Lensing by (the most) massive structures in the universe

- today: basics of lensing
- tomorrow: how can we investigate the matter content of galaxy clusters using lensing (strong and weak)
- Wednesday: cosmology with galaxy clusters

Lecture 1: a concise introduction to gravitational lensing

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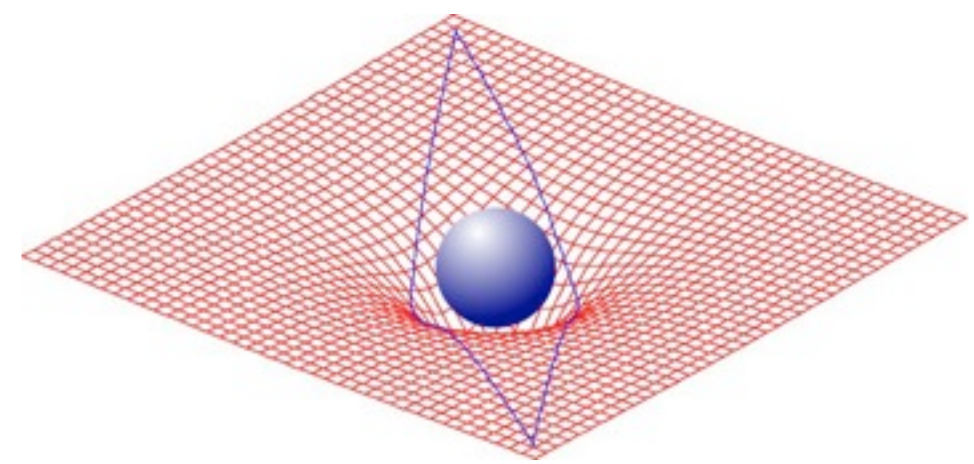
Gravitational lensing

- General relativity (GR): explains gravity in terms of assemblies of mass and energy curving the space-time



- photons feel gravity similarly to massive particles

- how can we formalize this?

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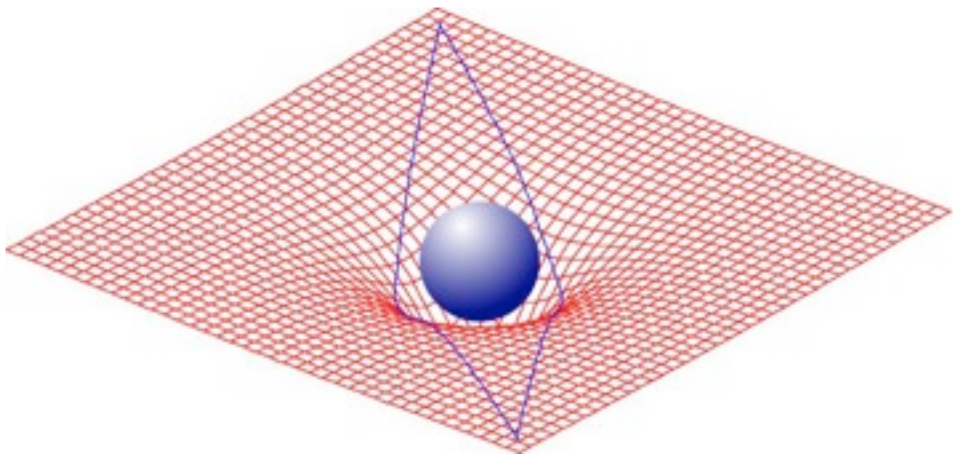

- how can we formalize this?

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = (\mathrm{d}x^0)^2 - (\mathrm{d}\vec{x})^2 = c^2\mathrm{d}t^2 - (\mathrm{d}\vec{x})^2$$

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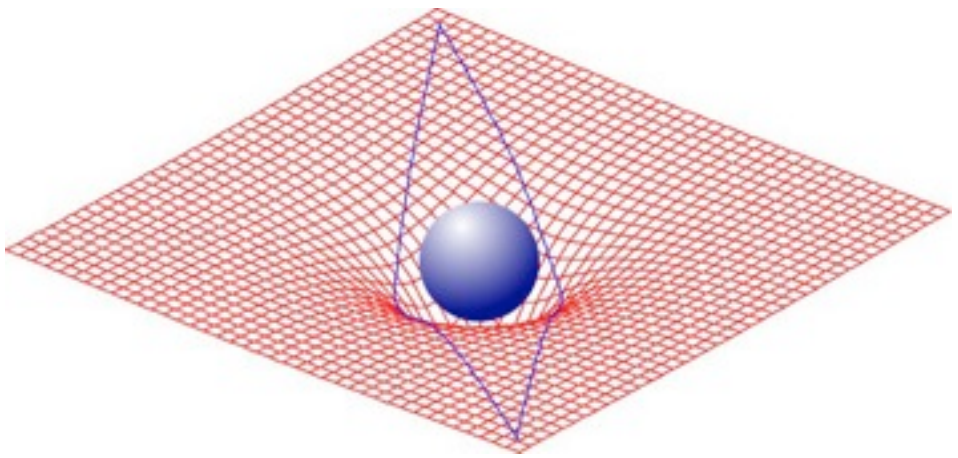

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$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

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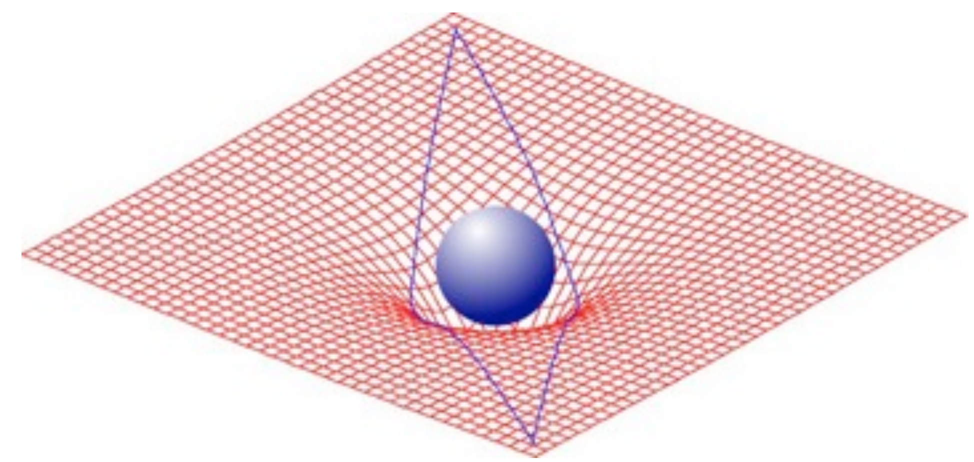

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$$\eta_{\mu\nu} \rightarrow g_{\mu\nu} = \begin{pmatrix} 1 + \frac{2\Phi}{c^2} & 0 & 0 & 0 \\ 0 & -(1 - \frac{2\Phi}{c^2}) & 0 & 0 \\ 0 & 0 & -(1 - \frac{2\Phi}{c^2}) & \\ 0 & 0 & 0 & -(1 - \frac{2\Phi}{c^2}) \end{pmatrix}$$

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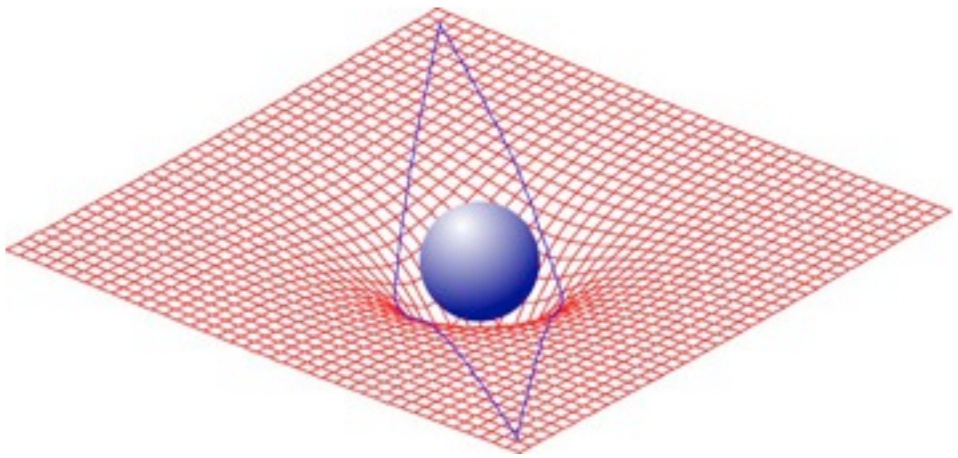

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \left(1 + \frac{2\Phi}{c^2}\right) c^2 \mathrm{d}t^2 - \left(1 - \frac{2\Phi}{c^2}\right) (\mathrm{d}\vec{x})^2$$

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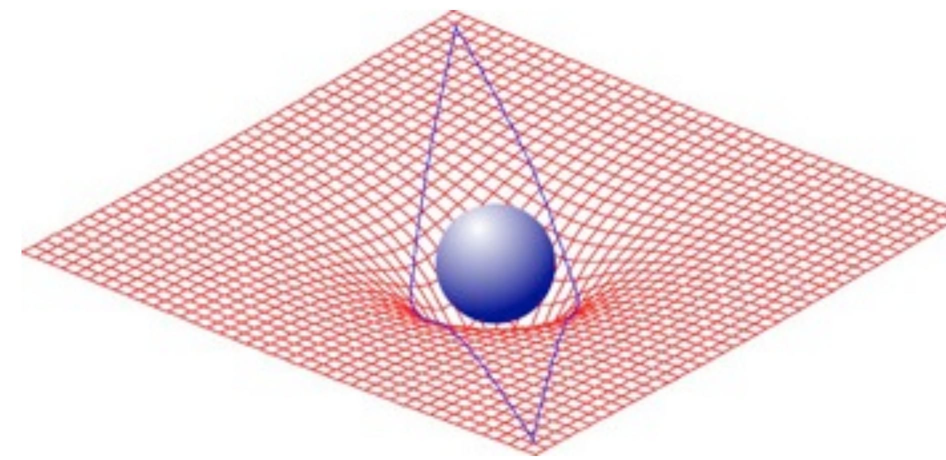

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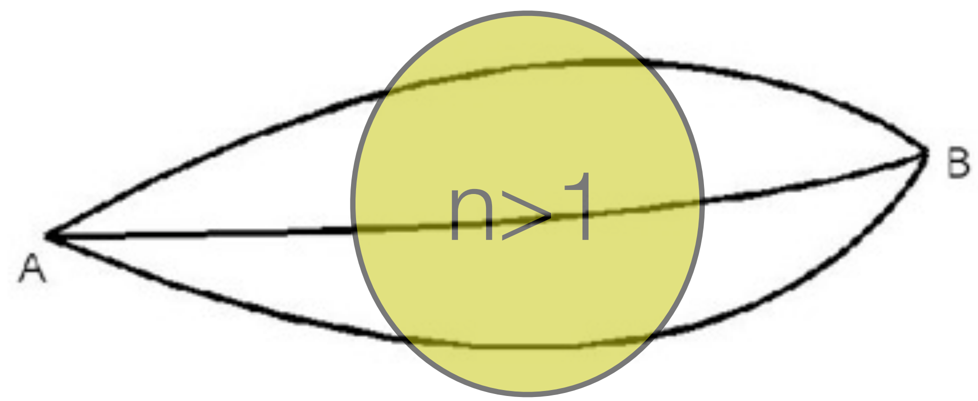
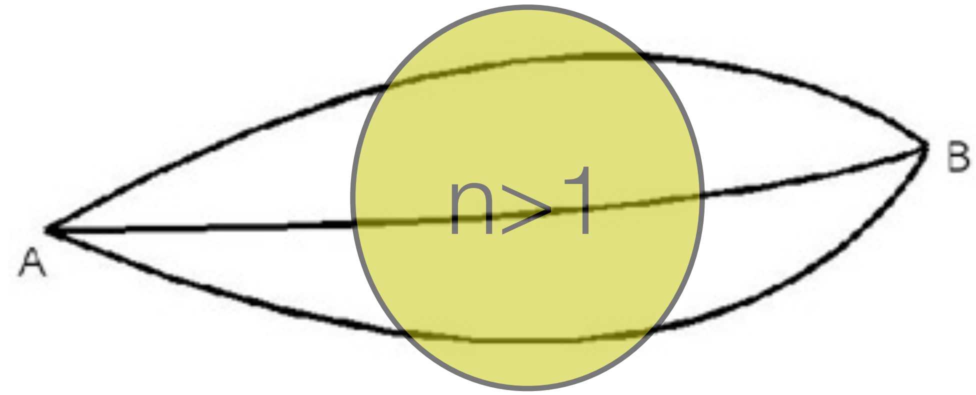
$$\left(1 + \frac{2\Phi}{c^2}\right) c^2 \mathrm{d}t^2 = \left(1 - \frac{2\Phi}{c^2}\right) (\mathrm{d}\vec{x})^2$$

$$c' = \frac{|\mathrm{d}\vec{x}|}{\mathrm{d}t} = c\sqrt{\frac{1 + \frac{2\Phi}{c^2}}{1 - \frac{2\Phi}{c^2}}} \approx c\left(1 + \frac{2\Phi}{c^2}\right) \qquad n = c/c' = \frac{1}{1 + \frac{2\Phi}{c^2}} \approx 1 - \frac{2\Phi}{c^2}$$

