Roman Konoplya, JGRG 22(2012)111602

# "Stability of black holes: summary of some results for the past 10 years"

## RESCEU SYMPOSIUM ON

## GENERAL RELATIVITY AND GRAVITATION

# JGRG 22

**November 12-16 2012**

**Koshiba Hall, The University of Tokyo, Hongo, Tokyo, Japan**

# Black holes' stability: A review

R. A. Konoplya

DAMTP, University of Cambridge, UK

Tokyo, Nov. 11 - Nov. 16, 2012
the 60th birthday of T. Futamase, H. Kodama, M. Sasaki

# Content:

Recent reviews on stability of BHs:
in $D > 4$ A. Ishibashi, H. Kodama, Prog. Theor. Phys. Suppl. 189 (2011) 165-209
$D \geq 4$ R. A. Konoplya, A. Zhidenko, Rev. Mod. Phys. 83 (2011) 793-836

Introduction

# Content:

Recent reviews on stability of BHs:
in $D > 4$ A. Ishibashi, H. Kodama, Prog. Theor. Phys. Suppl. 189 (2011) 165-209
$D \geq 4$ R. A. Konoplya, A. Zhidenko, Rev. Mod. Phys. 83 (2011) 793-836

# Content:

Recent reviews on stability of BHs:
in $D > 4$ A. Ishibashi, H. Kodama, Prog. Theor. Phys. Suppl. 189 (2011) 165-209
$D \geq 4$ R. A. Konoplya, A. Zhidenko, Rev. Mod. Phys. 83 (2011) 793-836

# Content:

Recent reviews on stability of BHs:
in $D > 4$ A. Ishibashi, H. Kodama, Prog. Theor. Phys. Suppl. 189 (2011) 165-209
$D \geq 4$ R. A. Konoplya, A. Zhidenko, Rev. Mod. Phys. 83 (2011) 793-836

# Content:

Recent reviews on stability of BHs:
in $D > 4$ A. Ishibashi, H. Kodama, Prog. Theor. Phys. Suppl. 189 (2011) 165-209
$D \geq 4$ R. A. Konoplya, A. Zhidenko, Rev. Mod. Phys. 83 (2011) 793-836

# Content:

Recent reviews on stability of BHs:
in $D > 4$ A. Ishibashi, H. Kodama, Prog. Theor. Phys. Suppl. 189 (2011) 165-209
$D \geq 4$ R. A. Konoplya, A. Zhidenko, Rev. Mod. Phys. 83 (2011) 793-836



We shall discuss *mainly* (but not only) linear dynamical (in)stabilities

# Motivations:

Two main motivations to study gravitational stability of black holes:

# Motivations:

Two main motivations to study gravitational stability of black holes:

• Criterium of existence (in $D = 4$ for alternative theories of gravity and in $D > 4$ owing to absence of uniqueness)

# Motivations:

Two main motivations to study gravitational stability of black holes:

• Criterium of existence (in $D = 4$ for alternative theories of gravity and in $D > 4$ owing to absence of uniqueness)

• gauge-gravity duality (instability corresponds to the phase transition in the dual theory)

# Motivations:

Two main motivations to study gravitational stability of black holes:

• Criterium of existence (in $D = 4$ for alternative theories of gravity and in $D > 4$ owing to absence of uniqueness)

• gauge-gravity duality (instability corresponds to the phase transition in the dual theory)

• scenarios with extra dimensions (though experimental data on LHC gives no optimism: no large total transverse energy so far at 8 TEV: CMS collaboration claims that semiclassical BHs with mass below 6.1 TeV are excluded)

# From linearized perturbations to a master wave equation

- Step 1: Perturbations can be written in the linear approximation in the form

$$g_{\mu\nu} = g^0_{\mu\nu} + \delta g_{\mu\nu}, \tag{1}$$

$$\delta R_{\mu\nu} = \kappa\,\delta\left(T_{\mu\nu} - \frac{1}{D-2}Tg_{\mu\nu}\right) + \frac{2\Lambda}{D-2}\delta g_{\mu\nu}. \tag{2}$$

Linear approximation means that in Eq. (2) the terms of order $\sim \delta g^2_{\mu\nu}$ and higher are neglected. The unperturbed space-time given by the metric $g^0_{\mu\nu}$ is called the background.

