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"The first law of binary black hole dynamics"

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Institut d'Astrophysique de Paris
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FIRST LAW OF BINARY BLACK HOLE DYNAMICS

dedicated to the 60$^{\text{th}}$ birthday of
Toshi Futamase, Hideo Kodama, Misao Sasaki

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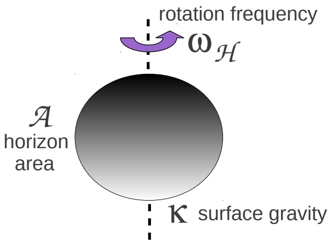
Gravitation et Cosmologie ($\mathcal{G}\mathbb{R}\varepsilon\mathbb{C}\mathcal{O}$)
Institut d'Astrophysique de Paris

15 novembre 2012

Four Laws of Black Hole Dynamics



ZEROTH LAW

Surface gravity κ is constant over the horizon of a stationary black hole

FIRST LAW

Mass M and angular momentum J of BH change according to [Bardeen, Carter & Hawking 1973]

$$\delta M - \omega_{\mathcal{H}}\,\delta J = \frac{\kappa}{8\pi}\delta\mathcal{A}$$

SECOND LAW

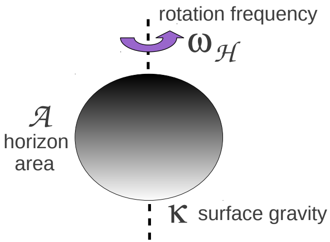
In any physical process involving one or several BHs with or without an environment [Hawking 1971]

$$\delta A \geqslant 0$$

THIRD LAW

It is impossible to achieve $\kappa = 0$ in any process

Four Laws of Black Hole Dynamics



ZEROTH LAW

Surface gravity κ is constant over the horizon of a stationary black hole

FIRST LAW

Mass M and angular momentum J of BH change according to [Christodoulou 1970, Smarr 1973]

$$M - 2\omega_{\mathcal{H}} J = \frac{\kappa}{4\pi}\mathcal{A}$$

SECOND LAW

In any physical process involving one or several BHs with or without an environment [Hawking 1971]

$$\delta A \geqslant 0$$

THIRD LAW

It is impossible to achieve $\kappa = 0$ in any process

Fourty Years of BH Thermodynamics [Bekenstein 1972, Hawking 1976]

- Using arguments involving a piece of matter with entropy thrown into a BH, Bekenstein derived the BH entropy

$$S_{\text{BH}} = \alpha\,\mathcal{A}$$

This would require $T_{\text{BH}} = \frac{\kappa}{8\pi\alpha}$ but the thermodynamic temperature of a *classical* BH is absolute zero since a BH is a perfect absorber

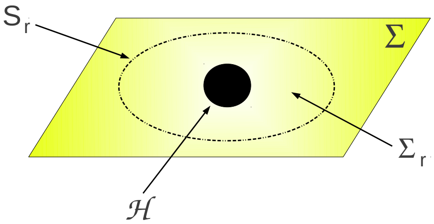
- However Hawking proved that *quantum* particle creation effects near a BH result in a black body temperature $T_{\text{BH}} = \frac{\kappa}{2\pi}$. This leads to the famous

Bekenstein-Hawking entropy of a stationary black hole

$$S_{\text{BH}} = \frac{c^3 k}{\hbar G}\frac{\mathcal{A}}{4}$$

The analogy between BH dynamics and the laws of thermodynamics is complete

Toward a Generalized First Law for a System of BHs



- The mass and the angular momentum of the BH are given by Komar surface integrals at spatial infinity

$$
\begin{aligned}
M &= -\frac{1}{8\pi}\lim_{r\to\infty}\oint_{S_r}\nabla^\mu t^\nu \, \mathrm{d}S_{\mu\nu} \\
J &= \frac{1}{16\pi}\lim_{r\to\infty}\oint_{S_r}\nabla^\mu \phi^\nu \, \mathrm{d}S_{\mu\nu}
\end{aligned}
$$

