Scalar perturbations induced by the second-order effects of primordial gravitational waves

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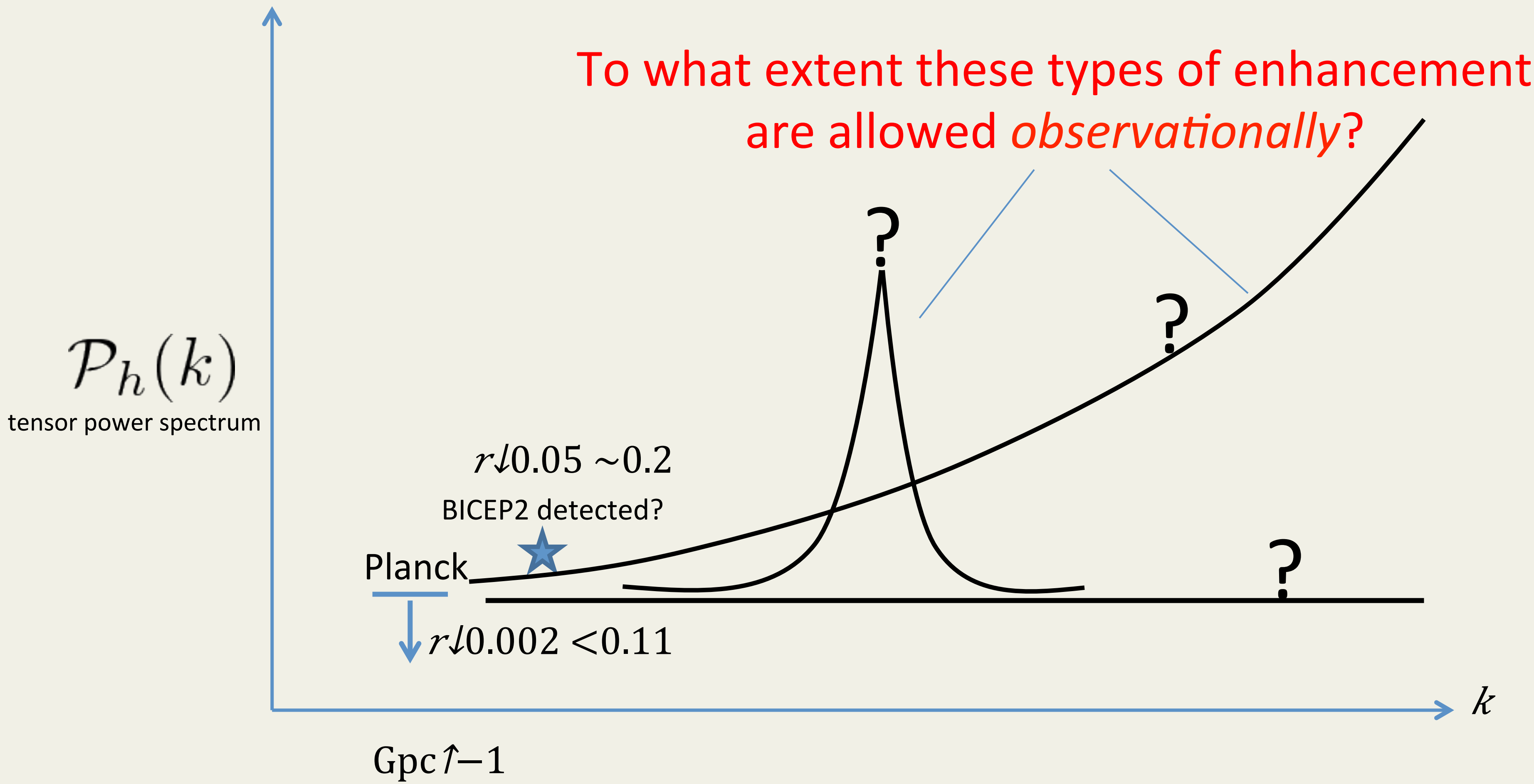
Motivation:
Investigating primordial tensor perturbations on small scales



Related works: Ota et al. (2014),
Chluba et al. (2014)

Induced scalar perturbations

- Assume tensor pert. >> scalar pert. on small scales.
- Then scalar perturbations are generated due to the second order effects of tensor pert.
- If tensor pert. is sufficiently large, induced scalar pert. becomes large so that PBHs are overproduced.
- We can place upper bounds on tensor pert. requiring PBHs are not overproduced.

cf. Saito Yokoyama 2009

PBHs formed from collapse of primordial inhomogeneities during R.D.