# From linearized perturbations to a master wave equation

• Step 1: Perturbations can be written in the linear approximation in the form

$$g_{\mu\nu} = g^0_{\mu\nu} + \delta g_{\mu\nu}, \tag{1}$$

$$\delta R_{\mu\nu} = \kappa\, \delta\left(T_{\mu\nu} - \frac{1}{D-2} T g_{\mu\nu}\right) + \frac{2\Lambda}{D-2}\delta g_{\mu\nu}. \tag{2}$$

Linear approximation means that in Eq. (2) the terms of order $\sim \delta g^2_{\mu\nu}$ and higher are neglected. The unperturbed space-time given by the metric $g^0_{\mu\nu}$ is called the background.

• Step 2: decomposition of the perturbed space-time into scalar, vector and tensor parts

# From linearized perturbations to a master wave equation

• Step 1: Perturbations can be written in the linear approximation in the form

$$g_{\mu\nu} = g^0_{\mu\nu} + \delta g_{\mu\nu}\,, \tag{1}$$

$$\delta R_{\mu\nu} = \kappa\,\delta\left(T_{\mu\nu} - \frac{1}{D-2}Tg_{\mu\nu}\right) + \frac{2\Lambda}{D-2}\delta g_{\mu\nu}\,. \tag{2}$$

Linear approximation means that in Eq. (2) the terms of order $\sim \delta g^2_{\mu\nu}$ and higher are neglected. The unperturbed space-time given by the metric $g^0_{\mu\nu}$ is called the background.

• Step 2: decomposition of the perturbed space-time into scalar, vector and tensor parts

• Step 3: using the gauge invariant formalism (or fixing the gauge)

# From linearized perturbations to a master wave equation

• Step 1: Perturbations can be written in the linear approximation in the form

$$g_{\mu\nu} = g^0_{\mu\nu} + \delta g_{\mu\nu}, \tag{1}$$

$$\delta R_{\mu\nu} = \kappa\, \delta \left( T_{\mu\nu} - \frac{1}{D-2} T g_{\mu\nu} \right) + \frac{2\Lambda}{D-2} \delta g_{\mu\nu}. \tag{2}$$

Linear approximation means that in Eq. (2) the terms of order $\sim \delta g^2_{\mu\nu}$ and higher are neglected. The unperturbed space-time given by the metric $g^0_{\mu\nu}$ is called the background.

• Step 2: decomposition of the perturbed space-time into scalar, vector and tensor parts

• Step 3: using the gauge invariant formalism (or fixing the gauge)

• Step 4: Reducing the perturbation equations (after separation of angular variables) to a second order partial differential equation, termed *master wave equation*. For example, for static and some stationary BHs the master wave equation has the form:

$$-\frac{d^2 R}{dr_*^2} + V(r,\omega) R = \omega^2 R, \tag{3}$$

# Criteria of stability: analytical vs numerical

• If the effective potential $V_{eff}$ in the wave equation (3) is positive definite, the differential operator

$$A = -\frac{\partial^2}{\partial r_*^2} + V_{eff} \tag{4}$$

is a positive self-adjoint operator in the Hilbert space of square integrable functions $L^2(r_*, dr_*)$. Then, there are no negative (growing) mode solutions that are normalizable, i. e., for a well-behaved initial data (smooth data of compact support), all solutions are bounded all of the time.

# Criteria of stability: analytical vs numerical

• If the effective potential $V_{eff}$ in the wave equation (3) is positive definite, the differential operator

$$A = -\frac{\partial^2}{\partial r_*^2} + V_{eff} \tag{4}$$

is a positive self-adjoint operator in the Hilbert space of square integrable functions $L^2(r_*, dr_*)$. Then, there are no negative (growing) mode solutions that are normalizable, i. e., for a well-behaved initial data (smooth data of compact support), all solutions are bounded all of the time.