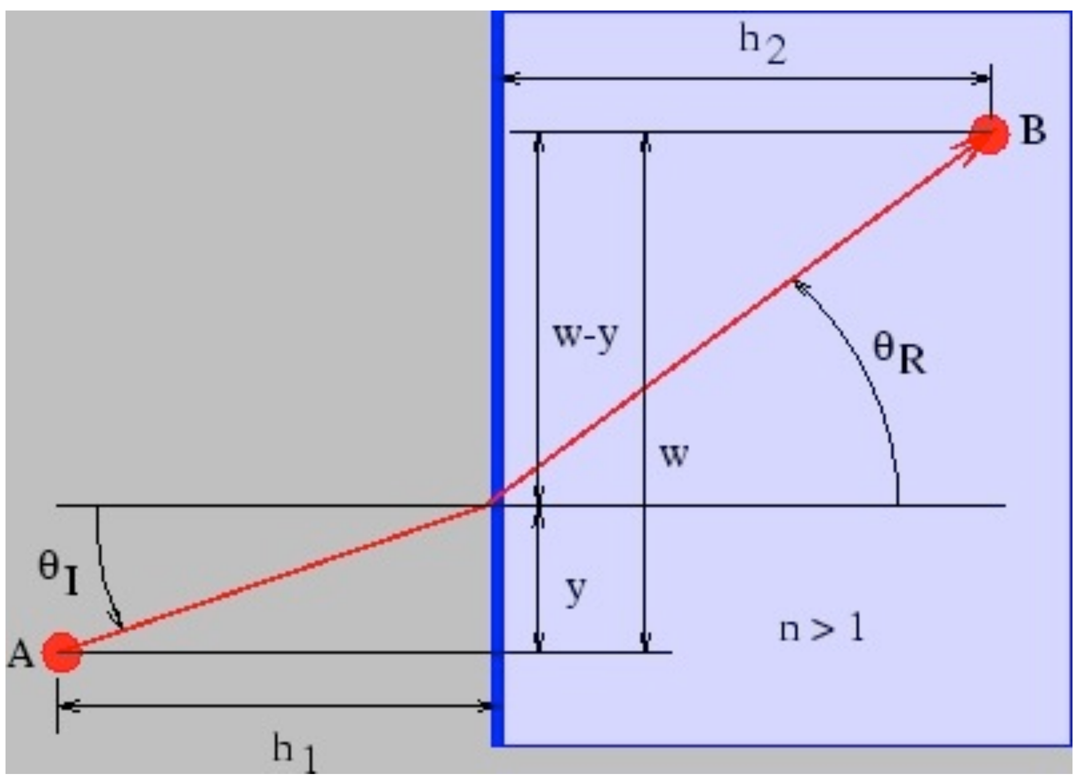
Fermat principle



Fermat principle

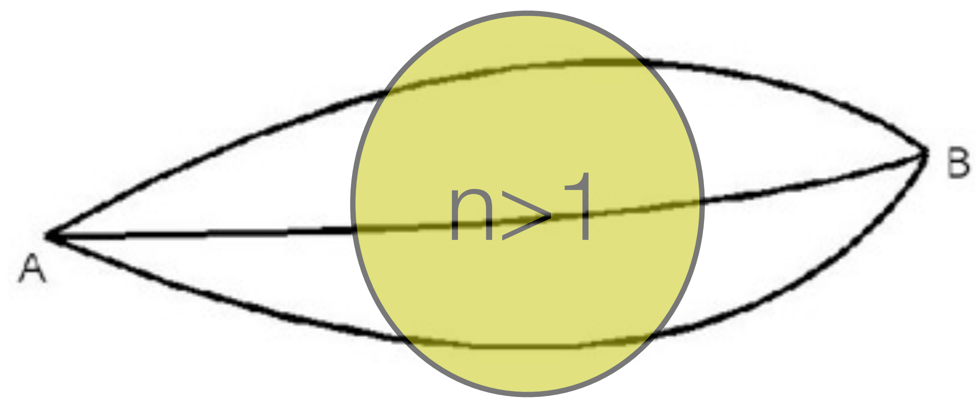


Classical optics: Snell law



$$\sin\theta_I = n\sin\theta_R$$

Fermat principle



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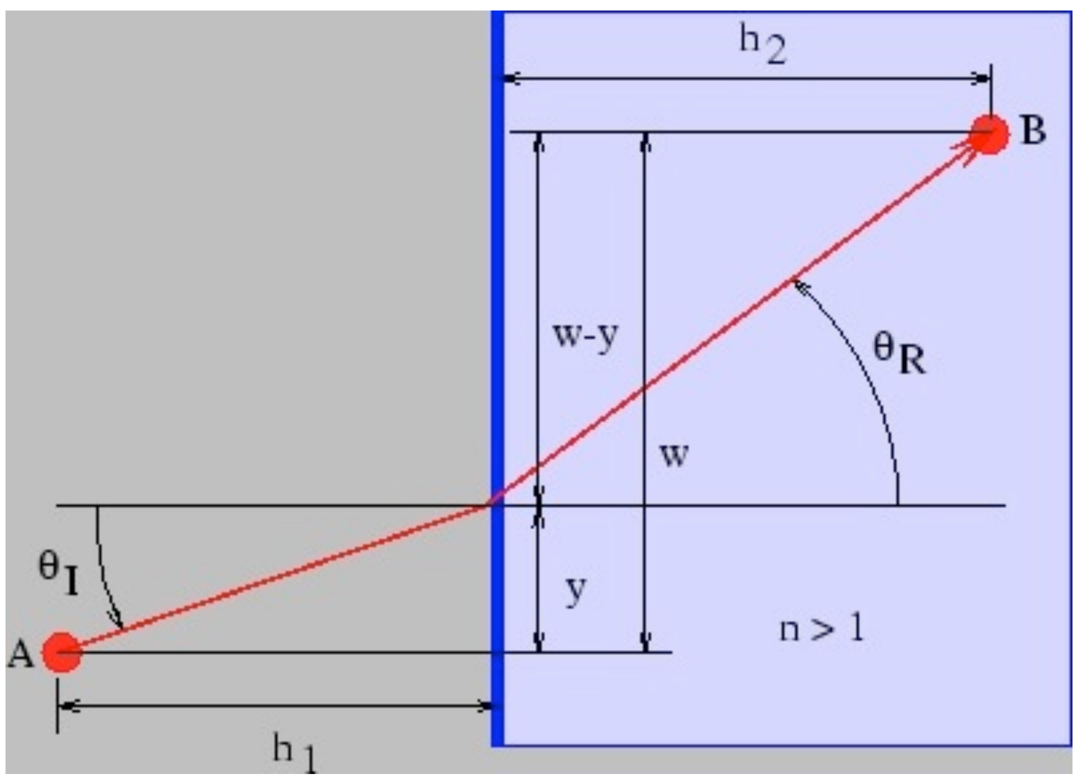


$$\sin\theta_I = n\sin\theta_R$$

General relativity: deflection angle

$$\vec{\hat{\alpha}} = -\int \vec{\nabla}_\perp n\, dl = \frac{2}{c^2}\int \vec{\nabla}_\perp \Phi\, dl$$

$$\vec{\hat{\alpha}}(b) = \frac{2}{c^2}\int_{-\infty}^{+\infty} \vec{\nabla}_\perp \phi \mathrm{d}z$$

Deflection angle for a point mass

$$\Phi = -\frac{GM}{r} \qquad r = \sqrt{x^2 + y^2 + z^2} = \sqrt{b^2 + z^2} \qquad b = \sqrt{x^2 + y^2}$$

$$\vec{\nabla}_\perp \Phi = \begin{pmatrix} \partial_x \Phi \\ \partial_y \Phi \end{pmatrix} = \frac{GM}{r^3} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{aligned} \hat{\vec{\alpha}}(b) &= \frac{2GM}{c^2} \begin{pmatrix} x \\ y \end{pmatrix} \int_{-\infty}^{+\infty} \frac{dz}{(b^2 + z^2)^{3/2}} \\ &= \frac{4GM}{c^2} \begin{pmatrix} x \\ y \end{pmatrix} \left[\frac{z}{b^2 (b^2 + z^2)^{1/2}} \right]_0^\infty = \frac{4GM}{c^2 b} \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \end{aligned}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = b \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}$$

Deflection angle for the general lens

For an ensamble of point masses:

$$\hat{\vec{\alpha}}(\vec{\xi}) = \sum_i \hat{\vec{\alpha}}_i(\vec{\xi} - \vec{\xi}_i) = \frac{4G}{c^2} \sum_i M_i \frac{\vec{\xi} - \vec{\xi}_i}{|\vec{\xi} - \vec{\xi}_i|^2}$$

For a more general three-dimensional distribution of matter, using the *thin screen approximation*:

$$\Sigma(\vec{\xi}) = \int \rho(\vec{\xi}, z)\ \mathrm{d}z$$

$$\hat{\vec{\alpha}}(\vec{\xi}) = \frac{4G}{c^2} \int \frac{(\vec{\xi} - \vec{\xi'})\Sigma(\vec{\xi'})}{|\vec{\xi} - \vec{\xi'}|^2}\ \mathrm{d}^2\xi'$$

Lensing potential

$$\hat{\Psi}(\vec{\theta}) = \frac{D_{\mathsf{LS}}}{D_{\mathsf{L}} D_{\mathsf{S}}} \frac{2}{c^2} \int \Phi(D_{\mathsf{L}}\vec{\theta}, z)\mathrm{d}z$$

$$\begin{aligned} \vec{\nabla}_\theta \hat{\psi} &= D_L \vec{\nabla}_\xi \hat{\psi} \\ &= \frac{2}{c^2} \frac{D_{LS}}{D_S} \int \vec{\nabla}_\perp \Phi dz \\ &= \vec{\alpha} \end{aligned}$$

$$\triangle\Phi = 4\pi G\rho$$

$$\Sigma(\vec{\theta}) = \frac{1}{4\pi G} \int_{-\infty}^{+\infty} \triangle\Phi \mathrm{d}z$$

$$\Sigma_{\mathsf{cr}} = \frac{c^2}{4\pi G} \frac{D_{\mathsf{S}}}{D_{\mathsf{L}} D_{\mathsf{LS}}}$$

$$\kappa(\vec{\theta}) = \frac{1}{c^2} \frac{D_{\mathsf{L}} D_{\mathsf{LS}}}{D_{\mathsf{S}}} \int_{-\infty}^{+\infty} \triangle\Phi \mathrm{d}z$$

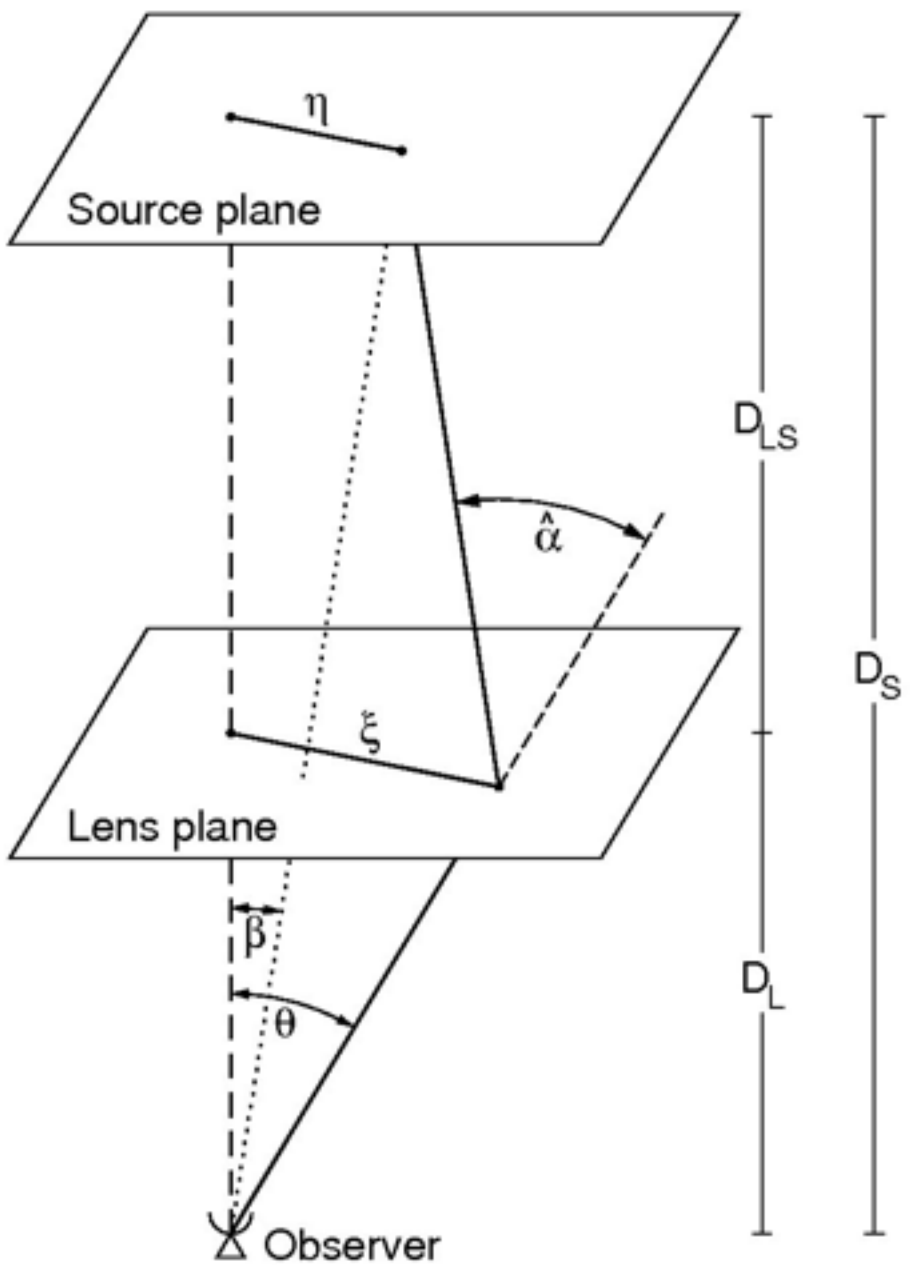
$$\vec{\alpha} = \vec{\nabla}_\theta \hat{\psi}$$