- If $1/3 \lesssim \bar{\delta}_{\mathrm{r,cr}}$ in some region at horizon crossing, this region collapses to form a black hole (PBH) during R.D..

  (Carr 1975)

- The mass can take various values depending on when they were formed.

- They could be detected by microlensing or emit high-energy particles or form binaries and emit gravitational waves.

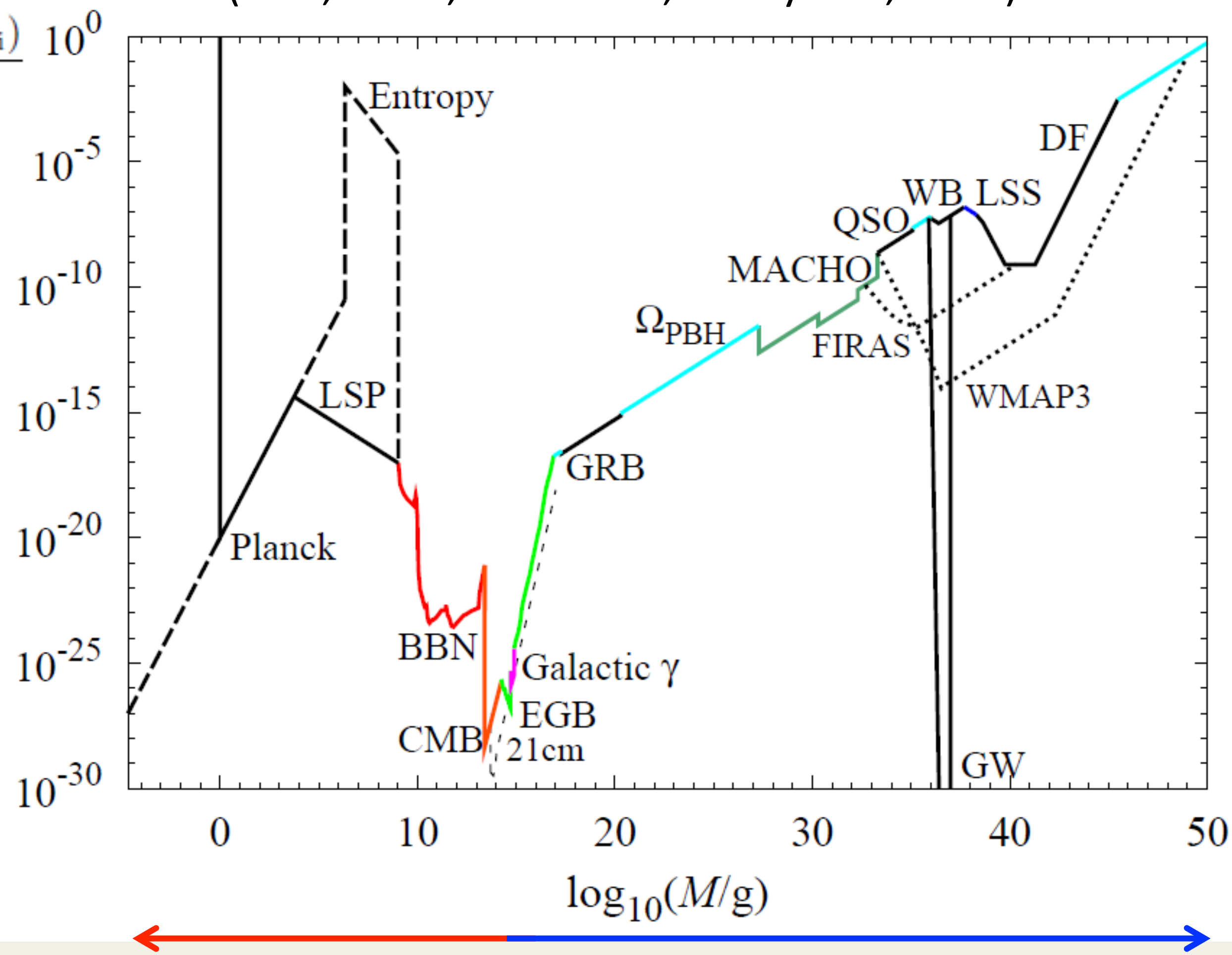
- So far only upper bounds of the abundance of PBHs have been obtained on various mass scales.

