RESCEU APCosPA Summer School on Cosmology and Particle Astrophysics

Teleparallel Gravity in Five Dimensional Theories

Reference: arXiv:1403.3161 [gr-qc]

(accepted by *Classical and Quantum Gravity*)

Ling-Wei Luo

Department of Physics, National Tsing Hua University, Taiwan

Collaborators: **Chao-Qiang Geng** (*NTHU/NCTS*), **Huan Hsin Tseng** (*NTHU*)

August 1, 2014 @ Matsumoto, Japan

Outline

Outline

Alternative Gravitational Theory

- Einstein's unified field theory: *Riemannian Geometry with Maintaining the Notion of Distant Parallelism* (Teleparallelism, *Einstein 1928*)
- Torsion scalar (*Einstein 1929*)
- Teleparallel Lagrangian is equivalent to the Riemann scalar (*Lanczos 1929*)
- Generalization: *New General Relativity* (NGR) (*Hayashi & Shirafuji 1979*)

EINSTEIN: RIEMANN-Geometrie mit Aufrechterhaltung d. Begriffes d. Fernparallelismus 217

RIEMANN-Geometrie mit Aufrechterhaltung des Begriffes des Fernparallelismus.

Von A. EINSTEIN.

Die RIEMANNsche Geometrie hat in der allgemeinen Relativitätstheorie zu einer physikalischen Beschreibung des Gravitationsfeldes geführt, sie liefert aber keine Begriffe, die dem elektromagnetischen Felde zugeordnet werden können. Deshalb ist das Bestreben der Theoretiker darauf gerichtet, natürliche Verallgemeinerungen oder Ergänzungen der RIEMANNschen Geometrie aufzufinden, welche begriffsreicher sind als diese, in der Hoffnung, zu einem logischen Gebäude zu gelangen, das alle physikalischen Feldbegriffe unter einem einzigen Gesichtspunkte vereinigt. Solche Bestrebungen haben mich zu einer Theorie geführt, welche ohne jeden Versuch einer physikalischen Deutung mitgeteilt werden möge, weil sie schon wegen der Natürlichkeit der eingeführten Begriffe ein gewisses Interesse beanspruchen kann.

Die RIEMANNsche Geometrie ist dadurch charakterisiert, daß die infinitesimale Umgebung jedes Punktes P eine euklidische Metrik aufweist, sowie dadurch, daß die Beträge zweier Linienelemente, welche den infinitesimalen Umgebungen zweier endlich voneinander entfernter Punkte P und Q angehören, miteinander vergleichbar sind. Dagegen fehlt der Begriff der Parallelität solcher zwei Linienelemente; der Richtungsbegriff existiert nicht für das Endliche. Die im folgenden dargelegte Theorie ist dadurch charakterisiert, daß sie neben der RIEMANNschen Metrik den der »Richtung« bzw. Richtungsgleichheit oder des »Parallelismus« für das Endliche einführt. Dem entspricht es, daß neben den Invarianten und Tensoren der RIEMANNschen Geometrie neue Invarianten und Tensoren auftreten.

What is Teleparallelism?

- Introduce the orthonormal frame (veirbein) in Weitzenböck geometry W_4

$$g_{\mu\nu} = \eta_{ij}\, e^i_\mu\, e^j_\nu\,, \qquad \eta_{ij} = \text{diag}(+1,-1,-1,-1)\,,$$

where $\mu,\nu,\rho,\ldots = 0,1,2,3$ and $i,j,k,\ldots = \hat{0},\hat{1},\hat{2},\hat{3}$.

- Metric compatible condition $\nabla\, g_{\mu\nu} = 0$:

$$\nabla\, e^i_\nu = 0\,, \qquad \omega_{ij} = -\,\omega_{ji}\,,$$

- Absolute parallelism for parallel vectors (*Cartan 1922/Eisenhart 1925*)

$$\nabla_\nu\, e^i_\mu = \partial_\nu e^i_\mu - e^i_\rho\, \Gamma^\rho_{\mu\nu} = 0\,.$$