$$\kappa(\theta) = \frac{1}{2} \triangle_\theta \hat{\Psi}$$

Some examples

Lens Model	$\psi(\theta)$	$\alpha(\theta)$
Point mass	$\frac{D_{\rm ds}}{D_{\rm s}} \frac{4GM}{D_{\rm d}c^2} \ln\|\theta\|$	$\frac{D_{\rm ds}}{D_{\rm s}} \frac{4GM}{c^2 D_{\rm d}\|\theta\|}$
Singular isothermal sphere	$\frac{D_{\rm ds}}{D_{\rm s}} \frac{4\pi\sigma^2}{c^2} \|\theta\|$	$\frac{D_{\rm ds}}{D_{\rm s}} \frac{4\pi\sigma^2}{c^2}$
Softened isothermal sphere	$\frac{D_{\rm ds}}{D_{\rm s}} \frac{4\pi\sigma^2}{c^2} \left(\theta_{\rm c}^2 + \theta^2\right)^{1/2}$	$\frac{D_{\rm ds}}{D_{\rm s}} \frac{4\pi\sigma^2}{c^2} \frac{\theta}{(\theta_{\rm c}^2 + \theta^2)^{1/2}}$
Constant density sheet	$\frac{\kappa}{2}\theta^2$	$\kappa\|\theta\|$

Lens equation

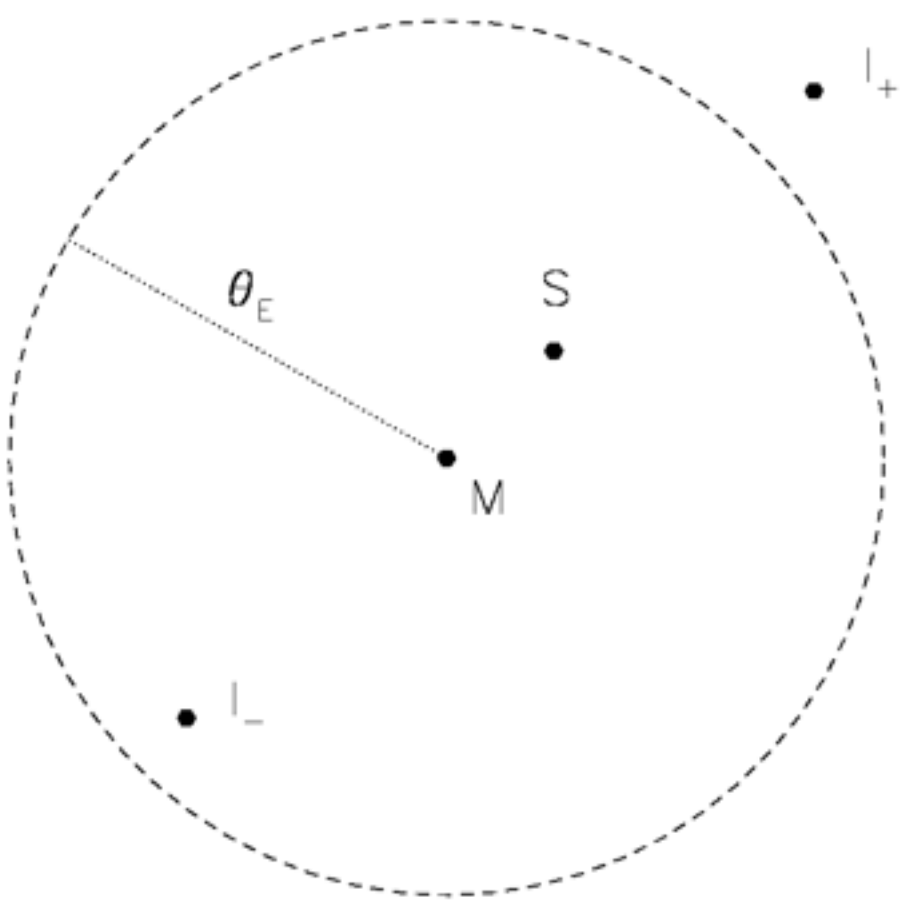


The lens equation links the true and the apparent positions of the source when it is lensed by a matter distribution.

$$\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta})$$

Point mass

$$\Psi(\theta) = \frac{D_{LS}}{D_S}\frac{4GM}{D_L c^2}\ln|\theta| \qquad \theta_E \equiv \sqrt{\frac{4GM}{c^2}\frac{D_{LS}}{D_L D_S}} \qquad \beta = \theta - \frac{\theta_E^2}{\theta}$$



multiple images and magnification!

$$\theta_\pm = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_E^2}\right)$$

More complicated lenses: time delay

Let's go back a few slides and consider again the concept of effective refraction index:

$$n = c/c' = \frac{1}{1+\frac{2\Phi}{c^2}} \approx 1 - \frac{2\Phi}{c^2}$$

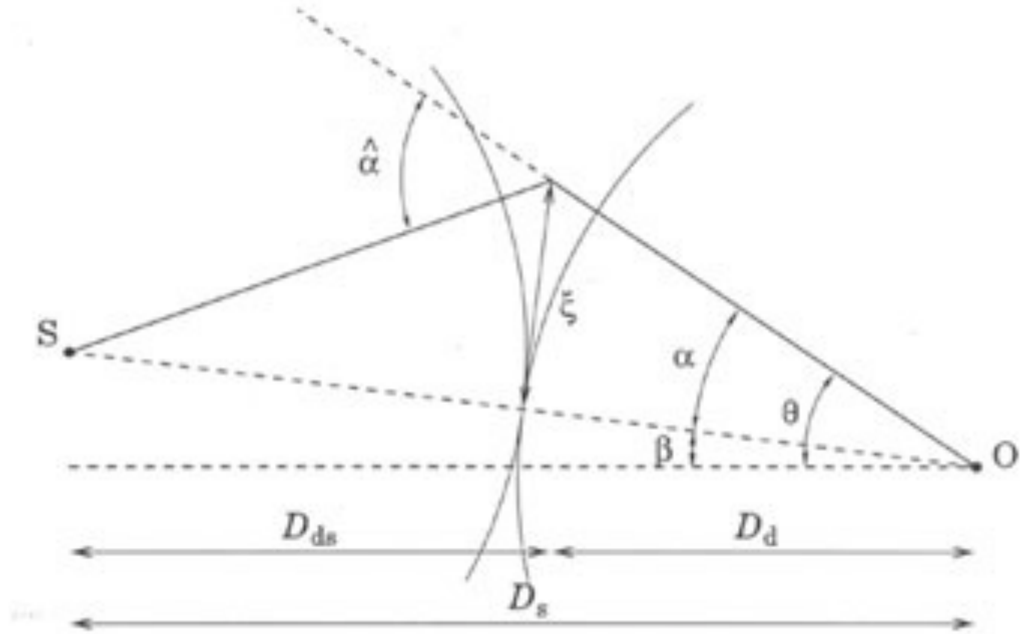
This concept implies that, when photons travel along a ray, they will appear to travel slower than in vacuum and will take an extra-time to pass by a massive object:

$$\Delta t_{grav} = -\frac{2}{c^3}\int \Phi dz$$

In a cosmological context:

$$\begin{aligned}\Delta t_{grav} &= -\frac{2}{c^3}(1+z_d)\int \Phi dz \\ &\propto (1+z_d)\Psi\end{aligned}$$

(Shapiro delay)



An additional contribution to the time delay comes from the extra geometrical path followed by light when it gets deflected towards the observer:

$$\Delta s = \vec{\xi}\frac{\hat{\vec{\alpha}}}{2}$$

Again, putting this in a cosmological context:

$$\begin{aligned}\Delta t_{geom} &= (1+z_d)\frac{D_d D_{ds}}{2D_s c}\hat{\alpha}^2 \\ &\propto (1+z_d)(\vec{\theta}-\vec{\beta})^2\end{aligned}$$

More complicated lenses: time delays

$$(\vec{\theta} - \vec{\beta}) - \vec{\nabla}_\theta \psi = 0 \qquad \vec{\nabla}_\theta \left[\frac{1}{2} (\vec{\theta} - \vec{\beta})^2 - \psi \right] = 0$$

More complicated lenses: time delays

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$$\begin{aligned} t(\vec{\theta}) &= \frac{(1 + z_L)}{c} \frac{D_L D_S}{D_{LS}} \left[\frac{1}{2} (\vec{\theta} - \vec{\beta})^2 - \psi(\vec{\theta}) \right] \\ &= t_{geom} + t_{grav} \end{aligned}$$

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The solutions of the lens equation correspond to the stationary points of the time delay function (maxima, minima, saddle points)

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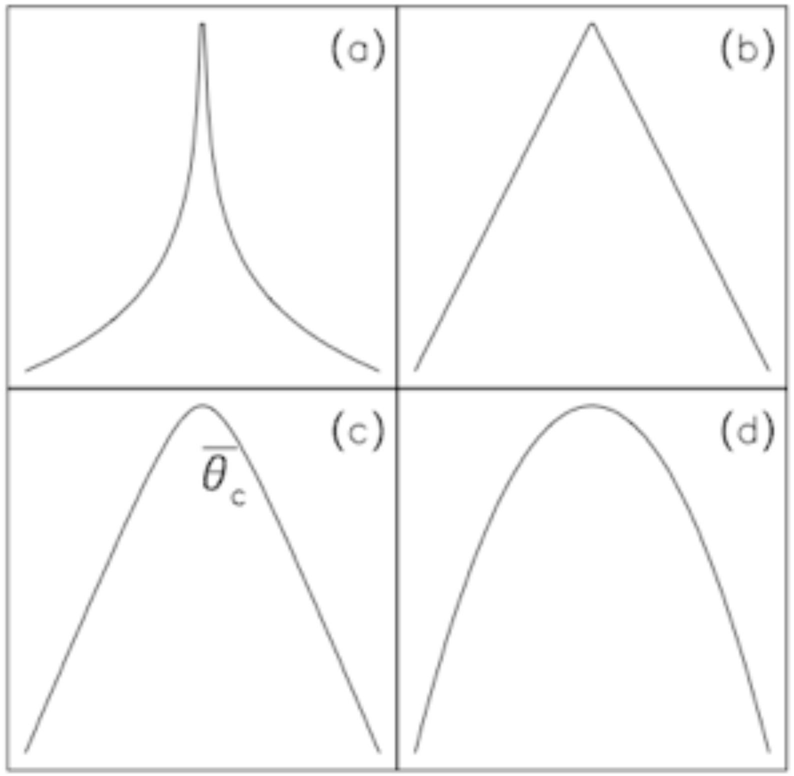
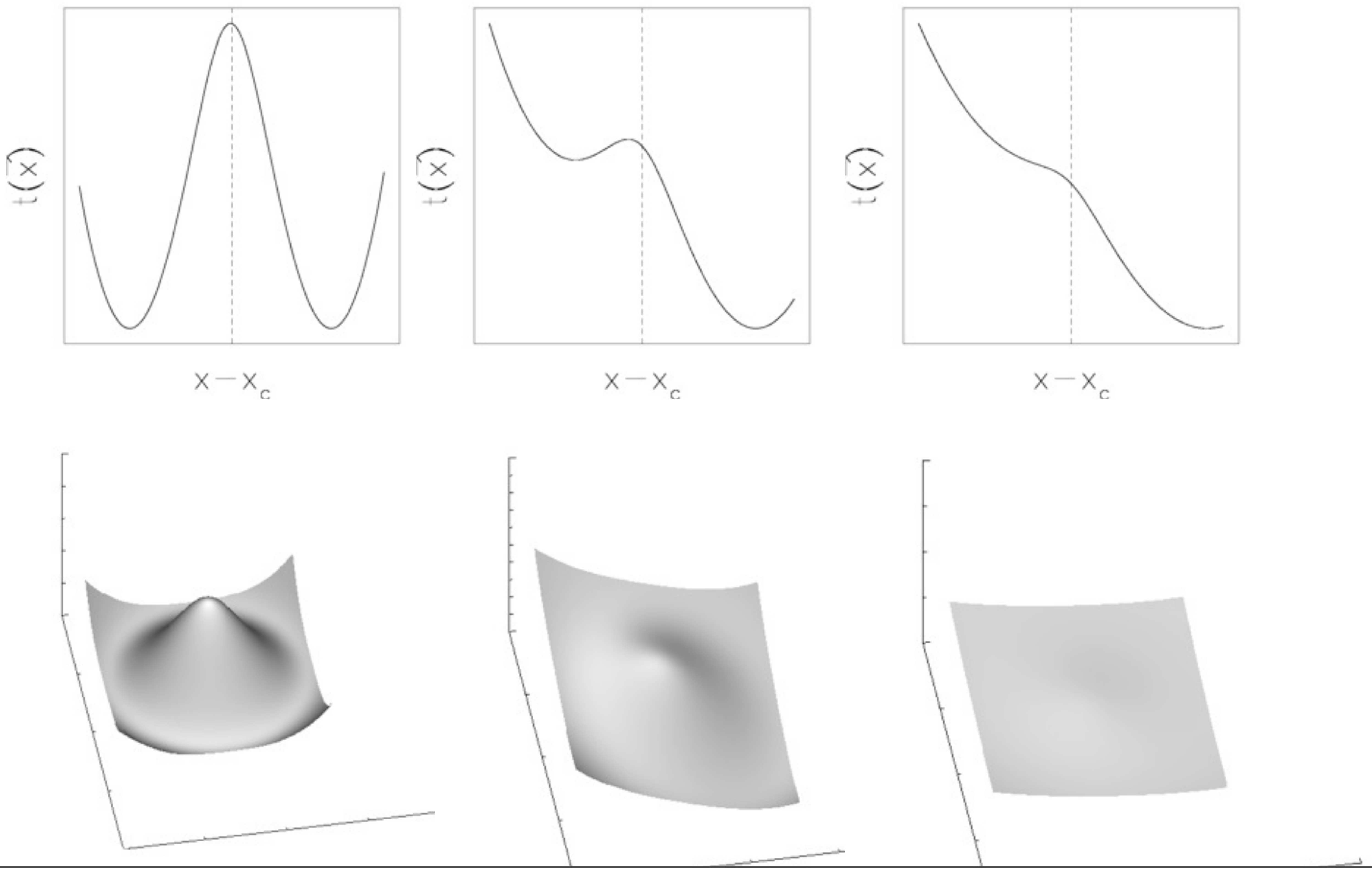


