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“The first law of binary black hole dynamics”

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FIRST LAW OF BINARY BLACK HOLE DYNAMICS

dedicated to the 60th birthday of
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**Four Laws of Black Hole Dynamics**

**ZERO TH LAW**
Surface gravity $\kappa$ is constant over the horizon of a stationary black hole.

**FIRST LAW**
Mass $M$ and angular momentum $J$ of BH change according to [Bardeen, Carter & Hawking 1973]

$$\delta M - \omega_{\mathcal{H}} \delta J = \frac{\kappa}{8\pi} \delta A$$

**SECOND LAW**
In any physical process involving one or several BHs with or without an environment [Hawking 1971]

$$\delta A \geq 0$$

**THIRD LAW**
It is impossible to achieve $\kappa = 0$ in any process.
Four Laws of Black Hole Dynamics

**ZEROTH LAW**
Surface gravity $\kappa$ is constant over the horizon of a stationary black hole.

**FIRST LAW**
Mass $M$ and angular momentum $J$ of BH change according to [Christodoulou 1970, Smarr 1973]

$$M - 2\omega \mathcal{H} J = \frac{\kappa}{4\pi} \mathcal{A}$$

**SECOND LAW**
In any physical process involving one or several BHs with or without an environment [Hawking 1971]

$$\delta \mathcal{A} \geq 0$$

**THIRD LAW**
It is impossible to achieve $\kappa = 0$ in any process.
Using arguments involving a piece of matter with entropy thrown into a BH, Bekenstein derived the BH entropy

\[ S_{\text{BH}} = \alpha A \]

This would require \( T_{\text{BH}} = \frac{\kappa}{8\pi\alpha} \) but the thermodynamic temperature of a classical BH is absolute zero since a BH is a perfect absorber.

However, Hawking proved that quantum particle creation effects near a BH result in a black body temperature \( T_{\text{BH}} = \frac{\kappa}{2\pi} \). This leads to the famous

**Bekenstein-Hawking entropy of a stationary black hole**

\[ S_{\text{BH}} = \frac{c^3 k A}{\hbar G 4} \]

The analogy between BH dynamics and the laws of thermodynamics is complete.
The mass and the angular momentum of the BH are given by Komar surface integrals at spatial infinity

\[ M = - \frac{1}{8\pi} \lim_{r \to \infty} \oint_{S_r} \nabla^\mu t^\nu \, dS_{\mu\nu} \]

\[ J = \frac{1}{16\pi} \lim_{r \to \infty} \oint_{S_r} \nabla^\mu \phi^\nu \, dS_{\mu\nu} \]

where \( t^\mu \) and \( \phi^\mu \) are the two stationary and axi-symmetric Killing vectors.
The first law of BH dynamics expresses the change
\[ \delta Q = \delta M - \omega_H \delta J \]
in the Noether charge \( Q \) between two nearby BH configurations, where \( Q \) is associated with the Killing vector
\[ K^\mu = t^\mu + \omega_H \phi^\mu \]
which is the null generator of the BH horizon.
A generalized First Law valid for systems of BHs can be obtained when the geometry admits a Helical Killing Vector (HKV)

\[ K^\mu \partial_\mu = \partial_t + \Omega \partial_\varphi \]

where \( \partial_t \) is time-like and \( \partial_\varphi \) is space-like (with closed orbits), even when \( \partial_t \) and \( \partial_\varphi \) are not separately Killing vectors.

This applies to the case of two Kerr BHs moving on exactly circular orbits with orbital frequency \( \Omega \).

The two BHs should be in corotation, so that \( \omega_H \) should approximately be equal to \( \Omega \). In particular the spins should be aligned with the orbital angular momentum.
Toward a Generalized First Law for a System of BHs

\[ \mathbf{S}_1 \] \quad \mathbf{L} \quad \mathbf{S}_2 \]

\[ \omega_\mathcal{H} = \Omega \]

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First law of binary black hole dynamics

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With the Helical Killing Vector $K^\mu \partial_\mu = \partial_t + \Omega \partial_\varphi$ the change in the associated Noether charge is given by

$$\delta Q = \delta M - \Omega \delta J$$

provided that the space-time is asymptotically flat [Friedman, Uryū & Shibata 2002].