- Weitzenböck connection $(\omega_{ij\mu} = 0)$ $\quad \Gamma^\rho_{\mu\nu} = e^\rho_i \partial_\nu e^i_\mu$
- Curvature-free $\quad R^\sigma{}_{\rho\mu\nu}(\Gamma) = e^\sigma_i e^j_\rho R^i{}_{j\mu\nu}(\omega) = 0$
- Torsion tensor

$$T^i{}_{\mu\nu} \equiv \Gamma^i_{\nu\mu} - \Gamma^i_{\mu\nu} = \partial_\mu e^i_\nu - \partial_\nu e^i_\mu\,.$$

- Contorsion tensor is defined as

$$K^\rho{}_{\mu\nu} = -\frac{1}{2}(T^\rho{}_{\mu\nu} - T_\mu{}^\rho{}_\nu - T_\nu{}^\rho{}_\mu) = -K_\mu{}^\rho{}_\nu\,.$$

Teleparallel Equivalent to GR

- The Weitzenböck connection can be decomposed as

$$\Gamma^{\rho}_{\mu\nu} = \left\{ {}^{\rho}_{\mu\nu} \right\} + K^{\rho}{}_{\mu\nu} \,,$$

where $\left\{ {}^{\rho}_{\mu\nu} \right\}(e)$ is the Levi-Civita connection

- Teleparallel Equivalent to GR ($\mathrm{GR}_{\|}$ or TEGR) in W_4 based on the the relation ($T_\mu := T^{\nu}{}_{\nu\mu}$)

$$R(\Gamma) = \tilde{R}(e) + T - 2\,\tilde{\nabla}_{\mu} T^{\mu} = 0 \quad \Rightarrow \quad -\tilde{R}(e) = T - 2\,\tilde{\nabla}_{\mu} T^{\mu}.$$

- The TEGR action is

$$S_{\mathrm{TEGR}} = \frac{1}{2\kappa} \int d^4x\, e\, T \qquad (e = \sqrt{-g})\,.$$

Torsion Scalar *(Einstein 1929)*

$$T \equiv \frac{1}{4} T^{\rho}{}_{\mu\nu} T_{\rho}{}^{\mu\nu} + \frac{1}{2} T^{\rho}{}_{\mu\nu} T^{\nu\mu}{}_{\rho} - T^{\nu}{}_{\mu\nu} T^{\sigma\mu}{}_{\sigma} = \frac{1}{2} T^{i}{}_{\mu\nu} S_{i}{}^{\mu\nu}$$

$S_{\rho}{}^{\mu\nu} \equiv K^{\mu\nu}{}_{\rho} + \delta^{\mu}_{\rho} T^{\sigma\nu}{}_{\sigma} - \delta^{\nu}_{\rho} T^{\sigma\mu}{}_{\sigma} = -S_{\rho}{}^{\nu\mu}$ is superpotential.

Outline

Hypersurface of GR

- The 5th-coordinate $y = x^5$ and the signature is $(+ - - - \varepsilon)$ with $\varepsilon = -1$.
- The unit normal vector of hypersurface is denoted by n and $\bar{g}_{55} = n \cdot n$
- The tensor $B_{MN} = -\tilde{\nabla}_M n_N$, $h_{MN} = \bar{g}_{MN} - \varepsilon\, n_M n_N$ with $M, N = 0, 1, 2, 3, 5$

$$\theta = h^{MN} B_{MN}\,, \quad \sigma_{MN} = B_{(MN)} - \frac{1}{3}\theta h_{MN}\,, \quad \omega_{MN} = B_{[MN]}$$

Gauss's equation

$$\bar{R}^{\mu}{}_{\nu\rho\sigma} = R^{\mu}{}_{\nu\rho\sigma} + \varepsilon (K^{\mu}{}_{\sigma} K_{\nu\rho} - K^{\mu}{}_{\rho} K_{\nu\sigma})$$

- Extrinsic curvature $K_{\mu\nu} = B_{(\mu\nu)} = -\varepsilon\, \tilde{\nabla}_\mu n \cdot e_\nu = -\varepsilon\, \frac{1}{2} \mathcal{L}_n g_{\mu\nu} = \varepsilon \left\{ {5 \atop \mu\nu} \right\} n_5$