• Yet, in majority of cases $A$ is not positive (negativeness of the effective potential in some regions, dependence of the potential on the complex frequencies $\omega$)

# Criteria of stability: analytical vs numerical

• If the effective potential $V_{eff}$ in the wave equation (3) is positive definite, the differential operator

$$A = -\frac{\partial^2}{\partial r_*^2} + V_{eff} \tag{4}$$

is a positive self-adjoint operator in the Hilbert space of square integrable functions $L^2(r_*, dr_*)$. Then, there are no negative (growing) mode solutions that are normalizable, i. e., for a well-behaved initial data (smooth data of compact support), all solutions are bounded all of the time.

• Yet, in majority of cases $A$ is not positive (negativeness of the effective potential in some regions, dependence of the potential on the complex frequencies $\omega$)

• Sometimes the situation can be remedied by the so-called S-deformation of the wave equation to the one with positive definite effective potential, in such a way that the lower bound of the energy spectrum does not change.

# Criteria of stability: analytical vs numerical

• If the effective potential $V_{eff}$ in the wave equation (3) is positive definite, the differential operator

$$A = -\frac{\partial^2}{\partial r_*^2} + V_{eff} \tag{4}$$

is a positive self-adjoint operator in the Hilbert space of square integrable functions $L^2(r_*, dr_*)$. Then, there are no negative (growing) mode solutions that are normalizable, i. e., for a well-behaved initial data (smooth data of compact support), all solutions are bounded all of the time.

• Yet, in majority of cases $A$ is not positive (negativeness of the effective potential in some regions, dependence of the potential on the complex frequencies $\omega$)

• Sometimes the situation can be remedied by the so-called S-deformation of the wave equation to the one with positive definite effective potential, in such a way that the lower bound of the energy spectrum does not change.

• Usually, it is difficult to find an ansatz for the S-deformation, so that numerical treatment of the wave equation is necessary.

• The numerical criteria of stability could be the evidence that all the proper oscillation frequencies of the black hole, termed the *quasinormal modes* are damped.

• The numerical criteria of stability could be the evidence that all the proper oscillation frequencies of the black hole, termed the *quasinormal modes* are damped.

• Quasinormal modes are eigenvalues of the master wave equation with appropriate boundary conditions: purely ingoing waves at the horizon and purely outgoing waves at infinity or de Sitter horizon. For AdS BHs boundary condition at infinity is dictated by AdS/CFT and is usually the Dirichlet one $\Psi = 0$, where $\Psi$ is some gauge inv. combination.

• The numerical criteria of stability could be the evidence that all the proper oscillation frequencies of the black hole, termed the *quasinormal modes* are damped.

• Quasinormal modes are eigenvalues of the master wave equation with appropriate boundary conditions: purely ingoing waves at the horizon and purely outgoing waves at infinity or de Sitter horizon. For AdS BHs boundary condition at infinity is dictated by AdS/CFT and is usually the Dirichlet one $\Psi = 0$, where $\Psi$ is some gauge inv. combination.

• Quasinormal modes normally have real and imaginary parts $\omega_{\ell,m,...} = Re\omega + iIm\omega$ and depend on a number of quantum numbers such as multipole number $\ell$, azimuthal number $m$, overtone numbers $n$. That is, there are infinite countable number of quasinormal modes. Potential difficulties: instability usually occurs at lower multipoles $\ell$, but also may happen at high $\ell$ (Gauss-Bonnet BH), an effective potential may have very cumbersome form, the imaginary part of the unstable mode may be very small, ....

• The numerical criteria of stability could be the evidence that all the proper oscillation frequencies of the black hole, termed the *quasinormal modes* are damped.

• Quasinormal modes are eigenvalues of the master wave equation with appropriate boundary conditions: purely ingoing waves at the horizon and purely outgoing waves at infinity or de Sitter horizon. For AdS BHs boundary condition at infinity is dictated by AdS/CFT and is usually the Dirichlet one $\Psi = 0$, where $\Psi$ is some gauge inv. combination.