Observational constraints on PBHs of various masses

abundance of PBHs
when they were formed

$\sim \beta(M) \equiv \frac{\rho_{\mathrm{PBH}}(t_i)}{\rho(t_i)}$

(Carr, Kohri, Sendouda, Yokoyama, 2010)



have been evaporated by now
due to Hawking radiation

can still exist in our universe

large tensor pert. on small scales ⇒ induced scalar pert. ⇒ overproduction of PBHs

obtain upper bounds on initial tensor pert.
to avoid overproduction of PBHs.

Formulation

Metric

$$ds^2 = a^2[-(1+2\Phi)d\eta^2 - 2B_{,i}d\eta dx^i + ((1-2\Psi)\delta_{ij} - 2h_{ij})dx^i dx^j]$$

η: conformal time

The Einstein equations

$$\Delta\Psi - 3\mathcal{H}(\Psi' + \mathcal{H}\Phi) - \mathcal{H}\Delta B + S_1 = 4\pi G a^2 \delta\rho$$

$$(\Psi' + \mathcal{H}\Phi + S_2)_{,i} = 0$$

$$\Psi'' + \mathcal{H}(2\Psi + \Phi)' + (2\mathcal{H}' + \mathcal{H}^2)\Phi + \frac{1}{2}\Delta(\Phi - \Psi + B' + 2\mathcal{H}B) + S_3 + S_4 = 4\pi G a^2 \delta p$$

$$(\Phi - \Psi + B' + 2\mathcal{H}B - 2S_5)_{,ij} = 0$$

The conservation of energy-momentum tensor

$$\delta\rho' + 3\mathcal{H}(\delta\rho + \delta p) - (\rho + p)\Delta B - 3(\rho + p)\Psi' - 2(\rho + p)h^{ij}h'_{ij} = 0$$

$$\partial_i(\delta p + (\rho + p)\Phi) = 0$$

Source terms

$$S_1 \equiv -\frac{1}{4}h'_{ij}h^{ij\prime} - 2\mathcal{H}h_{ij}h^{ij\prime} + h_{ij}\Delta h^{ij} - \frac{1}{2}\partial_j h_{ik}\partial^k h^{ij} + \frac{3}{4}\partial_k h_{ij}\partial^k h^{ij},$$

$$\Delta S_2 = \partial^i S_i,$$

$$S_i = -h^{jk}\partial_k h'_{ij} + \frac{1}{2}h^{jk\prime}\partial_i h_{jk} + h^{jk}\partial_i h'_{jk}$$

$$S_3 \equiv \frac{3}{4}h'_{ij}h^{ij\prime} + h_{ij}h^{ij\prime\prime} + 2\mathcal{H}h_{ij}h^{ij\prime} - h_{ij}\Delta h^{ij} + \frac{1}{2}\partial_j h_{ik}\partial^k h^{ij} - \frac{3}{4}\partial_k h_{ij}\partial^k h^{ij}$$

$$\Delta S_4 = \frac{1}{2}(\Delta S^i{}_i - \partial^i\partial^j S_{ij}),$$

$$\Delta^2 S_5 = \frac{1}{2}(3\partial^i\partial^j S_{ij} - \Delta S^i_i),$$

$$S_{ij} \equiv -h_i{}^{k\prime}h'_{jk} - h_{ik}h_j{}^{k\prime\prime} - 2\mathcal{H}h_i{}^k h'_{jk} + h^{kl}\partial_k\partial_l h_{ij} + h_i{}^k\Delta h_{jk} - h^{kl}\partial_l\partial_i h_{jk} - h^{kl}\partial_l\partial_j h_{ik}$$
$$-\partial_k h_{jl}\partial^l h_i{}^k + \partial_l h_{jk}\partial^l h_i{}^k + \frac{1}{2}\partial_i h_{kl}\partial_j h^{kl} + h^{kl}\partial_i\partial_j h_{kl}.$$

scalar pert.