However exact solutions of the Einstein field equations with Helical Killing symmetry cannot be asymptotically flat since they are periodic which contradicts the decrease of the Bondi mass at $\mathcal{J}^+$ [Gibbons & Stewart 1983].
Toward a Generalized First Law for a System of BHs

Physical situation
no incoming radiation condition

Situation with the HKV
standing waves at infinity
One way to deal with the problem is to look at approximate solutions which are asymptotically flat. A possible solution is to suppress radiation degrees of freedom by imposing a condition of conformal flatness for the spatial metric [Isenberg & Nester 1980; Wilson & Mathews 1989]

Here we follow a different route which is to consider only the conservative part of the dynamics in a post-Newtonian (PN) expansion, neglecting the dissipative effects due to the emission of gravitational radiation.

Thus we derive the First Law for a class of conservative PN space-times admitting a HKV and describing point particles (possibly with spins) moving on an exactly circular orbit.
Two Point Particles on an Exactly Circular Orbit

particle's trajectories
light cylinder
time
space

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Internal acceleration of a matter system is written as a formal PN expansion

\[
\frac{dv}{dt} = A_N + \frac{1}{c^2} A_{1PN} + \frac{1}{c^4} A_{2PN} + \frac{1}{c^5} A_{2.5PN} + \frac{1}{c^6} A_{3PN} + \frac{1}{c^7} A_{3.5PN} + \frac{1}{c^8} A_{4PN} + \mathcal{O}\left(\frac{1}{c^9}\right)
\]

Naive split would be to say that conservative effects are those which carry an even power of \(1/c\), while dissipative effects, linked to gravitational radiation reaction, are those which carry an odd power of \(1/c\)
Conservative versus Dissipative Dynamics in PN theory

- This is correct at leading 2.5PN order where the force derives from a scalar in an appropriate gauge, $A_{2.5\text{PN}} = \nabla V_{2.5\text{PN}}$ with [Burke & Thorne]

$$V_{2.5\text{PN}}(\mathbf{x}, t) = -\frac{1}{5} x^i x^j I^{(5)}_{ij}(t)$$

This term would change sign if we change the prescription of retarded potentials to the advanced potentials.

- This is still correct at sub-leading order 3.5PN where the force involves both scalar and vector potentials given by [Blanchet 1997]

$$V_{3.5\text{PN}} = \frac{1}{189} x^i x^j x^k I_{ijk}^{(7)}(t) - \frac{1}{70} \mathbf{x}^2 x^i x^j I_{ij}^{(7)}(t)$$

$$V_{3.5\text{PN}}^i = \frac{1}{21} x^{\langle i} x^j x^k \rangle I_{jk}^{(6)}(t) - \frac{4}{45} \varepsilon_{ijk} x^j x^l J_{kl}^{(5)}(t)$$

which also change sign from retarded to advanced potentials.
However the naive split fails starting at 4PN order because of the appearance of tails in the radiation reaction force [Blanchet & Damour 1988]

\[
V_{4\text{PN}} = -\frac{4M}{5} x^i x^j \int_{-\infty}^{t} dt' I_{ij}^{(7)}(t') \ln \left( \frac{t-t'}{2r} \right)
\]

This term is not invariant when we go from retarded to advanced potentials.
Logarithms at 4PN order in the Conservative Dynamics

With the HKV we have at our disposal the binary’s orbital period \( P = \frac{2\pi}{\Omega} \). We split

\[
\ln \left( \frac{t - t'}{2r} \right) = -\ln \left( \frac{r}{P} \right) + \ln \left( \frac{t - t'}{2P} \right)
\]

Tails produce a conservative 4PN logarithmic term

\[
V_{4\text{PN}} = -\frac{4M^2}{5} x^i x^j \left[ -I_{ij}^{(6)}(t) \ln \left( \frac{r}{P} \right) + \int_{-\infty}^{t} dt' I_{ij}^{(7)}(t') \ln \left( \frac{t - t'}{2P} \right) \right]
\]

conservative 4PN log term
dissipative term (neglected)

We shall see appearing at 4PN and higher orders like 5PN some logarithmic contributions in the conservative part of the dynamics of binary black holes.
Short History of the PN Approximation

Equations of motion
- 1PN equations of motion [Lorentz & Droste 1917; Einstein, Infeld & Hoffmann 1938]
- Radiation-reaction controversy [Ehlers et al 1979; Walker & Will 1982]
- 2.5PN equations of motion and GR prediction for the binary pulsar [Damour & Deruelle 1982; Damour 1983]
- The “3mn” Caltech paper [Cutler, Flanagan, Poisson & Thorne 1993]
- Ambiguity parameters resolved [DJS 2001; BDE 2003]