• Quasinormal modes normally have real and imaginary parts $\omega_{\ell,m,...} = Re\omega + iIm\omega$ and depend on a number of quantum numbers such as multipole number $\ell$, azimuthal number $m$, overtone numbers $n$. That is, there are infinite countable number of quasinormal modes. Potential difficulties: instability usually occurs at lower multipoles $\ell$, but also may happen at high $\ell$ (Gauss-Bonnet BH), an effective potential may have very cumbersome form, the imaginary part of the unstable mode may be very small, ....

• How to show that all of the QNMs are damped?

• The numerical criteria of stability could be the evidence that all the proper oscillation frequencies of the black hole, termed the *quasinormal modes* are damped.

• Quasinormal modes are eigenvalues of the master wave equation with appropriate boundary conditions: purely ingoing waves at the horizon and purely outgoing waves at infinity or de Sitter horizon. For AdS BHs boundary condition at infinity is dictated by AdS/CFT and is usually the Dirichlet one $\Psi = 0$, where $\Psi$ is some gauge inv. combination.

• Quasinormal modes normally have real and imaginary parts $\omega_{\ell,m,...} = Re\omega + iIm\omega$ and depend on a number of quantum numbers such as multipole number $\ell$, azimuthal number $m$, overtone numbers $n$. That is, there are infinite countable number of quasinormal modes. Potential difficulties: instability usually occurs at lower multipoles $\ell$, but also may happen at high $\ell$ (Gauss-Bonnet BH), an effective potential may have very cumbersome form, the imaginary part of the unstable mode may be very small, ....

• How to show that all of the QNMs are damped?

• Ideally, either: to achieve an asymptotic regime in all numbers $\ell$, $m$, etc... and parameters in the *frequency domain* or to perform *time-domain* integration until asymptotic tails. Better - both.

# (In)stability of 3+1 dimensional BHs and wormholes

| Black hole solution (parameters) | Publication |
|---|---|
| Schwarzschild ($M$) | Regge, Wheeler 1957 |
| Reissner-Nordström ($M$, $Q$) | Moncrief; Alekseev 1974 |
| exactly extreme Reissner-Nordström ($M$, $Q$) | Aretakis 2011 |
| Schwarzschild-dS (M, $\Lambda > 0$) | Mellor, Moss 1989 |
| Schwarzschild-AdS (M, $\Lambda < 0$) | Cardoso, Lemos 2001 |
| Reissner-Nordström-dS (M, Q, $\Lambda$) | Mellor 1989 |
| Kerr (M, J) | Press 1973; Teukolsky 1974 |
| exactly extreme Kerr (M, J) | Aretakis 2011; Lucietti, Reall, 2012 |
| Kerr-dS (M, J, $\Lambda > 0$) | Suzuki 1999 |
| Kerr-AdS (M, J, $\Lambda < 0$) | Giammatteo 2005 |
| Kerr-Newmann (M, J, Q) | ? |
| Kerr-Newman-A(dS) (M, J, Q, $\Lambda$) | ? |
| Dilaton (M, Q, $\phi$) | Holzhey 1991; Ferrari, 2000 |
| Dilaton-axion (M, Q, J, $\phi$, $\psi$) | ? |
| Dilaton-GB (M, $\phi$) | Torii 1998, Pani 2009 |
| Born-Infeld (M, Q) | axial only, Fernando 2004 |
| Black universes (M, $\phi$) | Bronnikov, Konoplya, Zhidenko 2012 |
| BHs in the Chern-Simons theory (M, $\beta$) | Cardoso 2010 |

# (In)stability of 3+1 dimensional BHs and wormholes

• Non extreme four dimensional BHs are usually stable under linear gravitational perturbations, except black universes with the minimal area function and rotating asymptotically AdS BHs (superradiant modes).