Time delay surfaces



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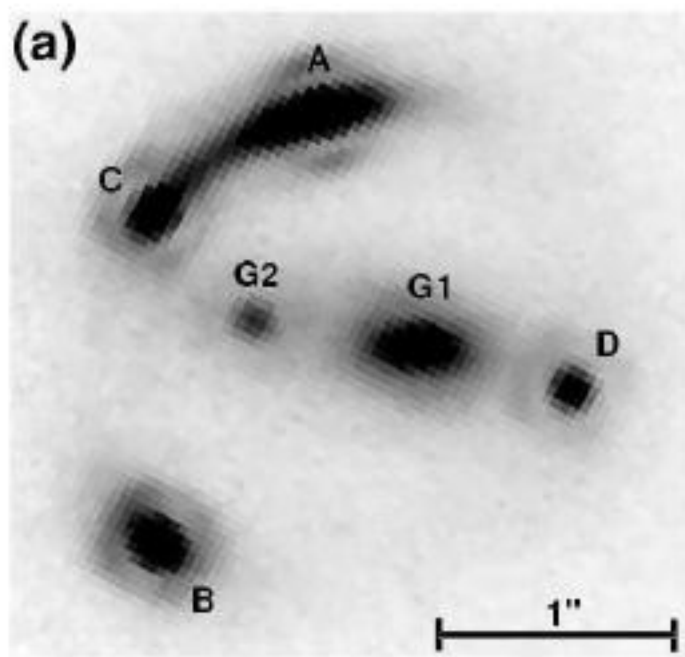

Koopmans et al. 2003

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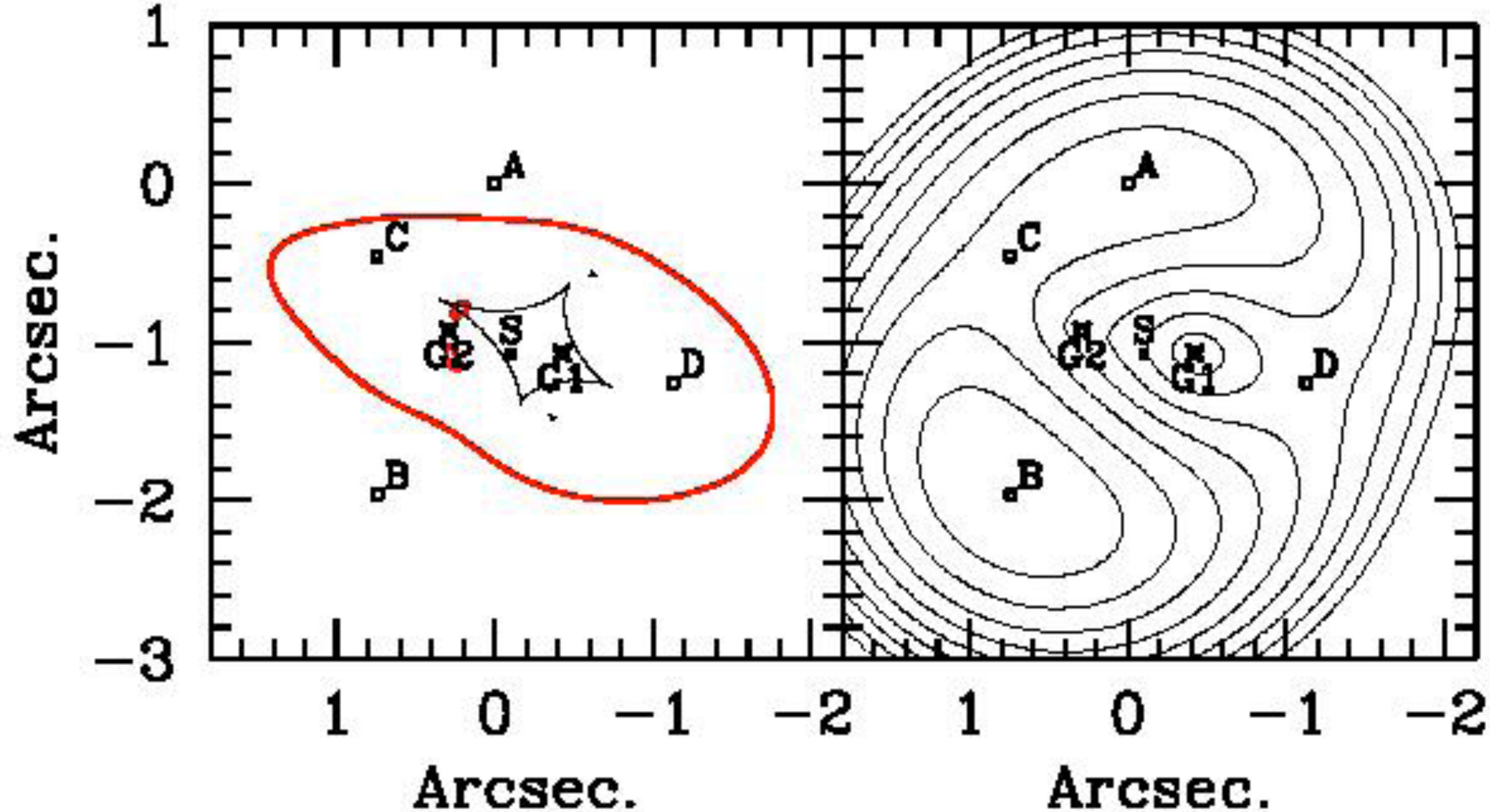


Koopmans et al. 2003

Lens mapping and distortions

Consider the limit of small deflections: in this case the lens equation can be written as:

$$\begin{aligned}\vec{\beta} &= \vec{\theta} - \vec{\alpha}(\vec{\theta}) \\ \vec{\beta} &= \frac{\partial \vec{\beta}}{\partial \vec{\theta}} \vec{\theta} = A\vec{\theta}\end{aligned}$$

$$\mathcal{A} \equiv \frac{\partial \vec{\beta}}{\partial \vec{\theta}} = \left(\delta_{ij} - \frac{\partial \alpha_i(\vec{\theta})}{\partial \theta_j} \right) = \left(\delta_{ij} - \frac{\partial^2 \psi(\vec{\theta})}{\partial \theta_i \partial \theta_j} \right)$$

Lens mapping and distortions

$$\frac{\partial^2 \psi}{\partial\theta_i \partial\theta_j} \equiv \psi_{ij}$$

convergence

$$\kappa = \frac{1}{2}(\psi_{11} + \psi_{22})$$

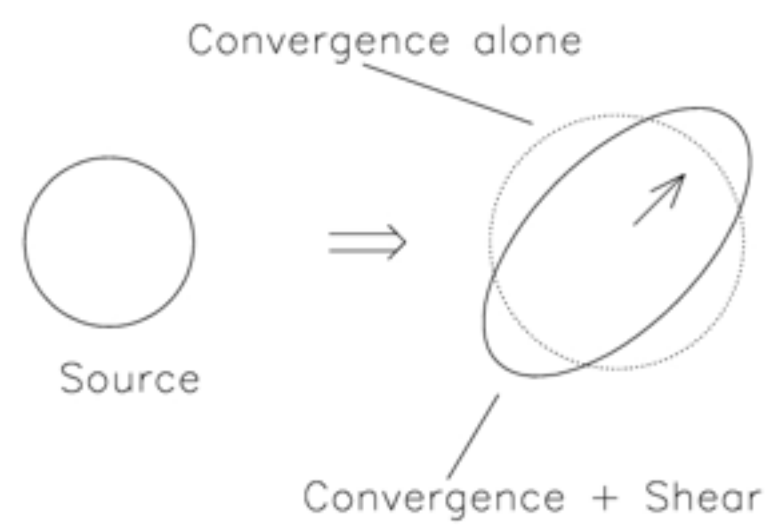
Shear

$$\gamma_1(\vec{\theta}) = \frac{1}{2}(\psi_{11} - \psi_{22}) \equiv \gamma(\vec{\theta}) \cos\left[2\phi(\vec{\theta})\right] ,$$
$$\gamma_2(\vec{\theta}) = \psi_{12} = \psi_{21} \equiv \gamma(\vec{\theta}) \sin\left[2\phi(\vec{\theta})\right] .$$

$$\mathcal{A} = \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix} = (1-\kappa)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \gamma \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix}$$

Eigenvalues

$$\lambda_t = 1 - \kappa - \gamma$$
$$\lambda_r = 1 - \kappa + \gamma$$



Critical lines and caustics

The inverse of the determinant Jacobian measures the magnification:

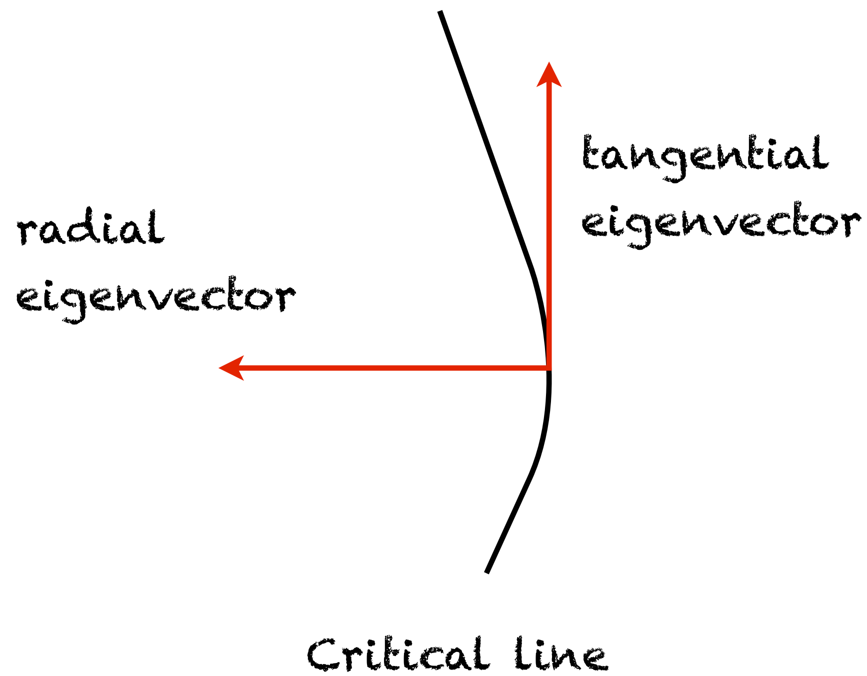
$$\frac{\delta\theta^2}{\delta\beta^2} = \det \mathcal{M} = \frac{1}{\det \mathcal{A}}$$

