Multi-disformal invariance of nonlinear primordial perturbations

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Introduction & goals

- anti-correlation of TE spectrum -> super-H curvature perturbations
- (future) BB spectrum -> super-H gravitational-wave perturbations
- The primordial **linear** tensor power spectrum from inflation can be always cast into the standard form at leading order in derivatives with suitable conformal and **disformal transformations** in EFT of inflation. [Creminelli et al 1407.8439]
- Invariance of the curvature perturbation under **disformal transformations** has been shown at linear order. [Minamitsuji 1409.1566, Tsujikawa 1412.6210]

- We extend the invariance of the curvature and GW perturbations to **fully nonlinear order**.
- We further show the invariance under a new type of disformal transformation, dubbed **multi-disformal transformation**, generated by a multi-component scalar field.
Decomposing spacetime

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -\alpha^2 dt^2 + \hat{\gamma}_{ij} (dx^i + \beta^i dt)(dx^j + \beta^j dt) \]

\( \alpha \) is the lapse function, \( \beta^i \) is the shift vector

\[ \hat{\gamma}_{ij} = a^2(t) e^{2\psi} \gamma_{ij}, \quad \det \gamma_{ij} = 1 \]

\[ \chi \equiv -\frac{3}{4} \triangle^{-1} \left\{ \partial^i \left[ e^{-3\psi} \partial^j \left( e^{3\psi} (\gamma_{ij} - \delta_{ij}) \right) \right] \right\} \]

\( \triangle \) is the flat 3-dimensional Laplacian and \( \partial^i = \delta^{ij} \partial_j \)
Decomposing spacetime and curvature pert’n

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -\alpha^2 dt^2 + \hat{\gamma}_{ij} (dx^i + \beta^i dt) (dx^j + \beta^j dt) \]

\[ \alpha \text{ is the lapse function, } \beta^i \text{ is the shift vector} \]

\[ \hat{\gamma}_{ij} = a^2(t) e^{2\psi} \gamma_{ij}, \quad \det \gamma_{ij} = 1 \]

\[ \chi \equiv -\frac{3}{4} \Delta^{-1} \left\{ \partial^i \left[ e^{-3\psi} \partial^j \left( e^{3\psi} (\gamma_{ij} - \delta_{ij}) \right) \right] \right\} \]

\[ \Delta \text{ is the flat 3-dimensional Laplacian and } \partial^i = \delta^{ij} \partial_j \]

**The uniform } \phi \text{ slicing**

(comoving slicing if } \phi \text{ dominates):

\[ \phi = \phi(t) \]

\[ \mathcal{K}_c \equiv \psi_c + \chi_c / 3 \]

\( \mathcal{K}_c \) is the (comoving) curvature perturbation at linear order and \( \mathcal{K}_c \) its non-linear generalization
Decomposing spacetime and GW pert’n

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -\alpha^2 dt^2 + \hat{\gamma}_{ij} (dx^i + \beta^i dt) (dx^j + \beta^j dt) \]

\( \alpha \) is the lapse function, \( \beta^i \) is the shift vector

\[ \hat{\gamma}_{ij} = a^2(t) e^{2\psi} \gamma_{ij}, \quad det \gamma_{ij} = 1 \]

\[ \chi \equiv -\frac{3}{4} \Delta^{-1} \left\{ \partial^i \left[ e^{-3\psi} \partial^j \left( e^{3\psi} (\gamma_{ij} - \delta_{ij}) \right) \right] \right\} \]

\( \Delta \) is the flat 3-dimensional Laplacian and \( \partial^i = \delta^{ij} \partial_j \)

The uniform \( \phi \) slicing: \( \phi = \phi(t) \)

\[ \partial^j \gamma_{ij}^{TT} = 0 \]

\( \gamma_{ij}^{TT} \) is independent of the time-slicing condition at linear order but is slice-dependent at higher orders.
General disformal transformation

\[ \tilde{g}_{\mu\nu} = A(\phi, X) g_{\mu\nu} + B(\phi, X) \partial_\mu \phi \partial_\nu \phi, \quad X \equiv -g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi / 2 \]