$$\Delta\Psi - 3\mathcal{H}(\Psi' + \mathcal{H}\Phi) - \mathcal{H}\Delta B + S_1 = 4\pi G a^2 \delta\rho$$

source ~ $O(h_{ij}^2)$

$$S_1 \equiv -\frac{1}{4}h'_{ij}h^{ij\prime} - 2\mathcal{H}h_{ij}h^{ij\prime} + h_{ij}\Delta h^{ij} - \frac{1}{2}\partial_j h_{ik}\partial^k h^{ij} + \frac{3}{4}\partial_k h_{ij}\partial^k h^{ij}$$

prime: ∂/∂

Scalar pert. are generated due to the source terms.

Specifying the initial condition

$$h_{ij}(\eta, \boldsymbol{x}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^{3/2}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \left(h^{+}(\eta, \boldsymbol{k}) e_{ij}^{+}(\boldsymbol{k}) + h^{\times}(\eta, \boldsymbol{k}) e_{ij}^{\times}(\boldsymbol{k})\right)$$

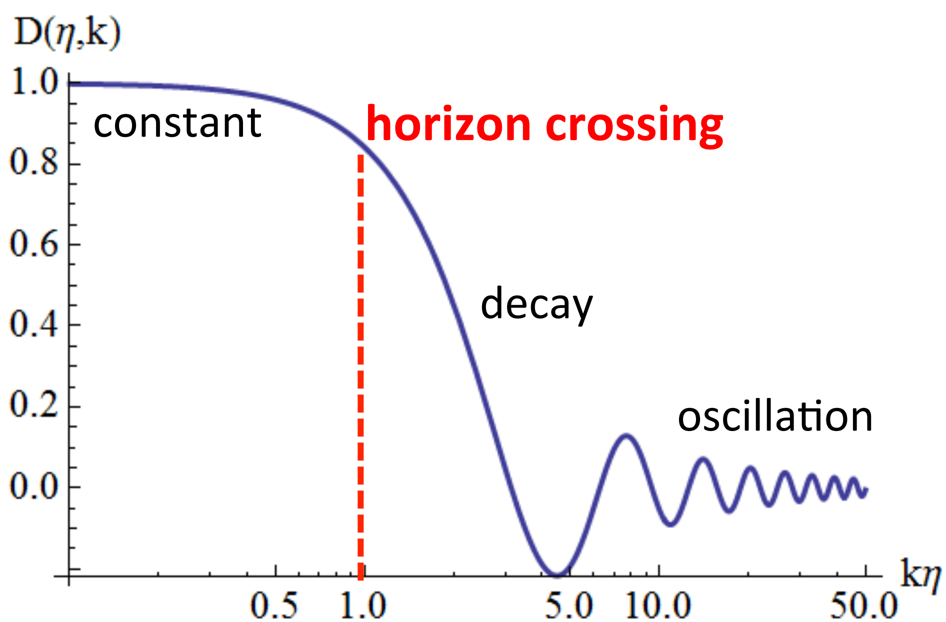
+ or ×

initial amplitude

$$h^r(\eta, \boldsymbol{k}) = D(\eta, k) h^r(\boldsymbol{k})$$

Growth factor: $\sin k\eta / k\eta$

$(\leftarrow h''_{ij} + 2\mathcal{H}h'_{ij} - \Delta h_{ij} = 0)$



Specifying the initial condition

$$h_{ij}(\eta, \boldsymbol{x}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^{3/2}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} (h^+(\eta, \boldsymbol{k}) e^+_{ij}(\boldsymbol{k}) + h^\times(\eta, \boldsymbol{k}) e^\times_{ij}(\boldsymbol{k}))$$

r: + or ×

$$h^r(\eta, \boldsymbol{k}) = D(\eta, k) h^r(\boldsymbol{k})$$

$h^r(\boldsymbol{k})$: initial amplitude

- The definition of the initial power spectrum:

$$\langle h^r(\boldsymbol{k}) h^s(\boldsymbol{K}) \rangle = \frac{2\pi^2}{k^3} \delta(\boldsymbol{k} + \boldsymbol{K}) \delta_{rs} \mathcal{P}_h(k)$$