The magnification diverges where the eigenvalues of A vanish:

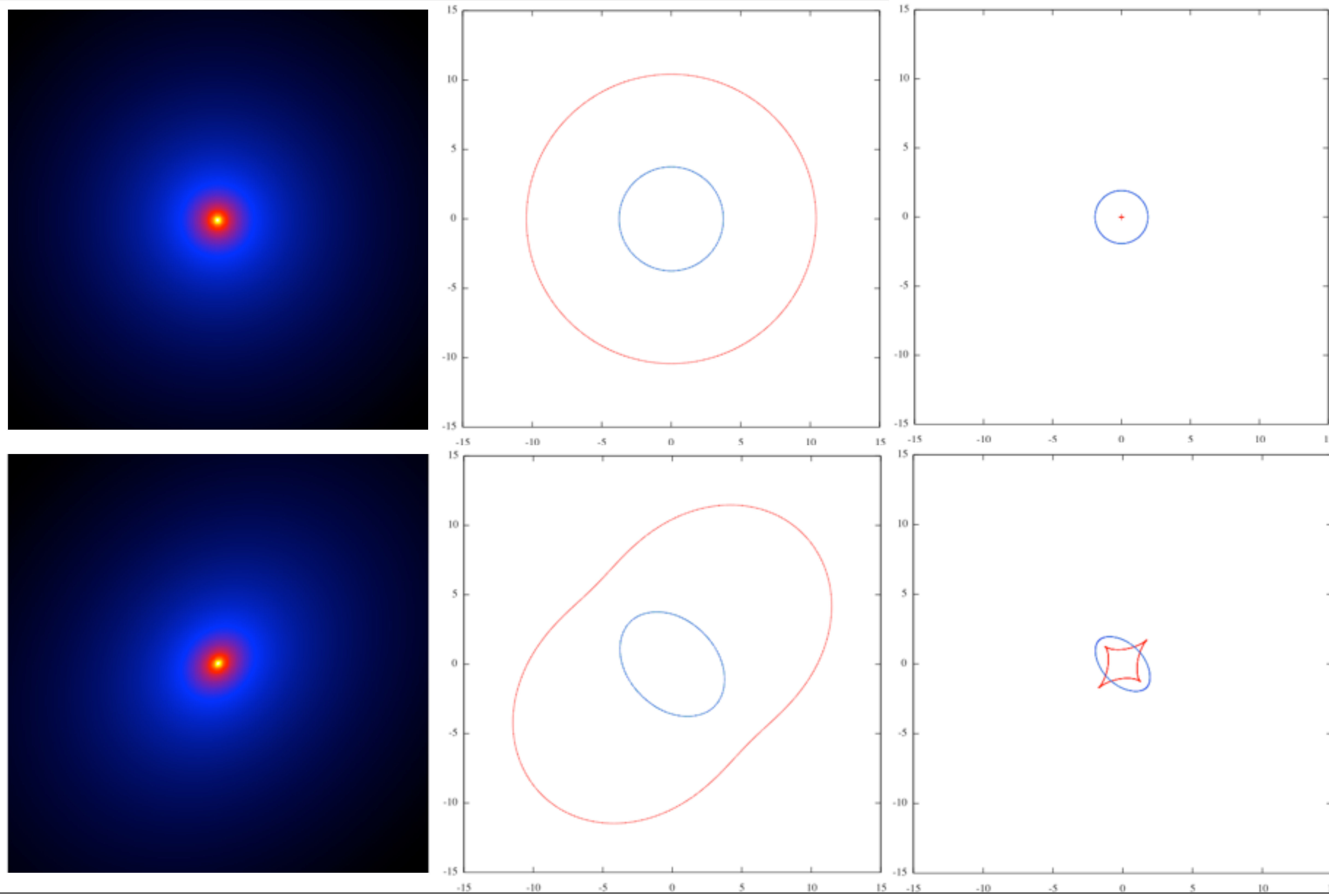
$$\begin{aligned} \lambda_t &= 1 - \kappa - \gamma \\ \lambda_r &= 1 - \kappa + \gamma \\ \Rightarrow & \quad \det A = 0 \end{aligned}$$

These two conditions define the critical lines

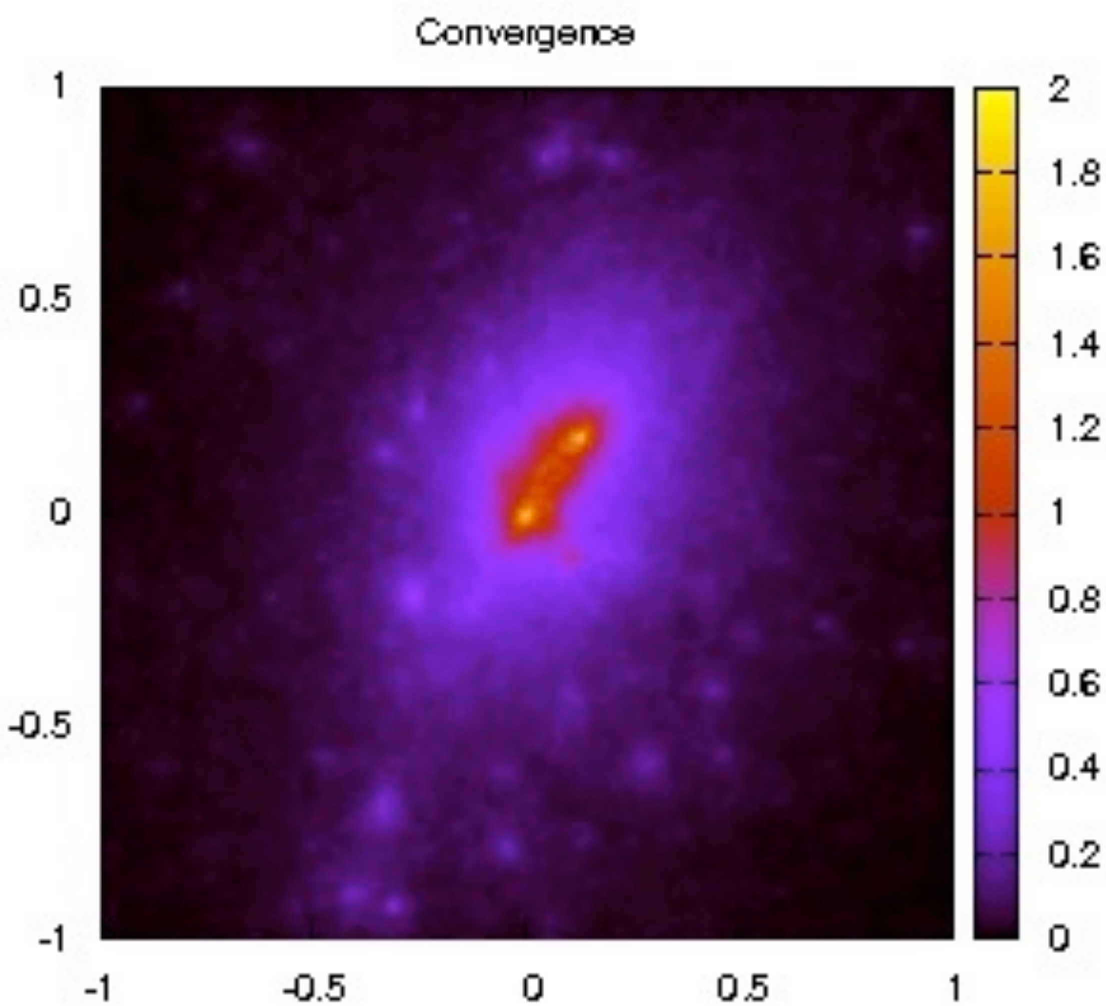
These are lines on the lens plane which are mapped on the caustics on the source plane



Examples



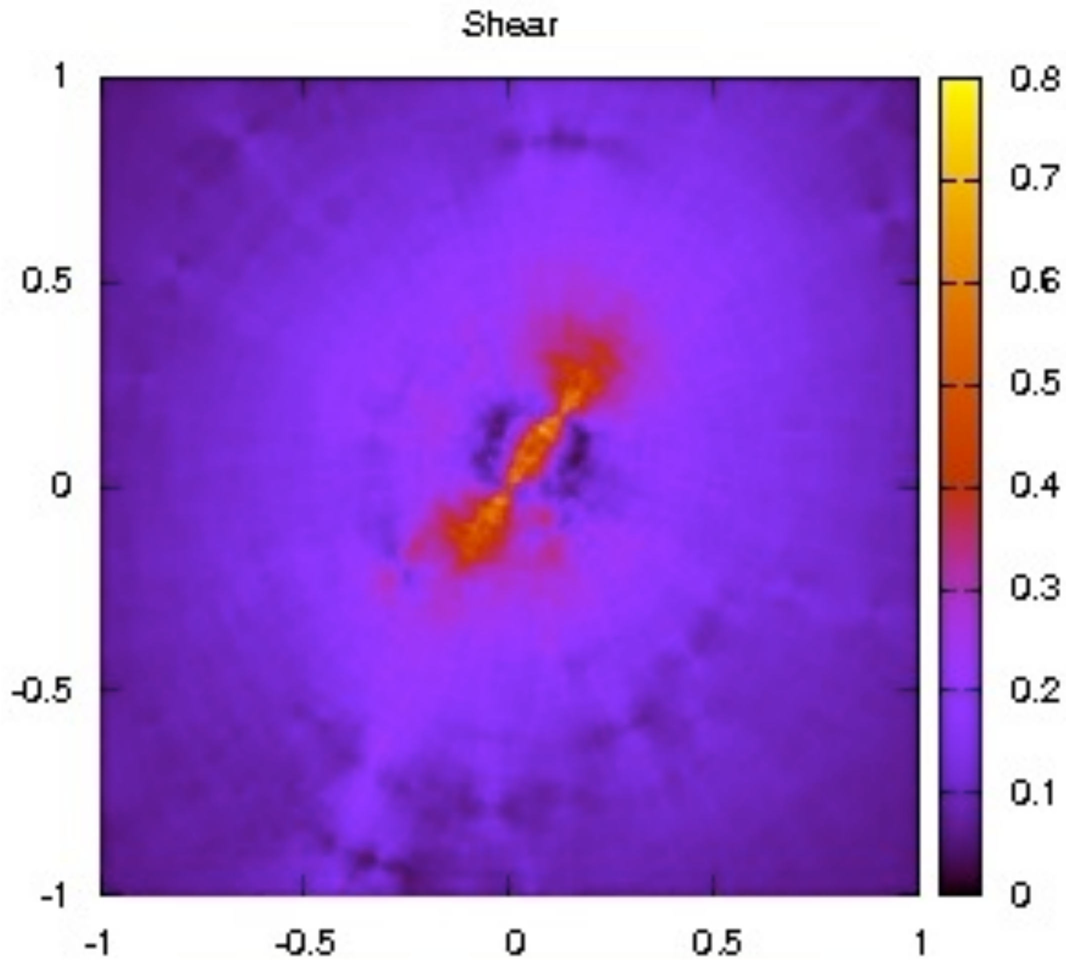
Convergence

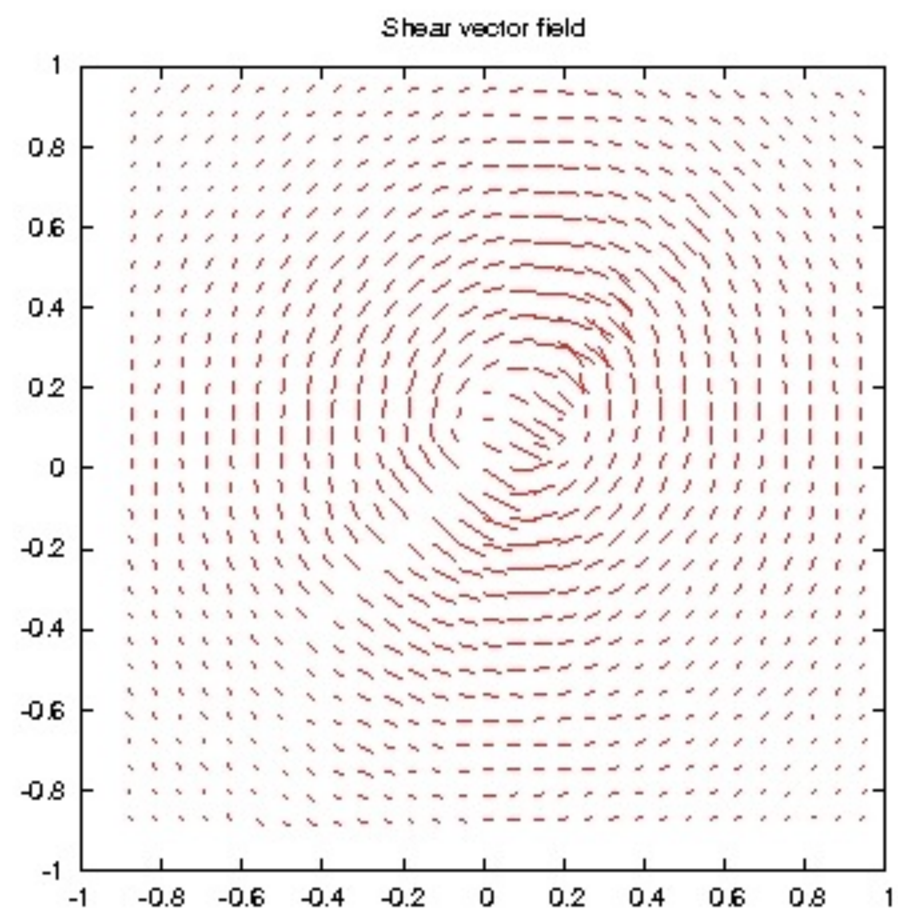


$$\kappa(\vec{x}) \equiv \frac{\Sigma(\vec{x})}{\Sigma_{\mathsf{cr}}} \quad \text{with} \quad \Sigma_{\mathsf{cr}} = \frac{c^2}{4\pi G}\frac{D_{\mathsf{S}}}{D_{\mathsf{L}} D_{\mathsf{LS}}}$$

