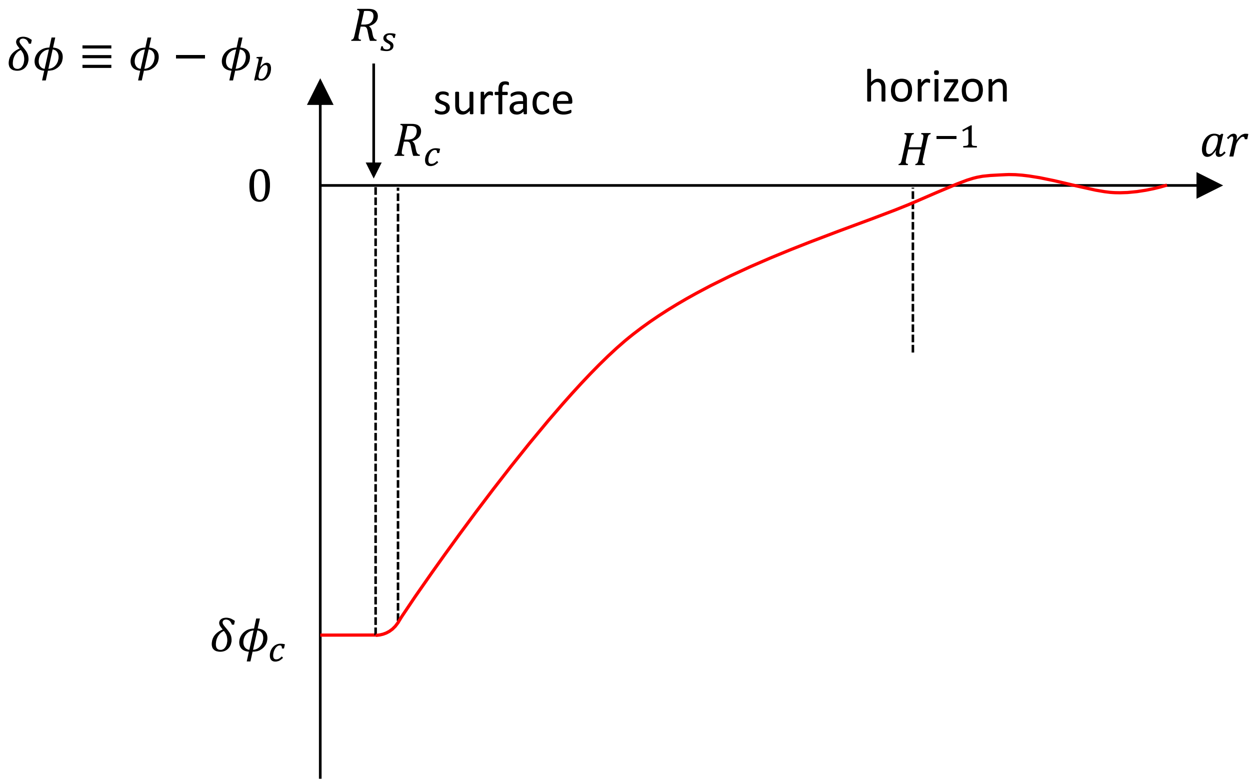
$$\begin{cases} \varphi_\alpha^{(1)}(u) = {}_2F_1\left(\frac{3+2i\alpha}{4}, \frac{3-2i\alpha}{4}; \frac{3}{2}; u^2\right), \\ \varphi_\alpha^{(2)}(u) = \frac{1}{u}\,{}_2F_1\left(\frac{1+2i\alpha}{4}, \frac{1-2i\alpha}{4}; \frac{1}{2}; u^2\right) \end{cases}$$

$\boxed{{}_2F_1\text{: hypergeometric function}}$

The constant before $\varphi_\alpha^{(1)}(u)$ is chosen so that $g_\alpha(u)$ does not diverge at $u = 1$.

➢ $w = -1$ case

- General form of the solution



- The ratio between the fifth force and the Newtonian force is of order ϵ_{th} as in an ordinary thin-shell solution.

$\rightarrow$ Small fifth force!

$w \neq -1$ case

- We choose the solution in $w = -1$ case

$$\delta\phi_{\text{out}} = -\frac{\beta\rho_c R_c^2}{M_{\text{Pl}}}\epsilon_{\text{th}} H R_c g_\alpha(\mathcal{H}r)$$

as a zeroth-order solution.

- Perturbative expansion with respect to $\epsilon = 1 + w$

$$\delta\phi_{\text{out}} = -\frac{\beta\rho_c R_c^2}{M_{\text{Pl}}}\epsilon_{\text{th}} H R_c [\underbrace{g_\alpha(\mathcal{H}r)}_{\gtrsim O(1)} + \epsilon \underbrace{A(\eta, r)}_{\text{should be } \lesssim O(1)}]$$

should be $\lesssim O(1)$
$\cdots$ checked later

- KG equation for the perturbative part

$$\epsilon\left[A'' + 2\mathcal{H}A' - \left(\nabla^2 - m_b^2 a^2\right)A\right]$$
$$= \frac{\epsilon}{\eta^2}\left[-\left(2C_\phi - 3\right)\mathcal{H}r g_\alpha'(\mathcal{H}r) - 2C_\phi g_\alpha(\mathcal{H}r)\right]$$

$$\boxed{\frac{\phi_b'}{\phi_b} \equiv \frac{C_\phi}{-\eta}\epsilon}$$

where C_ϕ is determined once a model is fixed.