5-Dimension Setting

- In normal coordinate, 5D metric is $\bar{g}_{MN} = \bar{\eta}_{IJ}\, e^I_M\, e^J_N$, $\bar{\eta}_{IJ} = \text{diag}(+1, -1, -1, -1, \varepsilon)$ with $\varepsilon = -1$.
- Coordinate frame
  $M, N = 0, 1, 2, 3, 5, \quad \mu, \nu = 0, 1, 2, 3, \quad \alpha, \beta = 1, 2, 3.$
- Othonormal frame (anholonomic frame)
  $A, B, I, J, K = \hat{0}, \hat{1}, \hat{2}, \hat{3}, \hat{5}, \quad i, j, k = \hat{0}, \hat{1}, \hat{2}, \hat{3}, \quad a, b = \hat{1}, \hat{2}, \hat{3}.$
- The 5D torsion scalar in the othonormal frame can be decomposed as

$$^{(5)}T = \underbrace{\bar{T}}_{\text{induced 4D torsion scalar}} + \frac{1}{2}\left(\bar{T}_{i\hat{5}j}\,\bar{T}^{i\hat{5}j} + \bar{T}_{i\hat{5}j}\,\bar{T}^{j\hat{5}i}\right) + 2\,\bar{T}^j{}_j{}^i\,\bar{T}^{\hat{5}}{}_{i\hat{5}} - \bar{T}^j{}_{\hat{5}j}\,\bar{T}^{k\hat{5}}{}_k\,.$$

- Metric is given by

$$\bar{g}_{MN} = \begin{pmatrix} g_{\mu\nu}(x^\mu, y) & 0 \\ 0 & \varepsilon\phi^2(x^\mu, y) \end{pmatrix} ,$$

$\Rightarrow$ the residual components of vierbein: $e_\mu^{\,i}$ and $e_5^{\,\hat{5}}$
$\Rightarrow$ $\bar{T}^\rho{}_{\mu\nu} = T^\rho{}_{\mu\nu}$

- The 5D torsion scalar in the coordinate frame:

$$^{(5)}T = \underbrace{T}_{\text{induced 4D torsion scalar}} + \frac{1}{2}\left(\bar{T}_{\rho 5\nu}\,\bar{T}^{\rho 5\nu} + \bar{T}_{\rho 5\nu}\,\bar{T}^{\nu 5\rho}\right) + 2\,\bar{T}^\sigma{}_\sigma{}^\mu\,\bar{T}^5{}_{\mu 5} - \bar{T}^\nu{}_{5\nu}\,\bar{T}^{\sigma 5}{}_\sigma\,.$$

Note:
In general, *induced torsion* $\bar{T}^\rho{}_{\mu\nu} = T^\rho{}_{\mu\nu} + \bar{C}^\rho{}_{\mu\nu}$, where

$$\bar{C}^\rho{}_{\mu\nu} = e_{\hat{5}}^{\,\rho}\overbrace{(\partial_\mu e_\nu^{\,\hat{5}} - \partial_\nu e_\mu^{\,\hat{5}})}^{\bar{C}^{\hat{5}}{}_{\mu\nu}}\,.$$

$\bar{C}^{\hat{5}}{}_{\mu\nu} = \Gamma^{\hat{5}}_{\nu\mu} - \Gamma^{\hat{5}}_{\mu\nu} = h_\mu^{\,M}\,h_\nu^{\,N}\,T^{\hat{5}}{}_{MN} \sim \omega_{\mu\nu}$ is related to the extrinsic torsion or twist $\omega_{\mu\nu}$.

Outline

Braneworld Theory

- Metric is given by

$$\bar{g}_{MN} = \begin{pmatrix} g_{\mu\nu}(x^\mu, y) & 0 \\ 0 & \varepsilon\phi^2(x^\mu, y) \end{pmatrix} .$$

- The induced torsion scalar $\bar{T} = T$ is a purely 4-dimensional object

- The bulk action in the othonormal frame is

$$S_{\text{bulk}} = \frac{1}{2\kappa_5} \int dvol^5 \left(T + \frac{1}{2}(\bar{T}_{i5j}\,\bar{T}^{i5j} + \bar{T}_{i5j}\,\bar{T}^{j5i}) + \frac{2}{\phi}\, e_i(\phi)\,\bar{T}^a - \bar{T}_5\,\bar{T}^5 \right),$$

where $\bar{T}_A := \bar{T}^b{}_{bA}$

- The bulk metric $\bar{g}$ is maximally symmetric 3-space with spatially flat $(k=0)$ and has the form

$$\bar{g}_{MN} = \text{diag}\left(1, -a^2(t,y), -a^2(t,y), -a^2(t,y), \varepsilon\,\phi^2(t,y)\right) ,$$

$$\bar{\vartheta}^{\hat{0}} = dt\,, \quad \bar{\vartheta}^{a} = a(t,y)\,dx^{\alpha}\,, \quad \bar{\vartheta}^{\hat{5}} = \phi(t,y)\,dy\,.$$