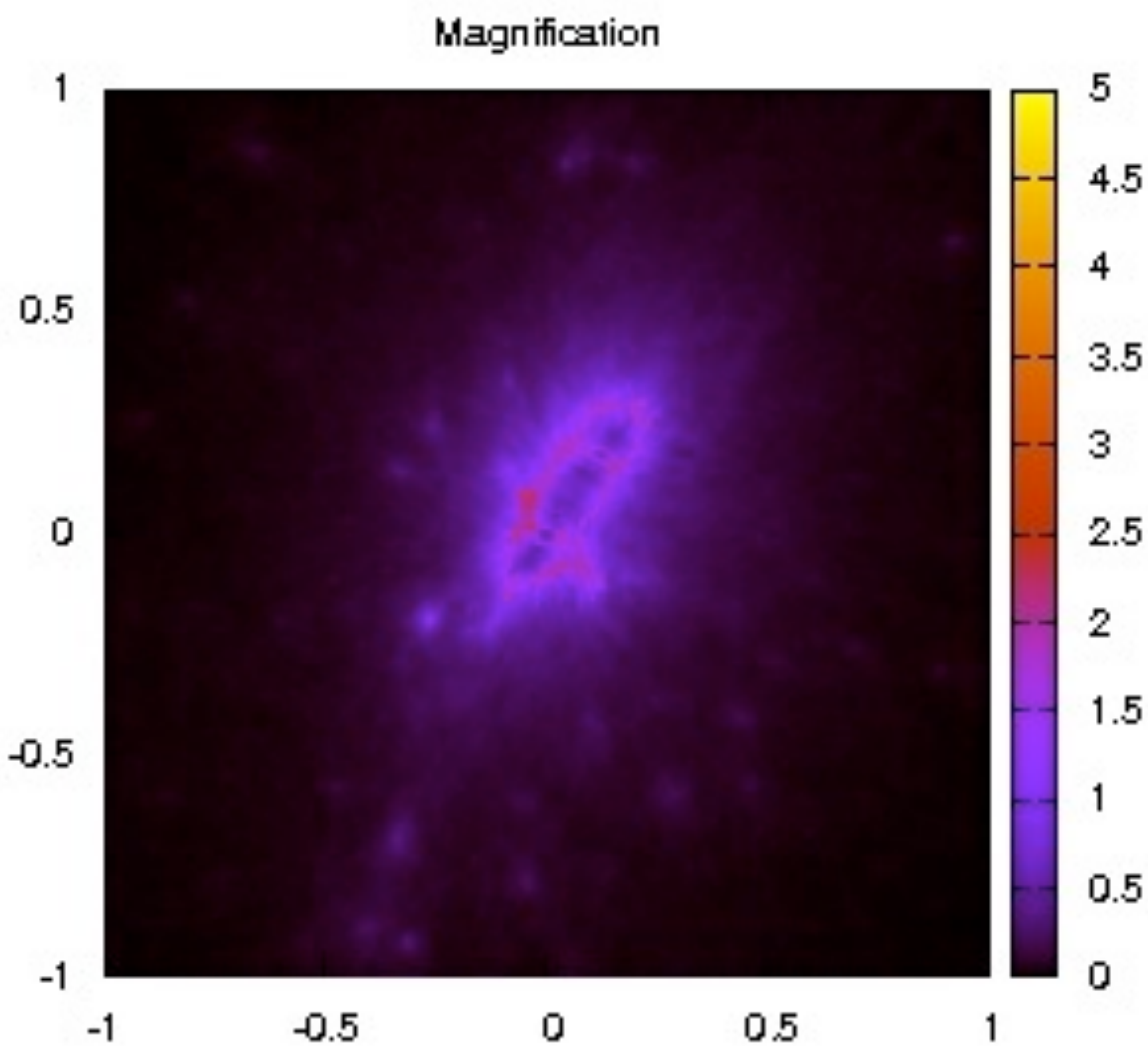
Shear

$$\gamma_1(\vec{x}) = \frac{1}{2}(\Psi_{11} - \Psi_{22})$$
$$\gamma_2(\vec{x}) = \Psi_{12} = \Psi_{21} \ ,$$



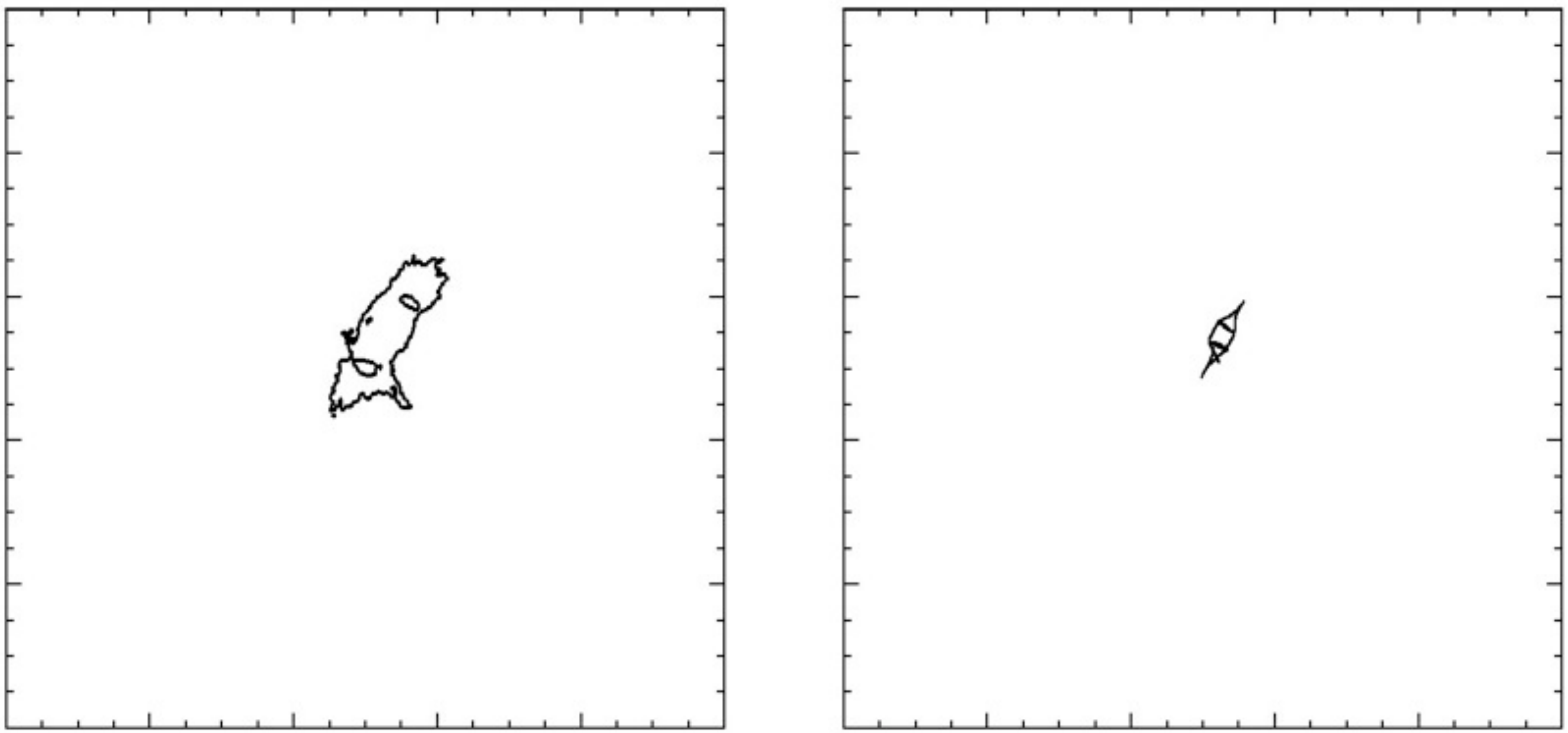


Magnification



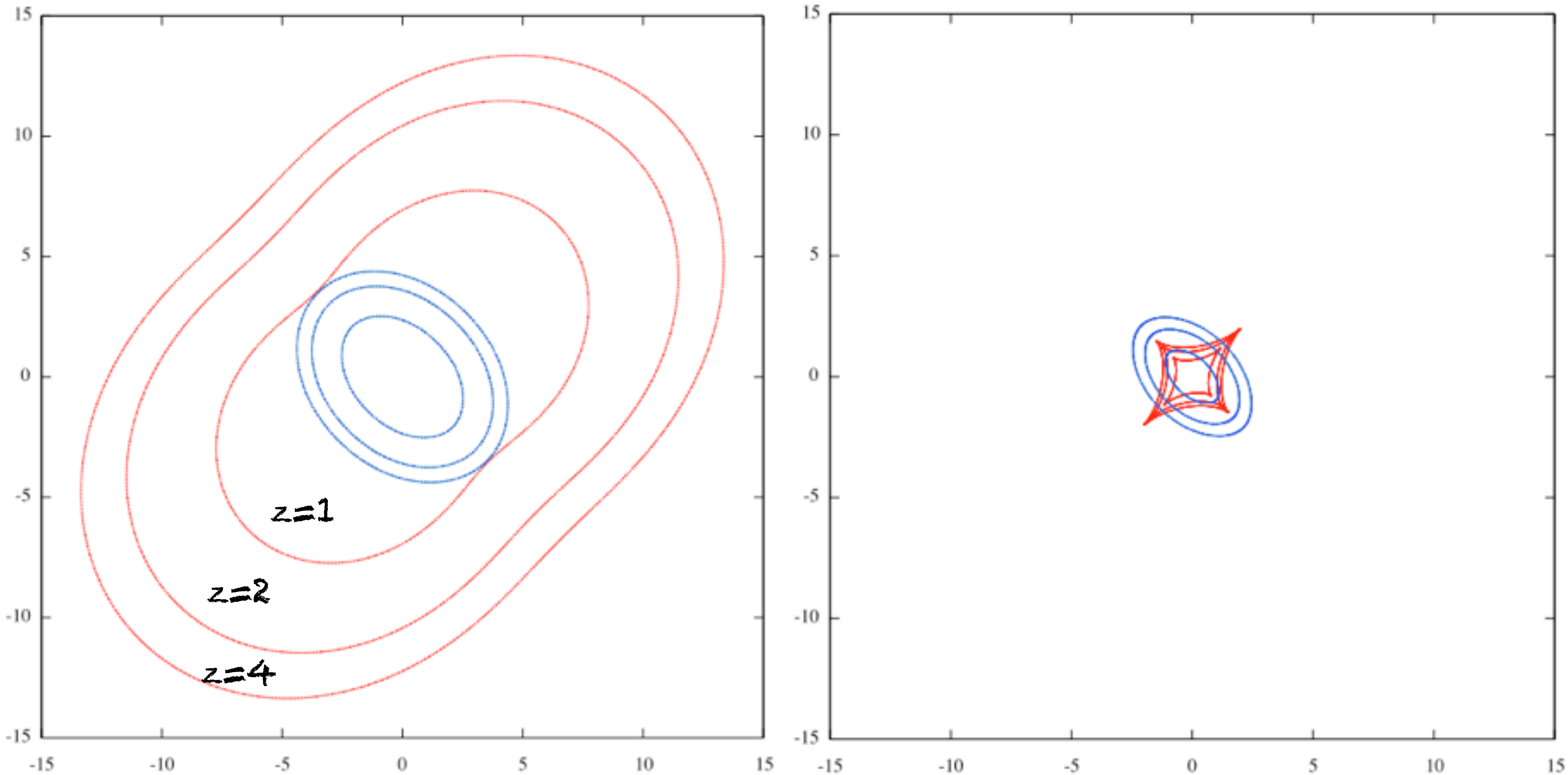
$$\mu \equiv \det M = \frac{1}{\det A} = \frac{1}{(1-\kappa)^2 - \gamma^2}$$