Radiation field
- 1918 Einstein quadrupole formula
- 1940 Landau-Lifchitz formula
- 1960 Peters-Mathews formula
- EW moments [Thorne 1980]
- BD moments and wave generation formalism [BD 1989; B 1995, 1998]
- 1PN phasing [Wagoner & Will 1976; BS 1989]
- Test-particle limit using BH perturbations [Tagoshi & Sasaki 1994]
- 2PN waveform [BDIWW 1995]
- 3.5PN phasing and 3PN waveform [BFIJ 2003, BFIS 2007]
- Ambiguity parameters resolved [BI 2004; BDEI 2004, 2005]
The waveform is obtained by matching a high-order post-Newtonian waveform describing the long inspiralling phase and a highly accurate numerical waveform describing the final merger and ringdown phases.
Explicit EOM for non-spinning compact binaries

\[ \frac{dv_1^i}{dt} = - \frac{Gm_2}{r_{12}^2} n_1^i \]

\[ + \frac{1}{c^2} \left\{ \left[ \frac{5G^2m_1m_2}{r_{12}^3} + \frac{4G^2m_2^2}{r_{12}^3} + \frac{Gm_2}{r_{12}^2} \left( \frac{3}{2} (n_{12}v_2)^2 - v_1^2 - 2v_2^2 \right) \right] n_{12}^i + \cdots \right\} \]

\[ + \left\{ \left[ \frac{1}{c^4} \cdots + \frac{1}{c^5} \cdots + \frac{1}{c^6} \cdots + \frac{1}{c^7} \cdots + \mathcal{O} \left( \frac{1}{c^8} \right) \right] \right\} \]

Spin effects arise at orders 1.5PN for the spin-orbit and 2PN for the spin-spin.
Mass and Angular Momentum of Compact Binaries

It is convenient to use the gauge invariant PN parameter

\[ x = \left( \frac{Gm\Omega}{c^3} \right)^{3/2} \]

with the mass parameters \( m = m_1 + m_2 \) and \( \nu = \frac{m_1m_2}{m^2} \).

**Conservative PN energy for circular orbits**

\[
E = -\frac{1}{2} m\nu \left\{ 1 + \left( -\frac{3}{4} - \frac{\nu}{12} \right) x + \left[ \cdots \right] x^2 + \left[ \cdots \right] x^3 + \left[ \cdots + \frac{448}{15} \nu \ln x \right] x^4 + \left[ \cdots + \left[ -\frac{4988}{35} - 6565\nu \right] \nu \ln x \right] x^5 + \mathcal{O}(x^6) \right\}
\]

The 4PN and 5PN conservative logarithmic terms have been computed recently [Blanchet, Detweiler, Le Tiec & Whiting 2010]
The angular momentum $J$ is checked to satisfy for all the terms up to 3PN order, and also for the 4PN and 5PN log terms, the thermodynamic relation valid for circular orbits

$$\frac{\partial M}{\partial \Omega} = \Omega \frac{\partial J}{\partial \Omega}$$

which constitutes the first ingredient in the First Law of binary black holes.

- The thermodynamic relation states that the flux of energy emitted in the form of gravitational waves is proportional to the flux of angular momentum.
- It is used in numerical computations of the binary evolution based on a sequence of quasi-equilibrium configurations [Gourgoulhon et al 2002]
The geometry has a Helical Killing Vector (HKV) asymptotically given by

\[ K^\mu \partial_\mu = \partial_t + \Omega \partial_\varphi \]

The four-velocity \( u_1^\mu \) of the particle must be proportional to the HKV at the location of the particle

\[ K_1^\mu = z_1 u_1^\mu \]

In suitable coordinate systems \( z_1 \) reduces to the inverse of the zeroth component of the particle’s velocity,

\[ z_1 = \frac{1}{u_1^t} = \sqrt{- (g_{\mu \nu})_1 v_1^\mu v_1^\nu} \]

The relation \( z_1(\Omega) \) is a well-defined observable which can be computed to high precision in PN theory.
The redshift observable was introduced in self-force computations of the motion of a particle around the black hole in the limit $m_1/m_2 \ll 1$

It represents the redshift of light rays emitted by the particle and received at infinity along the symmetry axis

$$z_1 = \frac{(k_\mu u^\mu)_{rec}}{(k_\mu u^\mu)_{em}} = \frac{1}{u^t_1}$$

This is also the Killing energy of the particle associated with the HKV
The PN metric is to be evaluated at the location of one of the particles:

\[ z_1 = \left[- \begin{pmatrix} g_{\mu\nu} \end{pmatrix}_1 v_1^\mu v_1^\nu \right]^{1/2} \]