A = 1, B ≠ 0: \textbf{Disformal transformation}
A ≠ 1, B = 0: \textbf{Conformal transformation}
Invariance of nonlinear perturbations

\[ \tilde{g}_{\mu\nu} = A(\phi, X) g_{\mu\nu} + B(\phi, X) \partial_\mu \phi \partial_\nu \phi, \quad X \equiv -g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi / 2 \]

A = 1, B ≠ 0: Disformal transformation

The uniform \( \phi \) slicing: \( \phi = \phi(t) \)

\[ \tilde{g}_{\mu\nu} = g_{\mu\nu} + B \dot{\phi}^2 \delta_{\mu}^0 \delta_{\nu}^0 \]

Only the lapse function is affected by the disformal transformation!

\[ \tilde{\alpha}^2 = \alpha^2 - B \dot{\phi}^2 \]

Thus, the spatial metric & shift vector are invariant to fully nonlinear order -> Invariance of curvature and GW perturbations
Invariance of nonlinear perturbations

\[ \tilde{g}_{\mu\nu} = A(\phi, X) g_{\mu\nu} + B(\phi, X) \partial_\mu \phi \partial_\nu \phi, \quad X \equiv -g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi / 2 \]

A ≠ 1, B = 0: Conformal transformation

The uniform \( \phi \) slicing: \( \phi = \phi(t) \)

\[ \tilde{g}_{\mu\nu} = A g_{\mu\nu} = \bar{A} g_{\mu\nu} + \delta A \bar{g}_{\mu\nu} + \delta A \delta g_{\mu\nu} \]

\( \mathcal{R} \rightarrow \mathcal{R} + \delta A / (2\bar{A}) + \cdots \) Unimodular part (GW) is invariant to fully nonlinear order

\( \delta A \) is sourced by \( \delta X \), but is vanishing on large scales b/c \( \dot{\mathcal{R}}_c \propto H \delta \alpha_c \)

\[ X_c = \frac{1}{2} \frac{\dot{\phi}^2(t)}{\alpha_c^2(t, x)} = \frac{1}{2} \dot{\phi}^2(t) - \delta \alpha_c(t, x) \dot{\phi}^2(t) + \cdots \]

\( \delta \alpha_c = \mathcal{O}(\varepsilon^2) \) \varepsilon represents the terms of 1st order in spatial deriv's. Curvature pert'n is nonlinearly invariant on super-H scales
Transformation of linear MS equations

GR:
\[
\frac{1}{z^2} \frac{1}{\alpha_0} \frac{d}{d\eta} \left( \frac{z^2}{\alpha_0} \frac{d}{d\eta} \mathcal{R}_c \right) + c_s^2 k^2 \mathcal{R}_c = 0 , \quad z \equiv \frac{\phi'}{\mathcal{H}}
\]

\( \alpha_0 \) is the background value of the lapse function and \( c_s \) is the sound velocity.

Under disformal transformation, \( \tilde{\alpha}^2 = \alpha^2 - B \phi^2 \), there are two ways to interpret:

\[
\Rightarrow \quad \frac{1}{\tilde{z}^2} \frac{1}{\tilde{\alpha}_0} \frac{d}{d\eta} \left( \frac{\tilde{z}^2}{\tilde{\alpha}_0} \frac{d}{d\eta} \mathcal{R}_c \right) + \tilde{c}_s^2 k^2 \mathcal{R}_c = 0 \quad \cdots \quad 1
\]

It takes the same form if \( d\tau = \alpha_0 dt \) and \( d\tilde{\tau} = \tilde{\alpha}_0 dt \) (\( dt = \alpha d\eta \)).

On the other hand

\[
\Rightarrow \quad \frac{1}{\tilde{z}^2} \frac{1}{\alpha_0} \frac{d}{d\eta} \left( \frac{\tilde{z}^2}{\alpha_0} \frac{d}{d\eta} \mathcal{R}_c \right) + \tilde{c}_s^2 k^2 \mathcal{R}_c = 0 , \quad \cdots \quad 2
\]

\[ \tilde{c}_s \equiv \frac{\tilde{\alpha}_0}{\alpha_0} c_s , \quad \tilde{z} \equiv \sqrt{\frac{\alpha_0}{\tilde{\alpha}_0}} z \]

It can be interpreted as the one in modified gravity with this redefinition of \( c_s \) and \( z \).
Transformation of **nonlinear** equations in spatial gradient expansion