- Torsion 2-forms are

$$\bar{T}^{\hat{0}} = \bar{d}\,\bar{\vartheta}^{\hat{0}} = 0, \quad \bar{T}^{a} = \frac{\dot{a}}{a}\,\bar{\vartheta}^{\hat{0}} \wedge \bar{\vartheta}^{a} + \frac{a'}{a\phi}\,\bar{\vartheta}^{\hat{5}} \wedge \bar{\vartheta}^{a}, \quad \bar{T}^{\hat{5}} = \frac{\dot{\phi}}{\phi}\,\bar{\vartheta}^{\hat{0}} \wedge \bar{\vartheta}^{\hat{5}}\,,$$

- Torsion 5-form reads

$$\bar{\mathcal{T}} = \left[T + \left(\frac{3+9\,\varepsilon}{\phi^2}\,\frac{a'^2}{a^2} + 6\,\frac{\dot{a}}{a}\,\frac{\dot{\phi}}{\phi}\right)\right] dvol^5\,.$$

- The equations of motion

$$\bar{H}_A := \frac{\delta\bar{\mathcal{T}}}{\delta\bar{T}^A} = (-2)\bar{\star}\left({}^{(1)}\bar{T}_A - 2\,{}^{(2)}\bar{T}_A - \frac{1}{2}\,{}^{(3)}\bar{T}_A\right) ,$$

$$\bar{E}_A := i_{\bar{e}_A}(\bar{\mathcal{T}}) + i_{\bar{e}_A}(\bar{T}^B) \wedge \bar{H}_B\,,$$

$$\bar{\Sigma}_A := \frac{\delta\bar{L}_{mat}}{\delta\bar{\vartheta}^A}\,.$$

- The equation of motion of the bulk ($A = \hat{0}, \hat{5}$ components):

$$\bar{D}\bar{H}_{\hat{0}} - \bar{E}_{\hat{0}} = 3\left[\left(\frac{\dot{a}^2}{a^2} + \frac{\dot{a}}{a}\frac{\dot{\phi}}{\phi}\right) + \frac{\varepsilon}{\phi^2}\left(\frac{a''}{a} - \frac{a'}{a}\frac{\phi'}{\phi}\right) - \left(\frac{1-\varepsilon}{2\phi^2}\right)\frac{a'^2}{a^2}\right]\bar{\star}\bar{\vartheta}_{\hat{0}} + \frac{3\varepsilon}{\phi}\left(\frac{\dot{a}'}{a} - \frac{a'}{a}\frac{\dot{\phi}}{\phi}\right)\bar{\star}\bar{\vartheta}_{\hat{5}} = \kappa_5\,\bar{\Sigma}_{\hat{0}}\,,$$

$$\bar{D}\bar{H}_{\hat{5}} - \bar{E}_{\hat{5}} = \frac{3}{\phi}\left(\frac{a'}{a}\frac{\dot{\phi}}{\phi} - \frac{\dot{a}'}{a}\right)\bar{\star}\bar{\vartheta}_{\hat{0}} + 3\left[\left(\frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2}\right) - \left(\frac{1-\varepsilon}{2\phi^2}\right)\frac{a'^2}{a^2}\right]\bar{\star}\bar{\vartheta}_{\hat{5}} = \kappa_5\,\bar{\Sigma}_{\hat{5}}\,.$$

- We have the 00-component equation

$$\left(\frac{\dot{a}^2}{a^2} + \frac{\dot{a}}{a}\frac{\dot{\phi}}{\phi}\right) - \frac{1}{\phi^2}\left(\frac{a''}{a} - \frac{a'}{a}\frac{\phi'}{\phi}\right) - \frac{1}{\phi^2}\frac{a'^2}{a^2} = \frac{\kappa_5}{3}\bar{T}_{00}\,.$$