# (In)stability of 3+1 dimensional BHs and wormholes

- Non extreme four dimensional BHs are usually stable under linear gravitational perturbations, except black universes with the minimal area function and rotating asymptotically AdS BHs (superradiant modes).

- Extreme Kerr and Reissner-Nordstrom BHs space-times are unstable.

# (In)stability of 3+1 dimensional BHs and wormholes

• Non extreme four dimensional BHs are usually stable under linear gravitational perturbations, except black universes with the minimal area function and rotating asymptotically AdS BHs (superradiant modes).

• Extreme Kerr and Reissner-Nordstrom BHs space-times are unstable.

• Wormholes and black holes supported by exotic matter, such as phantom fields, are usually unstable.

# (In)stability of 3+1 dimensional BHs and wormholes

• Non extreme four dimensional BHs are usually stable under linear gravitational perturbations, except black universes with the minimal area function and rotating asymptotically AdS BHs (superradiant modes).

• Extreme Kerr and Reissner-Nordstrom BHs space-times are unstable.

• Wormholes and black holes supported by exotic matter, such as phantom fields, are usually unstable.

• ...and from
1993 Gregory and Laflamme (black branes)
2003 Ishibashi and Kodama (black holes)

a higher dimensional story starts...

# Three types of instability:

• **Gregory-Laflamme instability**. It occurs for black strings at long wavelengthes in the extra dimension $z$ and is connected with inapplicability of the Birkhoff theorem. Gregory and Laflamme 1993. Gubser and Mitra instability (2001) for large highly charged RNdS BHs in $N = 8$ supergravity is of this kind. Developement of black string instability beyond linear order - nonuniform string - thermodynamic arguments and numerical computations by Choptuik and others, 2000th

# Three types of instability:

• **Gregory-Laflamme instability**. It occurs for black strings at long wavelengthes in the extra dimension $z$ and is connected with inapplicability of the Birkhoff theorem. Gregory and Laflamme 1993. Gubser and Mitra instability (2001) for large highly charged RNdS BHs in $N = 8$ supergravity is of this kind. Developement of black string instability beyond linear order - nonuniform string - thermodynamic arguments and numerical computations by Choptuik and others, 2000th

• **Superradiant instability of asymptotically AdS rotating BHs**. If

$$\frac{m\Omega}{\omega} > 1, \tag{5}$$

the reflected wave has larger amplitude than the incident one, a superradiance. This effect was predicted by Zeldovich and proved for Kerr BHs by A. Starobinsky in 1974. Super-radiance is connected with extraction of rotational energy of a black hole and occurs at positive ("co-rotating"with a black hole) $m$. Super-radiance + Dirichlet boundary conditions far from black hole = instability. Therefore, rotating AdS black holes are unstable in the superradiant regime.

# Three types of instability:

- **Gregory-Laflamme instability**. It occurs for black strings at long wavelengthes in the extra dimension $z$ and is connected with inapplicability of the Birkhoff theorem. Gregory and Laflamme 1993. Gubser and Mitra instability (2001) for large highly charged RNdS BHs in $N = 8$ supergravity is of this kind. Developement of black string instability beyond linear order - nonuniform string - thermodynamic arguments and numerical computations by Choptuik and others, 2000th

- **Superradiant instability of asymptotically AdS rotating BHs**. If

$$\frac{m\Omega}{\omega} > 1, \tag{5}$$

the reflected wave has larger amplitude than the incident one, a superradiance. This effect was predicted by Zeldovich and proved for Kerr BHs by A. Starobinsky in 1974. Super-radiance is connected with extraction of rotational energy of a black hole and occurs at positive ("co-rotating"with a black hole) $m$. Super-radiance + Dirichlet boundary conditions far from black hole = instability. Therefore, rotating AdS black holes are unstable in the superradiant regime.

- **"Non-Gregory-Laflamme"instability**. The "other"type of instability which is not connected with the inapplicability of the Birkhoff theorem or superradiance. It happens, for instance, for asymptotically de Sitter highly charged black holes in $D > 6$ space-times.