**GR [Takamizu et al]:**

\[
\frac{1}{z^2} \frac{1}{\alpha_0} \frac{\partial}{\partial \eta} \left( \frac{z^2}{\alpha_0} \frac{\partial}{\partial \eta} R_c \right) + \frac{c_s^2}{4} (3) R [e^{2\psi} \gamma_{ij}] = \mathcal{O}(\varepsilon^4)
\]

\[
\psi = R_c + \mathcal{O}(\varepsilon^2) \text{ and } (3) R \text{ is the spatial scalar curvature}
\]

By the same reasoning as in the linear case, it takes the same form if the proper time is rescaled, or is interpreted as the one in modified gravity with rescaled \( c_s \) and \( z \).

\[
\tilde{c}_s \equiv \frac{\tilde{\alpha}_0}{\alpha_0} c_s \quad \text{and} \quad \tilde{z} \equiv \sqrt{\frac{\alpha_0}{\tilde{\alpha}_0}} z
\]
Transformation of nonlinear equations in spatial gradient expansion

GR:

\[
\frac{1}{z_t^2} \frac{1}{\alpha_0} \frac{\partial}{\partial \eta} \left( \frac{z_t^2}{\alpha_0} \frac{\partial}{\partial \eta} \gamma_{ij}^{TT} \right) + \frac{1}{4} \left( e^{-2\psi} (3) R_{ij} \left[ e^{2\psi} \gamma_{ij} \right] \right)^{TT} = \mathcal{O}(\epsilon^4)
\]

\(z_t \equiv a\) and \((\cdots)^{TT}\) denotes the transverse-traceless projection.

By the same reasoning as in the linear case, it takes the same form if the proper time is rescaled, or is interpreted as the one in modified gravity with rescaled \(c\_s\) and \(z\).

\[
\tilde{c}_t \equiv \frac{\tilde{a}_0}{\alpha_0}, \quad \tilde{z}_t \equiv \sqrt{\frac{\alpha_0}{\tilde{a}_0}} z_t = \sqrt{\frac{\alpha_0}{\tilde{a}_0}} a
\]

\[
\frac{1}{\tilde{z}_t^2} \frac{1}{\alpha_0} \frac{\partial}{\partial \eta} \left( \frac{\tilde{z}_t^2}{\alpha_0} \frac{\partial}{\partial \eta} \gamma_{ij}^{TT} \right) + \frac{\tilde{c}_t^2}{4} \left( e^{-2\psi} (3) R_{ij} \left[ e^{2\psi} \gamma_{ij} \right] \right)^{TT} = \mathcal{O}(\epsilon^4)
\]
Suppose there are $N$ component scalar field, $\phi^I (I = 1, \cdots , N)$.

$$\tilde{g}_{\mu\nu} = A(\phi^I , X^{IJ}) g_{\mu\nu} + B_{KL}(\phi^I , X^{IJ}) \partial_\mu \phi^K \partial_\nu \phi^L ,$$

$$X^{IJ} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi^I \partial_\nu \phi^J$$

**Adiabatic limit:**

$$\phi^I = \phi^I (\varphi)$$

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + B_{KL} \left[ \phi^I (\varphi) , X^{IJ} (\varphi , \partial \varphi) \right] (\phi^K)' (\phi^L)' \partial_\mu \varphi \partial_\nu \varphi , \quad (\phi^I)' \equiv \frac{d\phi^I}{d\varphi}$$

**The uniform $\varphi$ slicing:**

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + B_{KL} \left[ \phi^I (\varphi) , X^{IJ} (\varphi , \dot{\varphi} / \alpha) \right] (\phi^K)' (\phi^L)' \dot{\varphi}^2 \delta_\mu^0 \delta_\nu^0$$

Since the multi-disformal transformation only affects the lapse function, we can apply the same argument as before!
Summary

- The curvature and tensor perturbations on the uniform $\phi$ slicing are fully non-linearly invariant under the disformal transformation.

- The same conclusion can be drawn for a multi-component extension of the disformal transformation, dubbed multi-disformal transformation, on the uniform $\varphi$ slicing in the adiabatic limit.

- Once a 2nd order differential eq. is obtained in modified gravity or EFT, one can map it into the same form as the one in GR by a suitable disformal transformation.