- The energy-momentum tensor is $\bar{\Sigma}_A = \bar{T}_A^B \bar{\star}\bar{\vartheta}_B$

$$\bar{T}_A^B(t,y) = \left(\bar{T}_A^B\right)_{\text{bulk}} + \left(\bar{T}_A^B\right)_{\text{brane}} ,$$
$$\left(\bar{T}_A^B\right)_{\text{brane}} = \frac{\delta(y)}{\phi} \, diag(\rho(t), -P(t), -P(t), -P(t), 0) ,$$
$$\left(\bar{T}_A^B\right)_{\text{bulk}} = \frac{\Lambda_5}{\kappa_5}\eta_A^B .$$

- The metric $g_{MN} = \theta(y) g_{MN}^{(+)} + \theta(-y) g_{MN}^{(-)}$
  FLRW background:

$$a(t,y) = \theta(y) a^{(+)}(t,y) + \theta(-y) a^{(-)}(t,y) .$$

- The discontinuity of the scale factor is

$$a''(t,y) = \delta(y) \, [a'](t,0) + \tilde{a}''(t,y) \text{ with } [a'] = a^{(+)\prime} - a^{(-)\prime} .$$

- The junction condition:

$$[a'](t,0) = -\frac{\kappa_5}{3\varepsilon}\rho\, a_0(t)\,\phi_0(t)\,.$$

- Impose the $\mathbb{Z}_2$ symmetry:

real-valued function $f(x) = -f(-x) \xRightarrow[\mathbb{Z}_2 \text{ symm.}]{} a'(t,0) = -\frac{\kappa_5}{6\varepsilon}\rho\, a_0(t)\,\phi_0(t)\,.$

- Modified Friedmann equation on the brane

$$\frac{\dot{a}_0^2(t)}{a_0^2(t)} + \frac{\ddot{a}_0(t)}{a_0(t)} = -\frac{\kappa_5^2}{36}\,\rho(t)(\rho(t)+3P(t)) - \frac{k_5}{3\phi_0^2(t)}\,\left(\bar{T}_{55}\right)_{\text{bulk}}\,.$$

$\Rightarrow$ Same as GR! (See *Binetruy, Deffayet and Langlois 2000*), but the *junction condition* comes from torsion itself!

Outline

KK Theory

- Low-energy effective gravitational theory
  $\Rightarrow$ consider the KK ansatz
  - Cylindrical condition (no y dependency)
  - Compactify to S^1 and only consider zero KK mode
- Setting the manifold is $M_4 \times S^1$, and the 5th-coordinate $y = r\,\theta$ with r radius of extra dimension
- The metric is reduced to

$$\bar{g}_{MN} = \begin{pmatrix} g_{\mu\nu}(x^\mu) & 0 \\ 0 & -\phi^2(x^\mu) \end{pmatrix} .$$

- The residual components are $T^\rho{}_{\mu\nu}$, and $\bar{T}^5{}_{\mu 5} = \partial_\mu \phi / \phi$
- The torsion scalar is ${}^{(5)}T = T + 2\, T^\sigma{}_\sigma{}^\mu\, \bar{T}^5{}_{\mu 5}$
- Integrating out the 5th-dimension $\int dy = \int_0^{2\pi} r\, d\theta = 2\pi r$
  $\Rightarrow$ the effective action is

$$S_{\text{eff}} = \frac{1}{2\,\kappa_4} \int d^4x\, e\, (\phi\, T + 2\, T^\mu\, \partial_\mu \phi)$$

GR vs. TEGR

- The effective Lagrangian of GR:

$$-\sqrt{-{}^{(5)}g}\,{}^{(5)}\tilde{R}$$

$$\rightarrow \quad -\sqrt{-g}\left(\phi\tilde{R}-\Box\phi\right).$$

- A specific case of Brans-Dicke theory (Brans-Dicke parameter $\omega_{\text{BD}}=0$.)

- The effective Lagrangian of TEGR:

$${}^{(5)}e\,{}^{(5)}T$$

$$\rightarrow \quad e\left(\phi\,T+2\,T^{\mu}\,\partial_{\mu}\phi\right).$$

- By substituting the curvature-torsion relation $-\tilde{R}(e)=T-2\,\tilde{\nabla}_{\mu}T^{\mu}$ into TEGR

$$\Rightarrow \quad \frac{-1}{2\kappa_4}\int d^4x\,e\left(\phi\tilde{R}(e)-2\,\tilde{\nabla}_{\mu}(\phi\,T^{\mu})\right),$$

which is equivalent to $\phi\tilde{R}$ up to the total derivative term.