- As an illustration, we consider a delta-function like power spectrum

$$\mathcal{P}_h(k) = \mathcal{A}^2 k \delta(k - k_p)$$

$\mathcal{A}$: amplitude

k_p: position of spike

Calculation of the power spectrum of the density perturbation

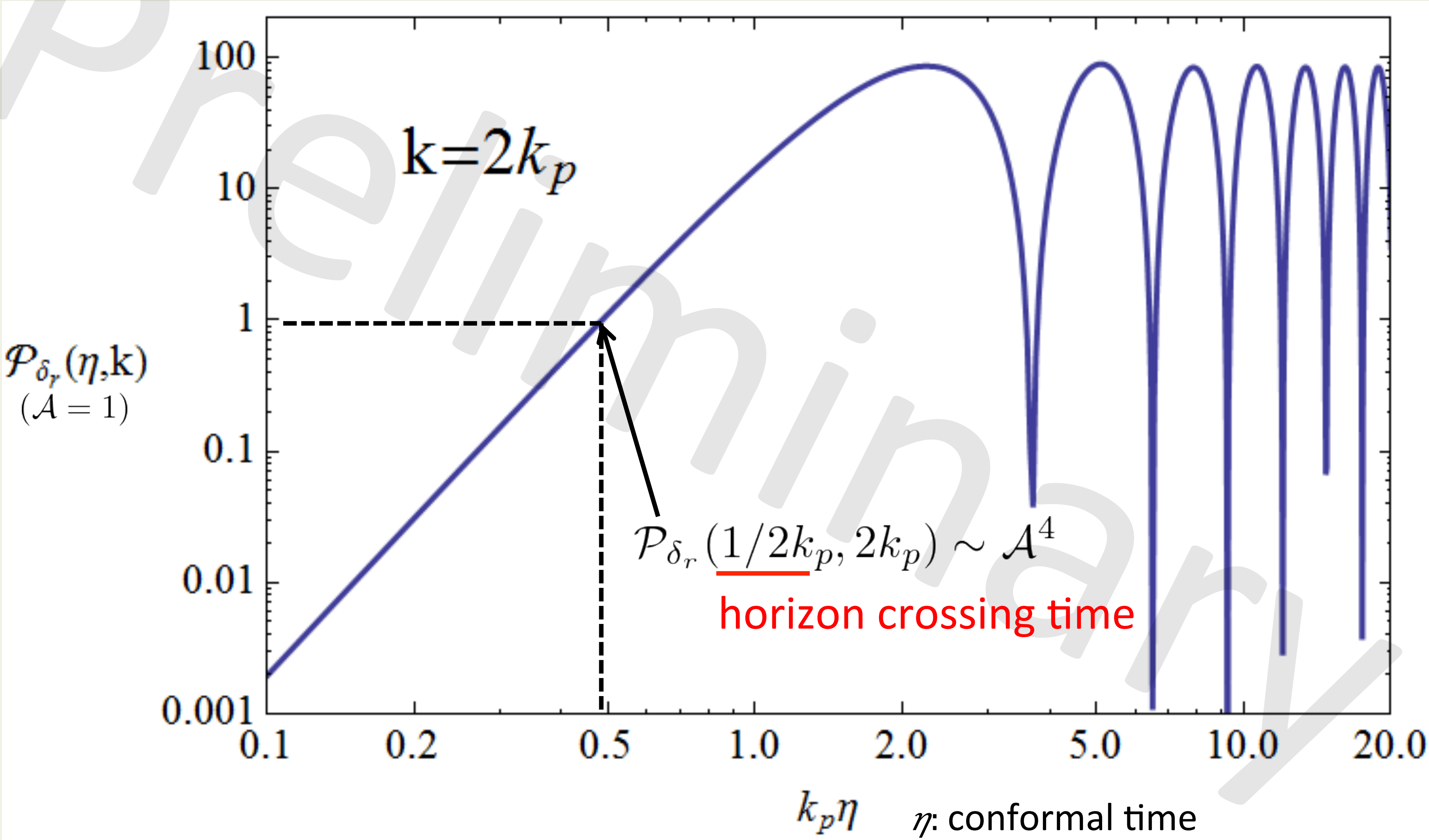
$$\mathcal{P}_h(k) = \mathcal{A}^2 k\delta(k - k_p)$$

$$\mathcal{P}_{\delta_r}(\eta, k) = \left(\frac{1 + c_s^2}{c_s^2}\right)^2 \mathcal{A}^4 \left(\frac{k}{k_p}\right)^2 \eta^2 \Theta\left(1 - \frac{k}{2k_p}\right) \sum_{rs} F_{rs}\left(\eta, k, k_p, \frac{k}{2k_p}\right)^2$$