A self-field regularization is required:

- Hadamard’s regularization [Hadamard 1932; Schwartz 1978] is convenient but has been shown to yield ambiguities at the 3PN order.
- Dimensional regularization [’t Hooft & Veltman 1972] is extremely powerful and is free of any ambiguity at 3PN order.
High-order PN result for the Redshift Observable
[Blanchet, Detweiler, Le Tiec & Whiting 2010ab]

- Posing $X_1 = m_1/m$ and still $x = (Gm\Omega/c^3)^{3/2}$, the redshift observable of particle 1 through 3PN order and augmented by 4PN and 5PN logarithmic contributions is

$$
z_1 = 1 + \left( -\frac{3}{2} X_1 + \frac{\nu}{2} \right) x + \left[ \cdots \right] x^2 + \left[ \cdots \right] x^3 + \left[ \cdots \right] x^4$$

$$
+ \left( \cdots + \left[ \cdots \right] \nu \ln x \right) x^5 + \left( \cdots + \left[ \cdots \right] \nu \ln x \right) x^6 + \mathcal{O}(x^7)
$$

- We can re-expand in the small mass-ratio limit $\nu = m_1m_2/m^2 \ll 1$ so that

$$
z_1 = z_{\text{Schw}} + \nu z_{\text{SF}} + \nu^2 z_{\text{PSF}} + \mathcal{O}(\nu^3)
$$

where

- $z_{\text{Schw}}$: Schwarzschild
- $z_{\text{SF}}$: Self-Force
- $z_{\text{PSF}}$: Post-Self-Force
The 3PN prediction agrees with the SF value with 7 significant digits

<table>
<thead>
<tr>
<th>3PN value</th>
<th>SF fit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_3 = -\frac{121}{3} + \frac{41}{32}\pi^2 = -27.6879026 \cdots$</td>
<td>$-27.6879034 \pm 0.0000004$</td>
</tr>
</tbody>
</table>

Post-Newtonian coefficients are fitted up to 7PN order

<table>
<thead>
<tr>
<th>PN coefficient</th>
<th>SF value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_4$</td>
<td>$-114.34747(5)$</td>
</tr>
<tr>
<td>$a_5$</td>
<td>$-245.53(1)$</td>
</tr>
<tr>
<td>$a_6$</td>
<td>$-695(2)$</td>
</tr>
<tr>
<td>$b_6$</td>
<td>$+339.3(5)$</td>
</tr>
<tr>
<td>$a_7$</td>
<td>$-5837(16)$</td>
</tr>
</tbody>
</table>
Comparison with the Self-Force Prediction

[Blanchet, Detweiler, Le Tiec & Whiting 2010]
We find by direct computation that the redshift observables $z_1$ and $z_2$ are related to the ADM mass and angular momentum by

$$\frac{\partial M}{\partial m_1} - \Omega \frac{\partial J}{\partial m_1} = z_1 \quad \text{and} \quad (1 \leftrightarrow 2)$$

Finally those relations can be summarized into the

**First law of binary point-particles mechanics**

$$\delta M - \Omega \delta J = z_1 \delta m_1 + z_2 \delta m_2$$

The first law tells how the ADM quantities change when the individual masses $m_1$ and $m_2$ of the particles vary (keeping the frequency $\Omega$ fixed).

An interesting consequence is the remarkably simple relation

**First integral of the first law**

$$M - 2\Omega J = z_1 m_1 + z_2 m_2$$
Agreement with the Generalized First Law of Mechanics

[Friedman, Uryū & Shibata 2002]

- Space-time generated by black holes and perfect fluid matter distributions
- Globally defined HKV field
- Asymptotic flatness

Generalized law of perfect fluid and black hole mechanics

\[
\delta M - \Omega \delta J = \int_{\Sigma} \left[ \bar{\mu} \Delta (dm) + \bar{T} \Delta (dS) + w^\mu \Delta (dC_\mu) \right] + \sum_n \frac{\kappa_n}{8\pi} \delta A_n
\]

where \( \Delta \) denotes the Lagrangian variation of the matter fluid, where \( dm \) is the conserved baryonic mass element, and where \( \bar{T} = zT \) and \( \bar{\mu} = z(h - T s) \) are the redshifted temperature and chemical potential.