Conformal Transformation

- By the conformal transformation ($\tilde{g}_{\mu\nu} = \Omega^2(x)\, g_{\mu\nu}$):

$$\begin{aligned} T &= \Omega^2\, \tilde{T} - 4\, \Omega\, \tilde{g}^{\mu\nu}\, \tilde{T}_\mu \partial_\nu \Omega - 6\, \tilde{g}^{\mu\nu}\, \partial_\mu \Omega\, \partial_\nu \Omega\,, \\ T_\mu &= \tilde{T}_\mu - 2\, \Omega^{-1}\, \partial_\mu \Omega\,. \end{aligned}$$

- Choosing $\phi = \Omega^2$, the action reads

$$S_{\text{eff}} = \int d^4x\, \tilde{e}\, \left[\frac{1}{2\,\kappa_4}\, \tilde{T} - \tilde{g}^{\mu\nu} \partial_\mu \psi\, \partial_\nu \psi \right]\,,$$

  where $\psi = (7/\kappa_4)^{1/2} \ln \Omega$ is the dilaton field.

- There exists an Einstein frame for such a non-minimal coupled effective Lagrangian in teleparallel gravity.

Equation of Motion

- The gravitational equation of motion

$$\frac{1}{2}\, e_i^{\mu}\Big(\phi\, T + 2\, T^{\sigma}\, \partial_{\sigma}\phi\Big) - e_i^{\rho}\Big(\phi\, T^{j}{}_{\rho\nu}\, S_j{}^{\mu\nu}\Big) - e_i^{\nu}\Big(\partial_{\sigma}\phi\, T^{\mu}{}_{\nu}{}^{\sigma} + \partial_{\nu}\phi\, T^{\mu} + \partial^{\mu}\phi\, T_{\nu}\Big) + \frac{1}{e}\,\partial_{\nu}\Big(e\,(\phi\, S_i{}^{\mu\nu} + e_i^{\mu}\,\partial^{\nu}\phi - e_i^{\nu}\,\partial^{\mu}\phi)\Big) = \kappa_4\,\Theta_i^{\mu}\,,$$

  where $\Theta_i^{\mu} = (-1/e)(\delta\,\mathcal{L}_m/\delta\, e_{\mu}^{i})$ with $\Theta_{\nu}^{\mu} = \mathrm{diag}(\rho, -P, -P, -P)$
- Consider flat FLRW universe, $g_{\mu\nu} = \mathrm{diag}(1, -a^2(t), -a^2(t), -a^2(t))$ and vierbein $e_{\mu}^{i} = \mathrm{diag}(1, a(t), a(t), a(t))$.
- The modified Friedmann equations are

$$3\,\phi\, H^2 + 3\, H\,\dot{\phi} = \kappa_4\,\rho\,,$$
$$3\,\phi\, H^2 + 2\,\dot{\phi}\, H + 2\,\phi\,\dot{H} + \ddot{\phi} = -\,\kappa_4\, P\,,$$

  where $H = \dot{a}/a$ is the Hubble parameter.

- The equation of motion of scalar field ϕ

$$T - 2\,\nabla_\mu T^\mu = 0 \xRightarrow[\Gamma^\mu_{\rho\mu}=0]{} T - 2\,\partial_0 T^0 = 0\,.$$

- Suppose the solution of $a(t)$ is proportional to $(t - t_r)^m$

$$a(t) = a_s + \frac{b}{t - t_r}\,.$$

- The constraint of the coefficient: $a_s\,b = 0$
- $b = 0$ case:
  - $a(t) = a_s \Rightarrow$ the static universe.
- $a_s = 0$ case:
  - The expansion condition of Hubble parameter $H = -1/(t - t_r) > 0$ $\Rightarrow t_r > t$, where t_r is future characteristic time of the scale factor.
  - The the acceleration of scale factor $\ddot{a} = 2\,b/(t - t_r)^3 > 0$ for $b < 0$ $\Rightarrow$ accelerated expanding universe.