# Instability of RNAdS BHs in supergravity and RNdS instability in EM theory

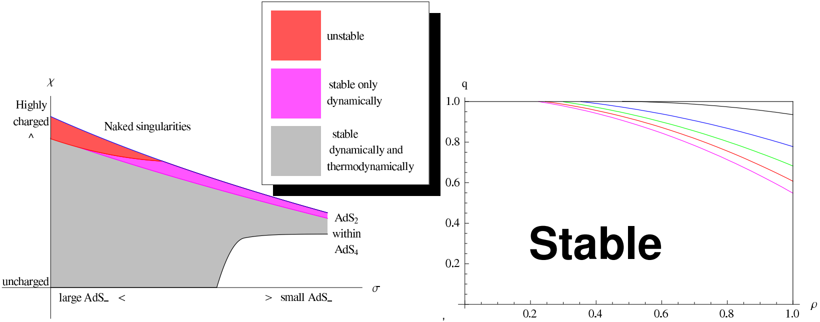


Рис.: **Left:** Gubser-Mitra instability of R-N-AdS in $N = 8$ supergravity; **Right:** R-N-dS instability, Konoplya, Zhidenko PRL, 2009. The parametric region of instability in the right upper corner of the square in the $\rho - q$ "coordinates" for $D = 7$ (top, black), $D = 8$ (blue), $D = 9$ (green), $D = 10$ (red), $D = 11$ (bottom, magenta). The units $r_+ = 1$ are used; $\rho = r_+/r_c = 1/r_c < 1$, $r_c$ is the cosmological horizon. The charge can be normalized by its extremal quantity $q = Q/Q_{ext} < 1$.

# Superradiant instability

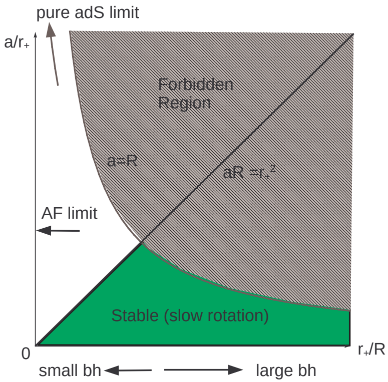


Рис.: The stable region in the parameter plane for the simply rotating higher-dimensional asymptotically AdS black hole. Kodama, Konoplya, Zhidenko 2009. The instability has the tiny growth rate which, apparently, will be suppressed by the intensive Hawking evaporation.

# (In)stability of higher-dimensional black holes

| Black hole solution (parameters) | Publication |
|---|---|
| Schwarzschild (M) | Stable for all $D$ Kodama, Ishibashi 2003 |
| non extreme R-N (M, Q) | Stable for $D = 5, 6, \dots, 11$ K.Z. 2009 |
| Schwarzschild-dS (M, Λ), (M, $\Lambda > 0$) | Stable for $D = 5, 6, \dots, 11$ K.Z. 2007 |
| Schwarzschild-AdS (M, Λ) (M, $\Lambda < 0$) | Stable in EM theory for $D = 5, 6, \dots, 11$ K.Z. 2008 |
| R-N -dS (M, Λ) (M, Q, $\Lambda > 0$) | Unstable for $D = 7, 8, \dots, 11$ K.Z. 2009 |
| R-N -AdS in E-M (M, Λ) (M, Q, $\Lambda < 0$) | stable in E-M K.Z. 2008 |
| R-N -AdS in supergravity (M, Λ) (M, Q, $\Lambda < 0$) | unstable Gubser, Mitra 2000 |
| Gauss-Bonnet (M, $\alpha$) | Unstable for large $\alpha$, Dotti 2006 |
| Lovelock (M, $\alpha$, $\beta$, ...) | Unstable for large $\alpha$, Soda, Takahashi |
| Myers-Perry and its generalizations (M, J) | ? Only particular types of perturbations |
| Dilaton (M, Q, $\phi$) | ? |
| Dilaton-axion (M, Q, J, $\phi$, $\psi$) | ? |
| Dilaton-Gauss-Bonnet (M, $\phi$, $\alpha$) | ? |
| squashed Kaluza-Klein black holes | stable, 2000th Soda, Ishihara and others |
| black strings and branes | Gregory and Laflamme 1993 |
| black rings, saturns, etc | ? (Heuristic arguments and analogies) |