Critical lines and caustics



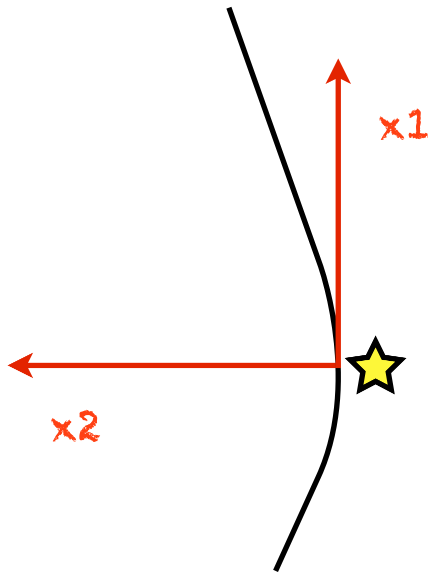
$$\mu_t = \frac{1}{\lambda_t} = \frac{1}{1-\kappa-\gamma}$$
$$\mu_r = \frac{1}{\lambda_r} = \frac{1}{1-\kappa+\gamma}$$

CL/CA vs source redshift



$$D = \frac{D_{\rm d} D_{\rm ds}}{D_{\rm s}}$$

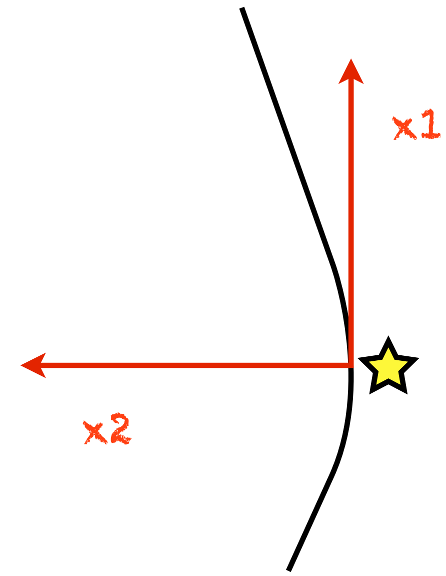
Tangential and radial distortions



Tangential
Critical line

$$\lambda_t = 1 - \kappa - \gamma = \delta \ll 1$$

$$A(0, x_c) = \begin{pmatrix} \delta & 0 \\ 0 & 1 - \kappa + \gamma \end{pmatrix}$$



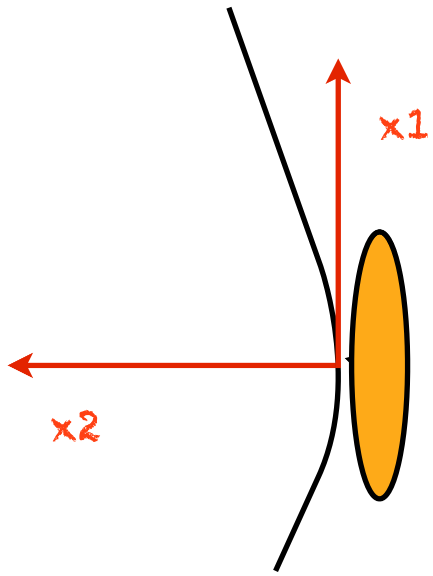
Radial
Critical line

$$\lambda_t = 1 - \kappa + \gamma = \delta \ll 1$$

$$A(0, x_c) = \begin{pmatrix} 1 - \kappa - \gamma & 0 \\ 0 & \delta \end{pmatrix}$$

If the source is circular, how is its elliptical image oriented?

Tangential and radial distortions



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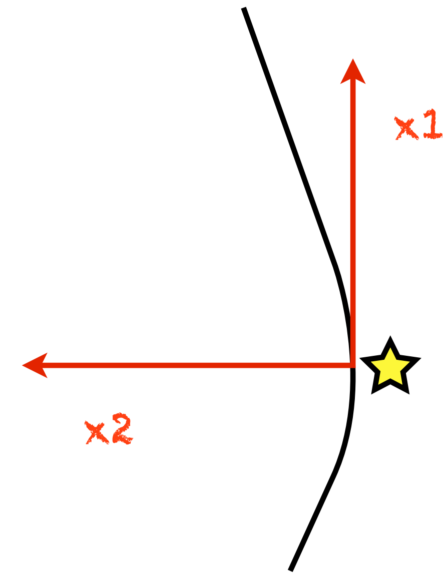


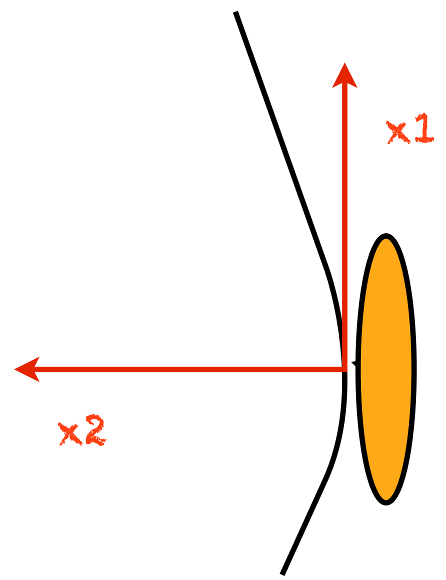
Radial
Critical line

$$\lambda_t = 1 - \kappa + \gamma = \delta \ll 1$$

$$A(0, x_c) = \begin{pmatrix} 1 - \kappa - \gamma & 0 \\ 0 & \delta \end{pmatrix}$$

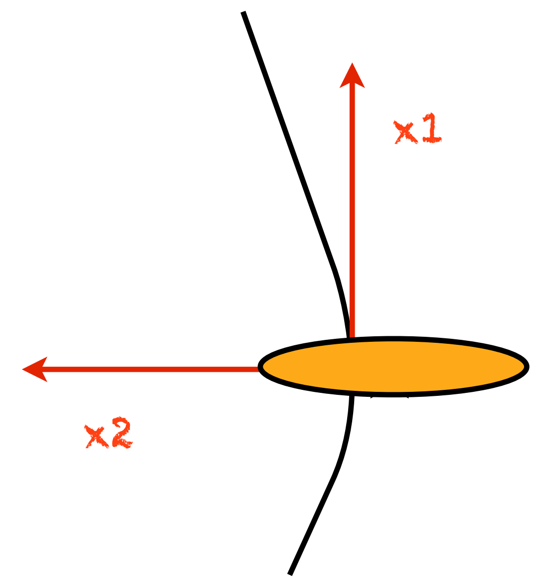
If the source is circular, how is its elliptical image oriented?

Tangential and radial distortions



$$\lambda_t = 1 - \kappa - \gamma = \delta \ll 1$$

$$A(0, x_c) = \begin{pmatrix} \delta & 0 \\ 0 & 1 - \kappa + \gamma \end{pmatrix}$$



$$\lambda_t = 1 - \kappa + \gamma = \delta \ll 1$$

$$A(0, x_c) = \begin{pmatrix} 1 - \kappa - \gamma & 0 \\ 0 & \delta \end{pmatrix}$$

If the source is circular, how is its elliptical image oriented?

Second order lensing

$$y_i \simeq \frac{\partial y_i}{\partial x_j} x_j + \frac{1}{2} \frac{\partial^2 y_i}{\partial x_j \partial x_k} x_j x_k$$

$$D_{ijk} = \frac{\partial^2 y_i}{\partial x_j \partial x_k} = \frac{\partial A_{ij}}{\partial x_k} \qquad y_i \simeq A_{ij} x_j + \frac{1}{2} D_{ijk} x_j x_k$$

$$D_{ij1} = \begin{pmatrix} -2\gamma_{1,1} - \gamma_{2,2} & -\gamma_{2,1} \\ -\gamma_{2,1} & -\gamma_{2,2} \end{pmatrix} \qquad D_{ij2} = \begin{pmatrix} -\gamma_{2,1} & -\gamma_{2,2} \\ -\gamma_{2,2} & 2\gamma_{1,2} - \gamma_{2,1} \end{pmatrix}$$

$$F = F_1 + iF_2 = (\gamma_{1,1} + \gamma_{2,2}) + i(\gamma_{2,1} - \gamma_{1,2})$$

First flexion

$$G = G_1 + iG_2 = (\gamma_{1,1} - \gamma_{2,2}) + i(\gamma_{2,1} + \gamma_{1,2})$$

Second flexion

$$\begin{aligned} \gamma_{1,1} &= \frac{1}{2}(\Psi_{111} - \Psi_{221}) \\ \gamma_{2,2} &= \Psi_{122} \\ \gamma_{2,1} &= \Psi_{121} \\ \gamma_{1,2} &= \frac{1}{2}(\Psi_{112} - \Psi_{222}) \end{aligned}$$

$$\begin{aligned} F_1 &= \frac{1}{2}(\Psi_{111} - \Psi_{221}) + \Psi_{122} = \frac{1}{2}(\Psi_{111} + \Psi_{221}) = \frac{\partial \kappa}{\partial x_1} \\ F_2 &= \frac{1}{2}(\Psi_{112} - \Psi_{222}) + \Psi_{121} = \frac{1}{2}(\Psi_{112} + \Psi_{222}) = \frac{\partial \kappa}{\partial x_2} \end{aligned}$$

$$\vec{F} = \vec{\nabla} \kappa$$

Second order lensing

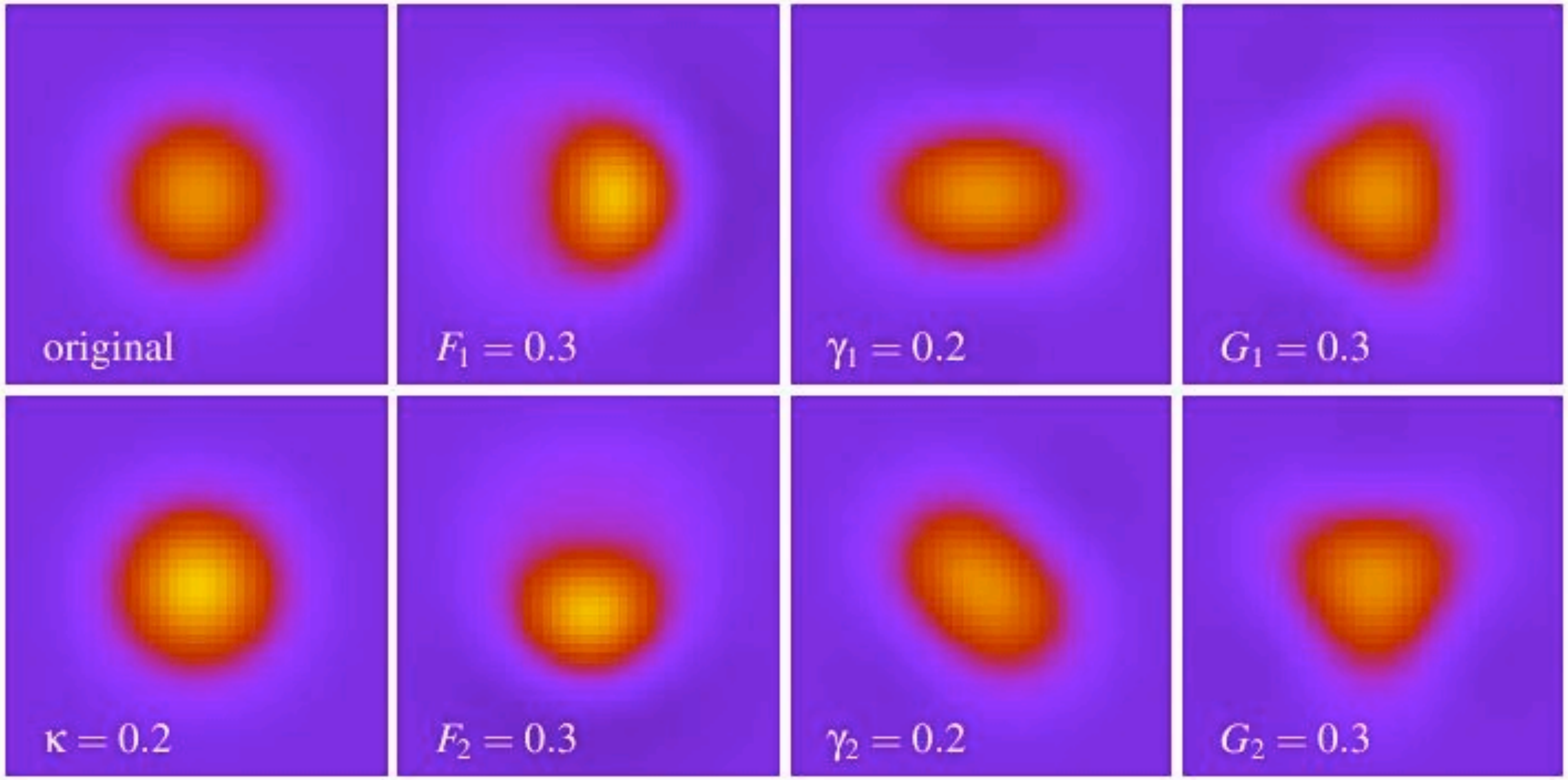
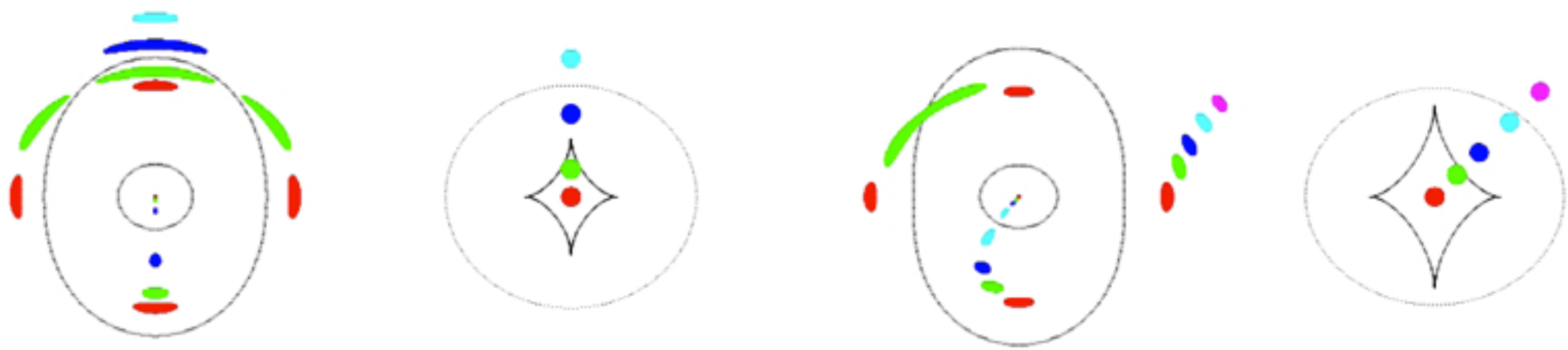