Outline

Summary

- In GR, the extrinsic curvature plays an important role to give the projected effect in the lower dimension.
- The effect on the lower dimensional manifold is totally determined by a higher dimensional geometry for TEGR.
- Braneworld theory of teleparallel gravity in the FLRW cosmology still provides an equivalent viewpoint as Einstein's general relativity.
- The KK reduction of telaparallel gravity generates an additional coupling term in the effective Lagrangian.
- There exists an Einstein frame for the non-minimal coupled teleparallel gravity by adding the additional coupling.
- If the time t is smaller than the future characteristic time t_r, it leads to an accelerated expanding universe.

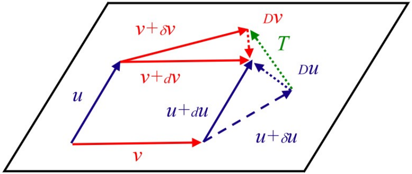
End

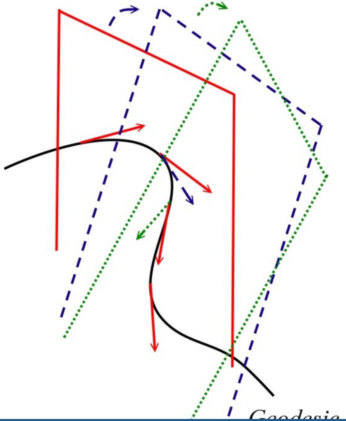
Thank You!!!

Outline

Geometrical Meaning of Torsion

- Torsion free: a tangent vector does not rotate when we parallel transport it. (*P.371, John Baez and Javier P. Muniain, "Gauge Fields, Knots and Gravity," 1994*)
- $T(u,v) = \nabla_u v - \nabla_v u - \underbrace{[u,v]}_{\text{vanished in coordinate space}}$





Riemann-Cartan Geometry U_4

- Einstein's general relativity was established in 1915 and described on Riemannian geometry V_4 with metric $g_{\mu\nu}$ and the metric-compatible Levi-Civita connection $\{{}^{\rho}_{\mu\nu}\} = \frac{1}{2}g^{\rho\lambda}(g_{\nu\lambda,\mu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda})$.
- The metric-compatible affine connection in U_4 is

$$\Gamma^{\rho}_{\mu\nu} = \{{}^{\rho}_{\mu\nu}\} + K^{\rho}{}_{\mu\nu}\,,$$

which can be decomposed into torsion and torsionless parts.

- Torsion: couple to spin angular momentum in gravity (*Élie Cartan 1922*.)
- Introducing the orthonormal frame $g_{\mu\nu} = \eta_{ij}e^{i}_{\mu}e^{j}_{\nu}$, and the relation of affine and *spin* connections in U_4 is

$$\Gamma^{\nu}_{\rho\mu} = e^{\nu}_{i}\partial_{\mu}e^{i}_{\rho} + e^{\nu}_{i}\,\omega^{i}{}_{j\mu}\,e^{j}_{\rho}.$$

Affine Connection and Spin Connection

- Consider noncoordinate basis (orthonormal frame)

$$\begin{aligned}
e_i^\nu D_\mu V^i &= e_i^\nu(\partial_\mu V^i + \omega^i{}_{j\mu} V^j) \\
&= e_i^\nu[\partial_\mu(e_\rho^i V^\rho) + \omega_\mu{}^i{}_j V^j] \\
&= e_i^\nu[(\partial_\mu e_\rho^i)V^\rho + e_\rho^i(\partial_\mu V^\rho) + \omega^i{}_{j\mu} e_\rho^j V^\rho] \\
&= (e_i^\nu \partial_\mu e_\rho^i)V^\rho + \underbrace{\delta_\rho^\nu \partial_\mu V^\rho}_{\partial_\mu V^\nu} + e_i^\nu \omega^i{}_{j\mu} e_\rho^j V^\rho] \\
&= \partial_\mu V^\nu + (e_i^\nu \partial_\mu e_\rho^i + e_i^\nu \omega^i{}_{j\mu} e_\rho^j)V^\rho \\
&\equiv \partial_\mu V^\nu + \Gamma^\nu_{\rho\mu} V^\rho = \nabla_\mu V^\nu.
\end{aligned}$$