K.Z. = Konoplya and Zhidenko

# Potential turbulent instability of AdS space-time

- Bizon and Rostworowski (2011) claimed that there is numerical evidence of a weakly turbulent instability of pure AdS space-time

# Potential turbulent instability of AdS space-time

• Bizon and Rostworowski (2011) claimed that there is numerical evidence of a weakly turbulent instability of pure AdS space-time

• This claim was supported by some intuitive arguments of O. Dias and collaborators (2011) borrowed from M. T. Anderson (2006): AdS boundary conditions act like a confining box. Any excitation added to the box, after some time, could explore a configuration consistent with the conserved quantities. The conjecture is that an excitation of AdS eventually finds itself inside its Schwarzschild radius and collapses to a black hole .

# Potential turbulent instability of AdS space-time

• Bizon and Rostworowski (2011) claimed that there is numerical evidence of a weakly turbulent instability of pure AdS space-time

• This claim was supported by some intuitive arguments of O. Dias and collaborators (2011) borrowed from M. T. Anderson (2006): AdS boundary conditions act like a confining box. Any excitation added to the box, after some time, could explore a configuration consistent with the conserved quantities. The conjecture is that an excitation of AdS eventually finds itself inside its Schwarzschild radius and collapses to a black hole .

• In concordance with this conjecture it was *argued* in a subsequent paper of Dias (2012) that a number of asymptotically AdS space-times including a SdS BH space-time apparently look like non-linearly stable

# Potential turbulent instability of AdS space-time

• Bizon and Rostworowski (2011) claimed that there is numerical evidence of a weakly turbulent instability of pure AdS space-time

• This claim was supported by some intuitive arguments of O. Dias and collaborators (2011) borrowed from M. T. Anderson (2006): AdS boundary conditions act like a confining box. Any excitation added to the box, after some time, could explore a configuration consistent with the conserved quantities. The conjecture is that an excitation of AdS eventually finds itself inside its Schwarzschild radius and collapses to a black hole .

• In concordance with this conjecture it was *argued* in a subsequent paper of Dias (2012) that a number of asymptotically AdS space-times including a SdS BH space-time apparently look like non-linearly stable

• By now, in my opinion, there is no clearness in this issue: alternative non-linear computations or possibly higher order perturbative calculations would help...

# Conclusions

By now, we know a lot of about various (in)stabilities of black holes and branes in four and higher dimensions in the linear regime, but far from everything....

# Conclusions

By now, we know a lot of about various (in)stabilities of black holes and branes in four and higher dimensions in the linear regime, but far from everything....

We know not much about possible nonlinear (in)stabilities in higher dimensions...

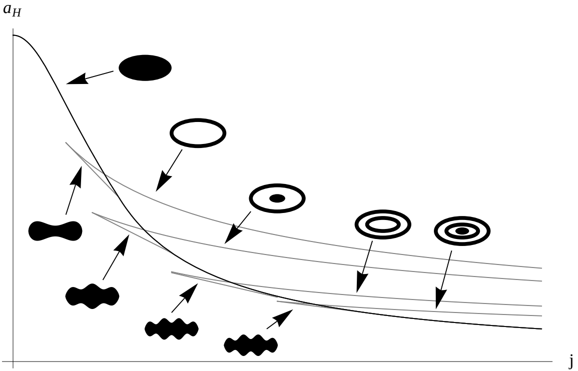


Рис.: The qualitative phase diagram for the black objects in $D \geq 6$ (taken from an Emparan's paper). The horizontal and vertical axes correspond, respectively, to the spin and area of a black object. If thermal equilibrium is not imposed, multirings are possible in the upper region of the diagram.