Image configurations



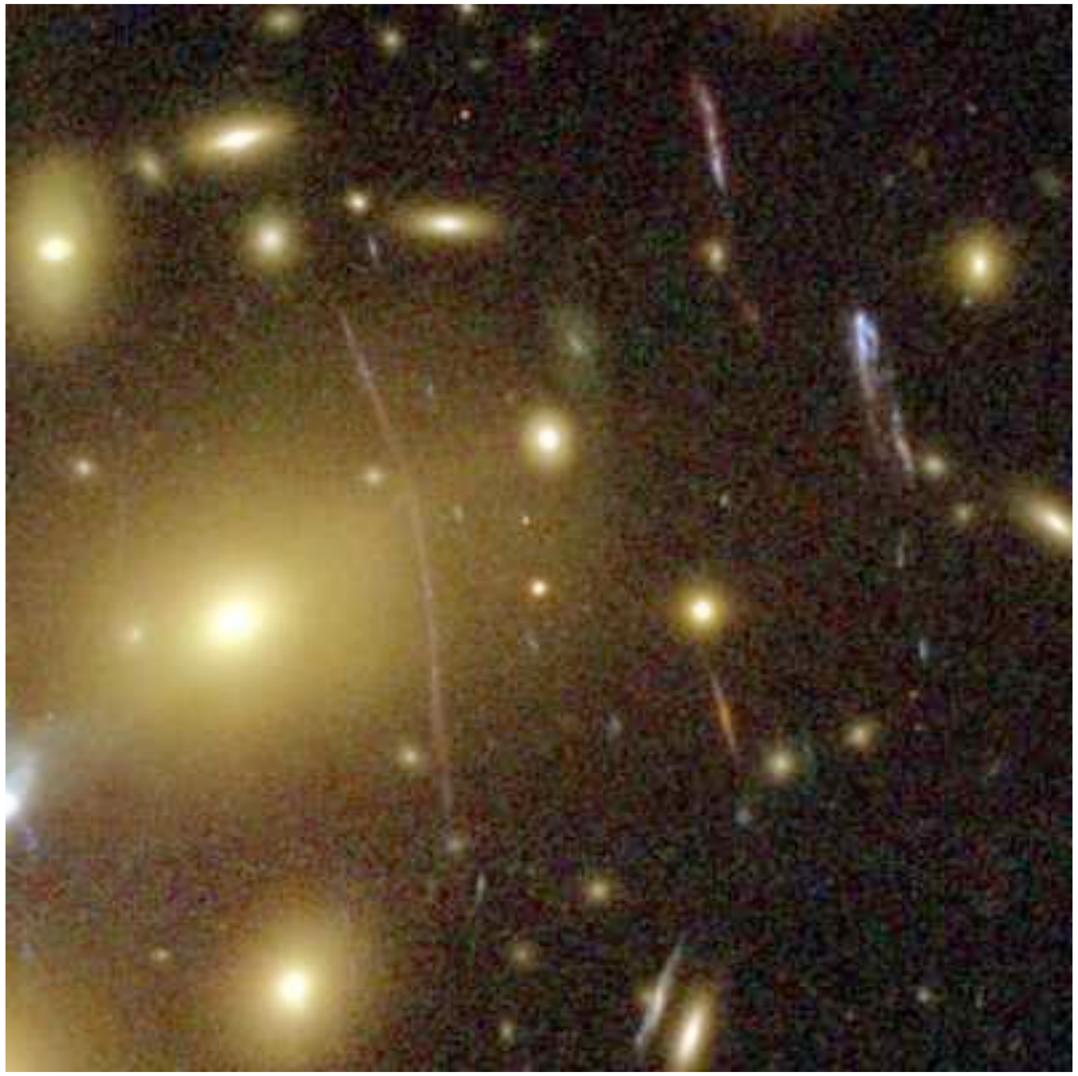
Note that multiple images exist only if the critical lines and the caustics exist!
This condition defines a "strong" lens.

Conclusions

From the fact that masses perturb the space-time, we expect:

1. deflections (displacements)
2. increased multiplicity
3. time delays
4. distortions (radial, tangential)

Now real observations...



Now real observations...

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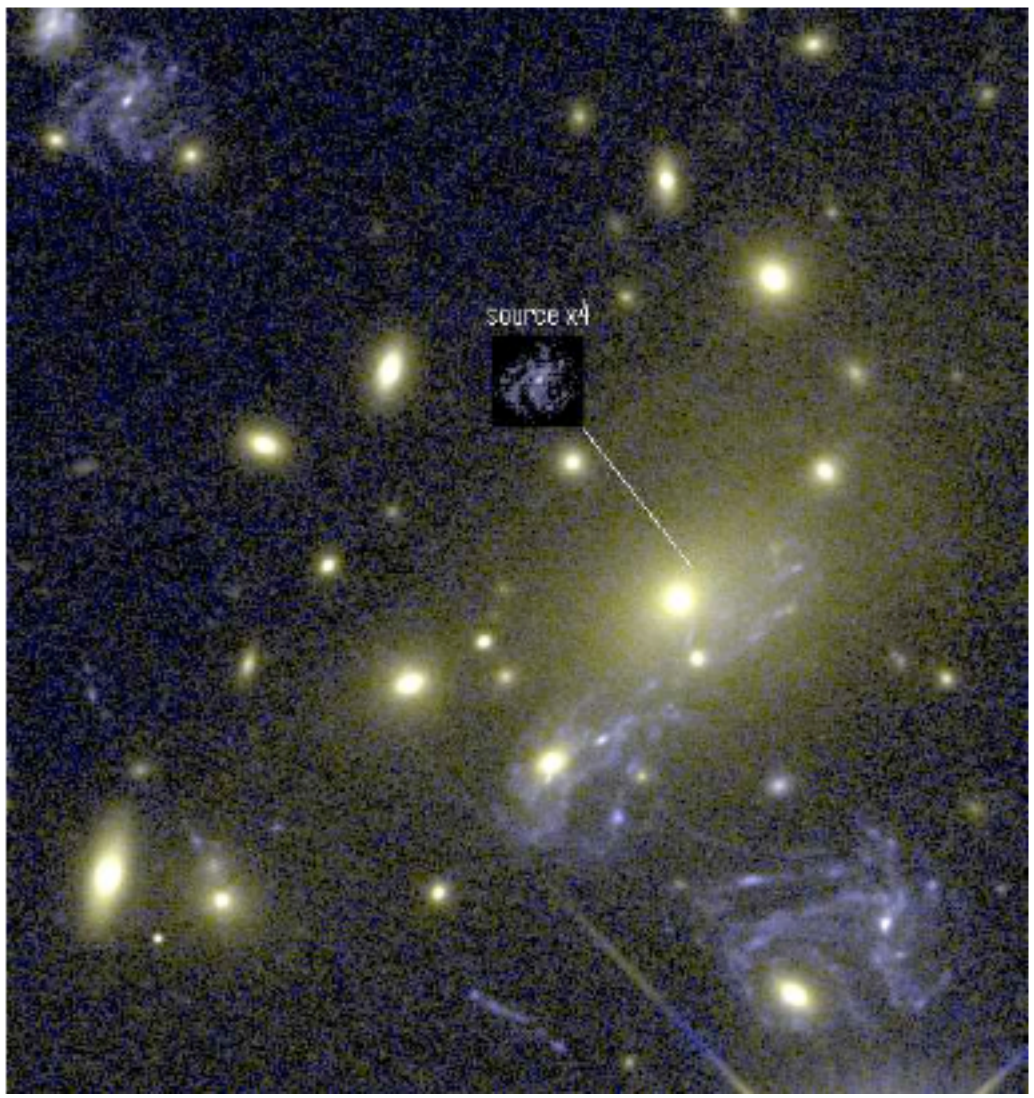

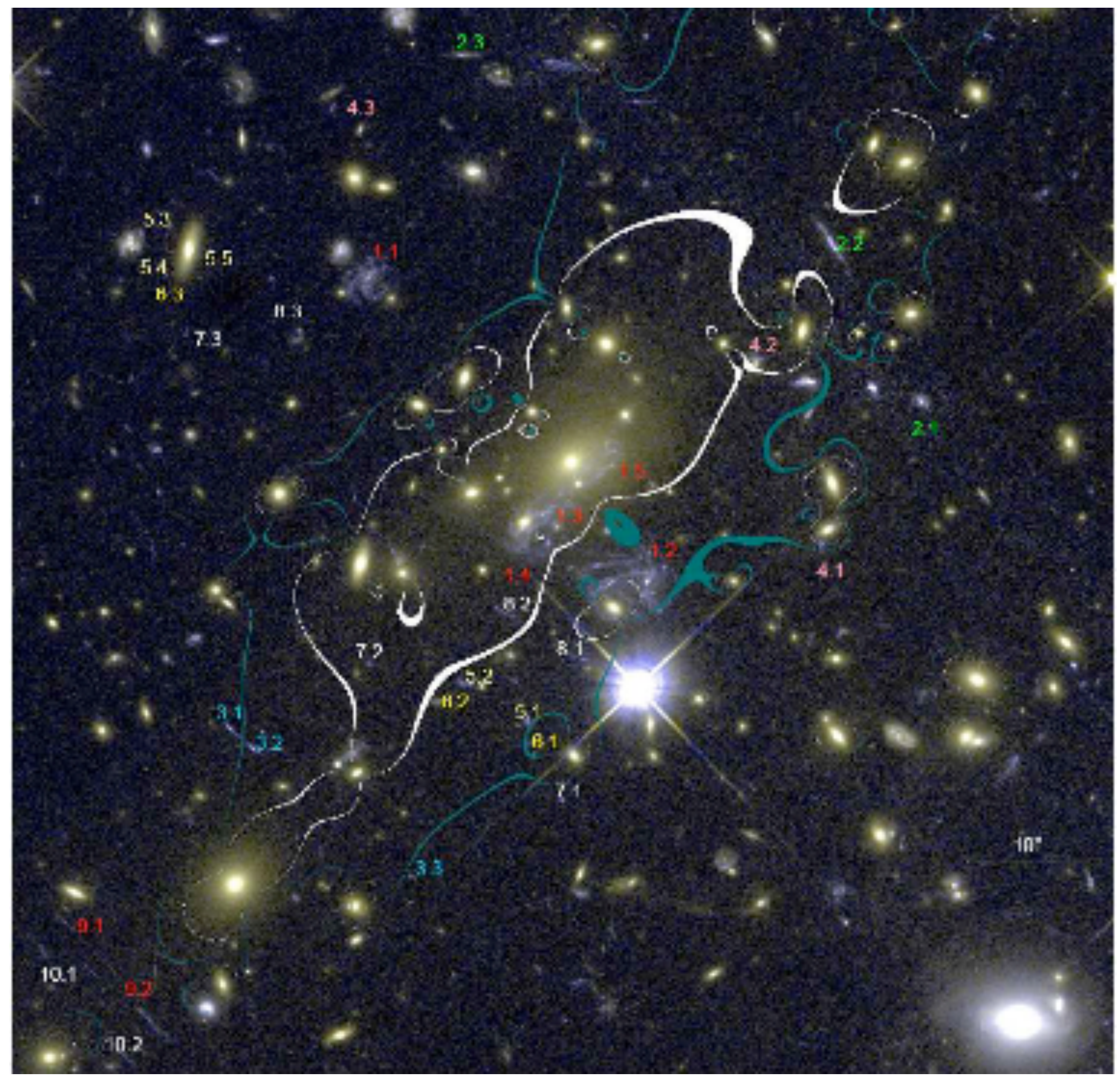

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Now real observations...