In the point-particle limit for the fluid bodies (without BHs) one recovers formally the PN result.
The spins must be aligned or anti-aligned with the orbital angular momentum.

**First law of binary point particles with spins**

\[ \delta M - \Omega \delta J = \sum_{n=1}^{2} \left[ z_n \delta m_n + (\Omega_n - \Omega) \delta S_n \right] \]

- The precession frequency \( \Omega_n \) of the spins obeys
  \[ \frac{dS_n}{dt} = \Omega_n \times S_n \]

- The total angular momentum is related to the orbital angular momentum by
  \[ J = L + S_1 + S_2 \]
Analogies with single and binary black holes

1. **Single black hole** [Bardeen et al 1972]

\[
\delta M - \omega H \, \delta J = \frac{\kappa}{8\pi} \delta A
\]

2. **Two black holes** [Friedman, Uryū & Shibata 2002]

\[
\delta M - \Omega \, \delta J = \sum_{n=1}^{2} \frac{\kappa_n}{8\pi} \delta A_n
\]

3. **Two point particles** [Le Tiec, LB & Whiting 2012]

\[
\delta M - \Omega \, \delta J = \sum_{n=1}^{2} z_n \delta m_n
\]

4. **Two spinning point particles** [LB, Buonanno & Le Tiec 2012]

\[
\delta M - \Omega \, \delta J = \sum_{n=1}^{2} \left[ z_n \, \delta m_n + (\Omega_n - \Omega) \, \delta S_n \right]
\]
Analogies with single and binary black holes

1. Single black hole [Smarr 1973]

\[ M - 2\omega H J = \frac{\kappa}{4\pi} A \]

2. Two black holes [Friedman, Uryū & Shibata 2002]

\[ M - 2\Omega J = \sum_{n=1}^{2} \frac{\kappa_n}{4\pi} A_n \]

3. Two point particles [Le Tiec, LB & Whiting 2012]

\[ M - 2\Omega J = \sum_{n=1}^{2} z_n m_n \]

4. Two spinning point particles [LB, Buonanno & Le Tiec 2012]

\[ M - 2\Omega J = \sum_{n=1}^{2} \left[ z_n m_n + 2(\Omega_n - \Omega) S_n \right] \]
For point particles which have no finite extension the notion of rotation frequency of the body is meaningless. Thus the First Law is valid for arbitrary aligned or anti-aligned spins.
To describe extended bodies such as black holes one must supplement the point particles with some internal constitutive relation of the type

\[ m_n = m_n (m_{nirr}, S_n) \]

where \( S_n \) is the spin and \( m_{nirr} \) is some “irreducible” constant mass.

We define the response coefficients associated with the internal structure

\[ c_n = \left( \frac{\partial m_n}{\partial m_{nirr}} \right)_{S_n}, \quad \omega_n = \left( \frac{\partial m_n}{\partial S_n} \right)_{m_{nirr}} \]

where in particular \( \omega_n \) is the rotation frequency of the body.

The First Law becomes

\[ \delta M - \Omega \delta J = \sum_{n=1}^{2} \left[ z_n c_n \delta m_{nirr} + (z_n \omega_n + \Omega_n - \Omega) \delta S_n \right] \]
The First Law for Binary Corotating Black Holes

Corotation condition for extended particles [LB, Buonanno & Le Tiec 2012]

\[ z_n \omega_n = \Omega - \Omega_n \]

The First Law is then in agreement with the first law of two black holes [Friedman, Uryū & Shibata 2002]

\[ \delta M - \Omega \delta J = \sum_{n=1}^{2} \frac{\kappa_n}{8\pi} \delta A_n \]

provided that we make the identifications

\[ \begin{align*}
    m_n^{\text{irr}} & \leftrightarrow \sqrt{\frac{A_n}{16\pi}} \\
    z_n c_n & \leftrightarrow 4m_n^{\text{irr}} \kappa_n
\end{align*} \]
Conclusions

1. Compact binary star systems are the most important source for gravitational wave detectors LIGO/VIRGO and LISA.

2. Post-Newtonian theory has proved to be the appropriate tool for describing the inspiral phase of compact binaries up to the ISCO.

3. For massive BH binaries the PN templates should be matched to the results of numerical relativity for the merger and ringdown phases.

4. The PN approximation is now tested against different approaches such as the SF and performs extremely well.

5. The conservative part of the dynamics of compact binaries exhibits a First Law which is the analogue of the First Law of black hole mechanics.