This reflects $\delta \sim O(h_{ij}^2)$

$$F_{rs}(\eta, \boldsymbol{k}, \boldsymbol{k}') \equiv \int d\tilde{\eta}(\tilde{\eta}/\eta)A_{rs}(\tilde{\eta}, \boldsymbol{k}, \boldsymbol{k}')(\partial_\eta - \mathcal{H})g_k(\eta, \tilde{\eta})$$
$$+ D(\eta, k')\left\{-\partial_\eta E_1^{rs} + \left(\frac{1}{2}\overleftarrow{\partial}_\eta + \partial_\eta\right)\left(1 - \frac{k'}{k}\mu\right)E_2^{rs}\right\} D(\eta, |\boldsymbol{k} - \boldsymbol{k}'|)$$

The time evolution of the power spectrum



Upper bound on the amplitude of primordial tensor perturbations

- PBH formation has to be sufficiently rare to be consistent with observation

$$10 \lesssim \frac{\text{threshold for PBH formation}}{\text{typical amplitude}} \sim \frac{1/3}{\sqrt{\mathcal{P}_{\delta_r}}} \sim \frac{1}{3\mathcal{A}^2}$$

$$\rightarrow \boxed{\mathcal{A}^2 \lesssim 0.03}$$

$$\mathcal{P}_{\delta_r}(1/2k_p, 2k_p) \sim \mathcal{A}^4$$

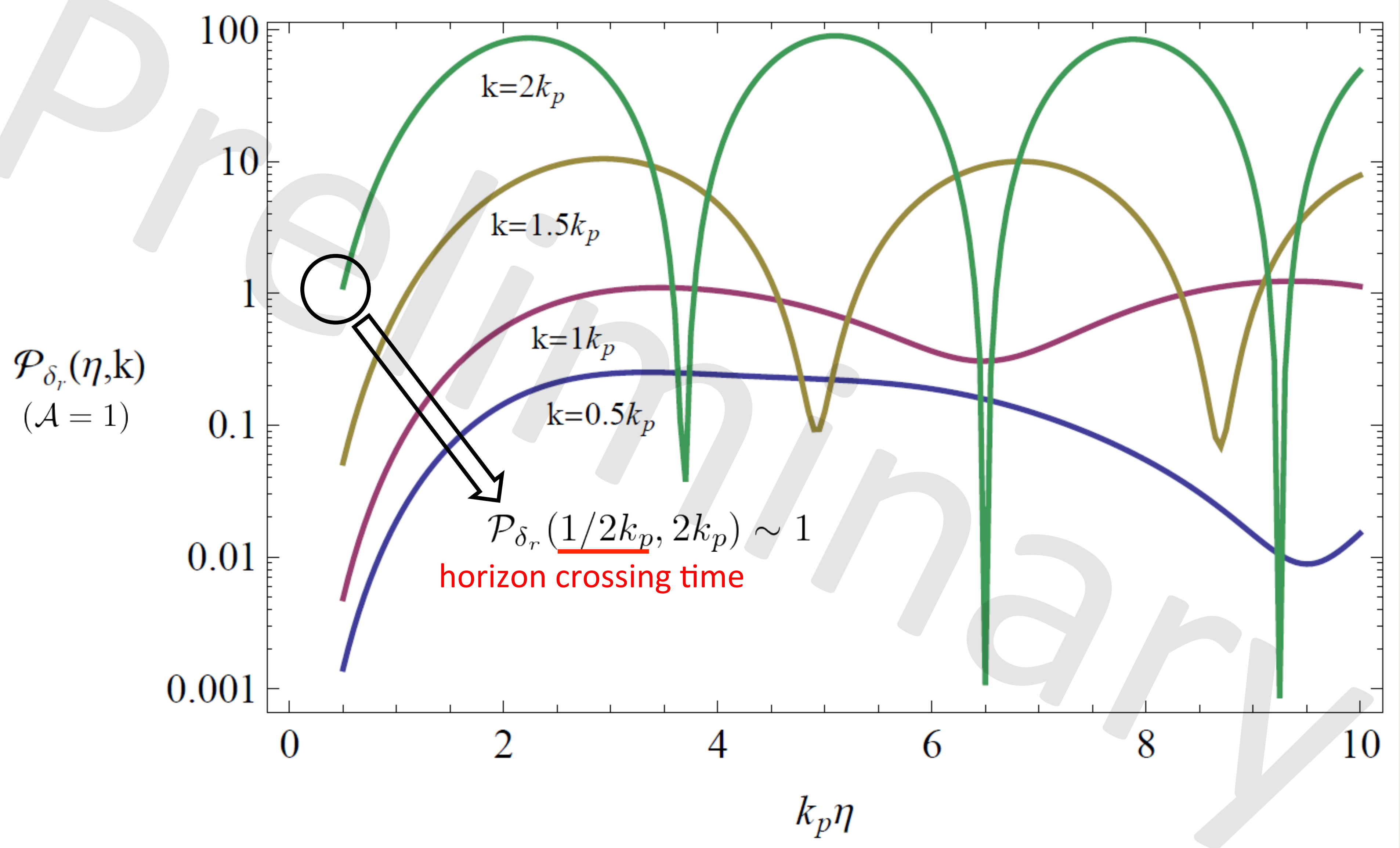
Summary

- Assume scalar pert. << tensor pert. on small scales.
- Then scalar perturbations are generated due to the second order effects of tensor pert.
- If tensor pert. is sufficiently large, induced scalar pert. becomes large so that PBHs are overproduced.
- We can place upper bounds on tensor pert. requiring it does not cause overproduction of PBHs.

$$\mathcal{P}_h(k) = \mathcal{A}^2 k \delta(k - k_p) \longrightarrow \mathcal{A}^2 \lesssim 0.03$$