➢ $w \neq -1$ case

- Again, assume a solution in the form of

$$A(\eta, r) \equiv B(u)$$

where

$$u \equiv \mathcal{H} r = H a r$$

The perturbed KG equation is rewritten as an ODE

$$\frac{d^2 B(u)}{du^2} + \frac{4u^2 - 2}{u(u^2 - 1)} \frac{dB(u)}{du} + \left(\frac{m_b}{H}\right)^2 \frac{B(u)}{u^2 - 1} = j(u)$$

$$j(u) \equiv -\frac{\left(2C_\phi - 3\right) u g_\alpha'(u) + 2C_\phi g_\alpha(u)}{u^2 - 1}$$

- The homogeneous solutions are already known ⋯ the solutions in $w = -1$ case

→ The inhomogeneous solutions can be obtained by the method of variation of parameters!

$w \neq -1$ case

- Basis of the homogeneous solutions

$$\begin{cases} B_1(u) \equiv \varphi_\alpha^{(1)}(u) \\ B_2(u) \equiv g_\alpha(u) \end{cases}$$

$$\boxed{W \equiv B_1 \frac{dB_2}{du} - B_2 \frac{dB_1}{du}}$$

- Inhomogeneous solution

$$B(u) = C_1 B_1(u) + C_2 B_2(u) - B_1(u) \int_0^u du' \frac{B_2}{W} j + B_2(u) \int_0^u du' \frac{B_1}{W} j$$

- We require that
  - B does not diverge at $u = 1$ ($ar = H^{-1}$)
  - $B = 0$ at $u = HR_c$ ($ar = R_c$)

$$\Rightarrow \boxed{B(p) = -B_1(u) \int_1^u du' \frac{B_2}{W} j + B_2(u) \left[\int_{HR_c}^u du' \frac{B_1}{W} j - \frac{B_1(HR_c)}{B_2(HR_c)} \int_{HR_c}^1 du' \frac{B_2}{W} j \right]}$$

$w \neq -1$ case

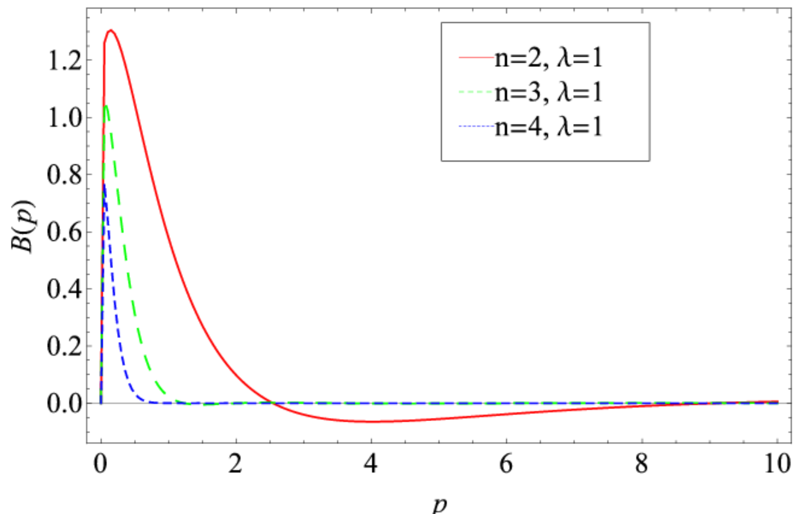
- The form of the solution (written again)

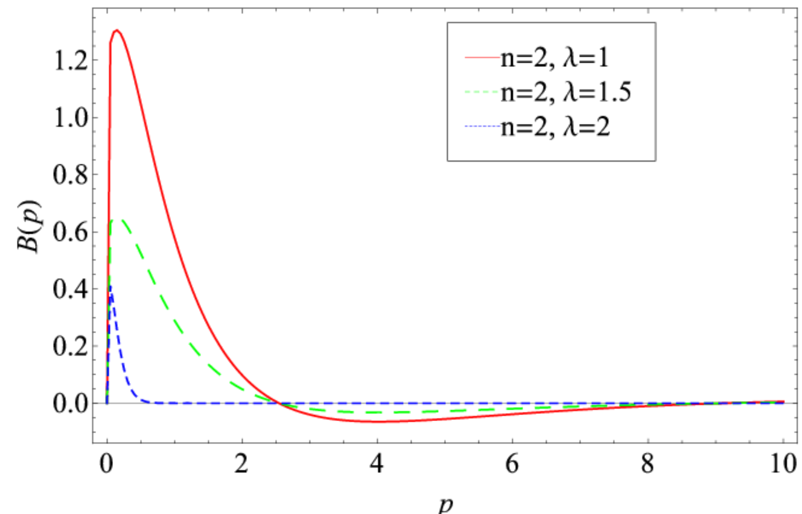
$$\delta\phi_{\mathrm{out}} = -\frac{\beta\rho_c R_c^2}{M_{\mathrm{Pl}}}\epsilon_{\mathrm{th}} H R_c[g_\alpha(\mathcal{H}r) + \epsilon A(\eta, r)]$$

$$A(\eta, r) \equiv B(u), \qquad u \equiv \mathcal{H}r = Har$$

$\epsilon \equiv 1 + w$

- Plots of $B(p)$ for various parameters of Starobinsky's model





$\lesssim O(1)$ for all parameters

- Here again the fifth force is small.

Summary

- The effective EoS parameter w deviates from -1 in $f(R)$ gravity.

- In the previous work, the thin-shell solution was naively used to a cosmological situation and it was concluded that w must be extremely close to -1. This is incorrect because the time derivative becomes important in a cosmological scale.

- We took time derivative into consideration and constructed a scalar configuration with small fifth force in the case where w deviates appreciably from -1.

$\rightarrow$ Models with $|1+w| \sim O(0.1)$ can not be excluded by the fifth-force constraint.