The relation between affine connection and spin connection

$$\Gamma^\nu_{\rho\mu} \equiv e_i^\nu \partial_\mu e_\rho^i + e_i^\nu \omega^i{}_{j\mu} e_\rho^j$$

- We have the definition of the total covariant derivative $\boldsymbol{\nabla}_\mu$

$$\Rightarrow \quad \partial_\mu e_\rho^i - \Gamma^\nu_{\rho\mu} e_\nu^i + \omega^i{}_{j\mu} e_\rho^j = 0$$

$$\Rightarrow \quad \boldsymbol{\nabla}_\mu e_\rho^i = 0 \text{ (vielbein postulate).}$$

Poincaré Gauge Theory (PGT)

- $\delta_0\phi = (\frac{1}{2}\,\omega(x)\cdot M + \xi(x)\cdot P)\phi$
- Gauge fields are $\omega^i{}_j = \omega^i{}_{j\mu}dx^\mu$ and $\theta^i = e^i_\mu dx^\mu$
- Fields strength is

$$D\circ D = \mathcal{R}^{ij}M_{ij} + \mathcal{T}^i P_i \quad \text{or} \quad [D_\rho, D_\sigma] = R^{ij}{}_{\rho\sigma}M_{ij} - T^i{}_{\rho\sigma}D_i$$

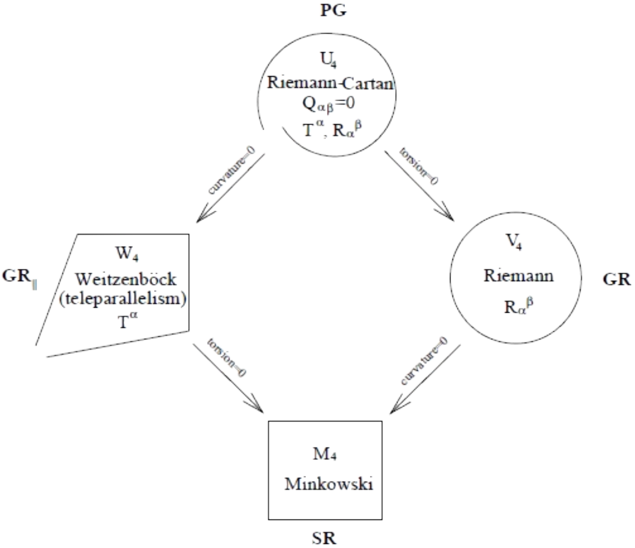
  where M_{ij} and D_i are *rotational* and *translational* generators, respectively.
- Cartan equations: $\mathcal{R}^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$ and $\mathcal{T}^i = D\theta^i$.

Einstein-Cartan-Sciama-Kibble (ECSK) Theory

The simplest Poincaré gauge theory:

$$S_{\text{ECSK}} = \int d^4x\; e\left[-\frac{1}{2\kappa}R(e,\omega)\right]$$

- ECSK extension: include *supersymmetry* and massless *Rarita-Schwinger field* (*Rarita & Schwinger 1941*) → 4D Supergravity.



Different gravitational theories with geometry (arXiv:9602013[gr-qc]).

Brief History of 5-Dimensional Theories

- Kaluza-Klein (KK) theory: to unify electromagnetism and gravity by gauge theory
  - Cylindrical condition (*Kaluza 1921*)
  - Compactification to a small scale (*Klein 1926*)
- Generalization of KK: induced-matter theory
  $\Rightarrow$ matter from the 5th-dimension (*Wesson 1998*)
- Large Extra dimension (*Arkani-Hamed, Dimopoulos and Dvali (ADD) 1998*)
  - Solving hierarchy problem
  - SM particles confined on the 3-brane

- Randall-Sundrum model in AdS_5 spacetime (*Randall and Sundrum 1999*)
  - RS-I (UV-brane and SM particles confined on IR-brane)
    $\Rightarrow$ solving hierarchy problem
  - RS-II (only one UV brane)
    $\Rightarrow$ compactification to generate 4-dimensional gravity
- DGP brane model (*Dvali, Gabadadze and Porrati 2000*)
  $\Rightarrow$ accelerating universe
- Universal Extra Dimension (*Appelquist, Cheng and Dobrescu 2001*)
  - Not only graviton but SM particles can propagate to the extra dimension $\Rightarrow$ low compactification scale: reach to the electroweak scale