- Other shapes of power spectrum, upper bounds from ultracompact minihalos,

The time evolution of the power spectrum



- Combining these equations yields the evolution equation for Ψ:

$$\Psi'' + 2\mathcal{H}\Psi' + c_s^2 k^2 \Psi = S,$$

$$S \equiv c_s^2 S_1 - S_3 - \hat{k}^i \hat{k}^j S_{ij} + 2c_s^2 \mathcal{H} h^{ij} h'_{ij}$$

- This can be formally solved as

$$\Psi(\eta, \boldsymbol{k}) = a^{-1}(\eta) \int d\tilde{\eta} g_k(\eta, \tilde{\eta}) a(\tilde{\eta}) S(\tilde{\eta}, \boldsymbol{k})$$

↑

Green's function

$$g_k'' + \left(c_s^2 k^2 - \frac{a''}{a} \right) g_k = \delta(\eta - \tilde{\eta})$$

- The energy density perturbation is given by

$$\delta_r = \frac{1 + c_s^2}{c_2^2 \mathcal{H}} (\Psi' + S_2)$$

$$S(\eta,\boldsymbol{k})=\sum_{rs}\int\frac{d^3\boldsymbol{k}'}{(2\pi)^{3/2}}h^r(\boldsymbol{k}')h^s(\boldsymbol{k}-\boldsymbol{k}')A_{rs}(\eta,\boldsymbol{k},\boldsymbol{k}'),$$

$$A_{rs}(\eta,\boldsymbol{k},\boldsymbol{k}')\equiv f_1(\eta,\boldsymbol{k},\boldsymbol{k}')E_1^{rs}+f_2(\eta,\boldsymbol{k},\boldsymbol{k}')E_2^{rs},$$

$$S \quad\rightarrow\quad \left\{\overleftarrow{\partial_\eta}\partial_\eta-\frac{1}{2}(3-c_s^2)k^2+3kk'\mu-k'^2\right\}E_1^{rs}+$$
$$\left\{-\frac{1}{4}(3+c_s^2)\overleftarrow{\partial_\eta}\partial_\eta+c_s^2\partial_\eta^2+2c_s^2\mathcal{H}\partial_\eta+\frac{1}{8}(1-3c_s^2)k^2-\frac{1}{2}k'\mu(k-k'\mu)+\frac{3}{4}(1+c_s^2)k'^2\right\}E_2^{rs}.$$