where t^μ and ϕ^μ are the two stationary and axi-symmetric Killing vectors

Toward a Generalized First Law for a System of BHs

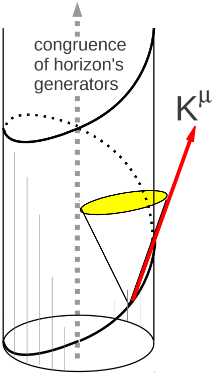
- The first law of BH dynamics expresses the change

$$\delta Q = \delta M - \omega_{\mathcal{H}}\, \delta J$$

in the Noether charge Q between two nearby BH configurations, where Q is associated with the Killing vector

$$K^\mu = t^\mu + \omega_{\mathcal{H}}\, \phi^\mu$$

which is the null generator of the BH horizon



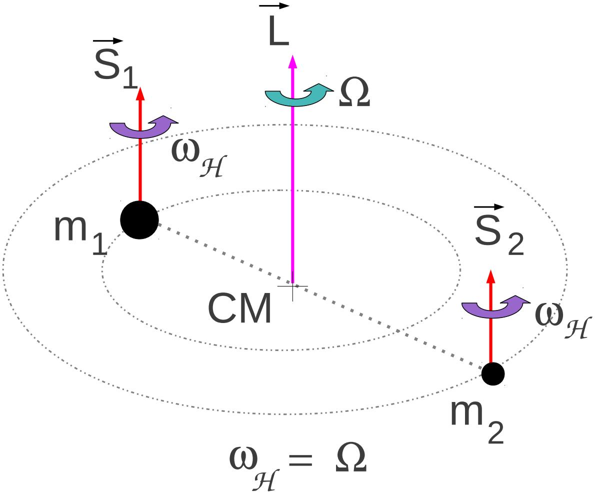
Toward a Generalized First Law for a System of BHs

- A generalized First Law valid for systems of BHs can be obtained when the geometry admits a Helical Killing Vector (HKV)

$$K^\mu \partial_\mu = \partial_t + \Omega\, \partial_\varphi$$

  where ∂_t is time-like and ∂_φ is space-like (with closed orbits), even when ∂_t and ∂_φ are not separately Killing vectors
- This applies to the case of two Kerr BHs moving on exactly circular orbits with orbital frequency Ω
- The two BHs should be in corotation, so that $\omega_{\mathcal{H}}$ should approximately be equal to Ω. In particular the spins should be aligned with the orbital angular momentum

Toward a Generalized First Law for a System of BHs



Toward a Generalized First Law for a System of BHs

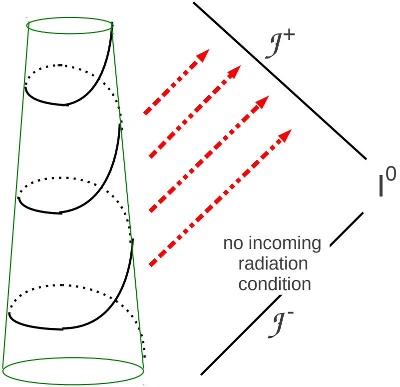
1. With the Helical Killing Vector $K^\mu \partial_\mu = \partial_t + \Omega\,\partial_\varphi$ the change in the associated Noether charge is given by

$$\delta Q = \delta M - \Omega\,\delta J$$

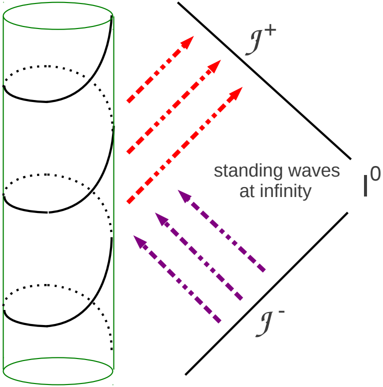
provided that the space-time is asymptotically flat [Friedman, Uryū & Shibata 2002]

2. However exact solutions of the Einstein field equations with Helical Killing symmetry cannot be asymptotically flat since they are periodic which contradicts the decrease of the Bondi mass at $\mathcal{J}^+$ [Gibbons & Stewart 1983]

Toward a Generalized First Law for a System of BHs



Physical situation

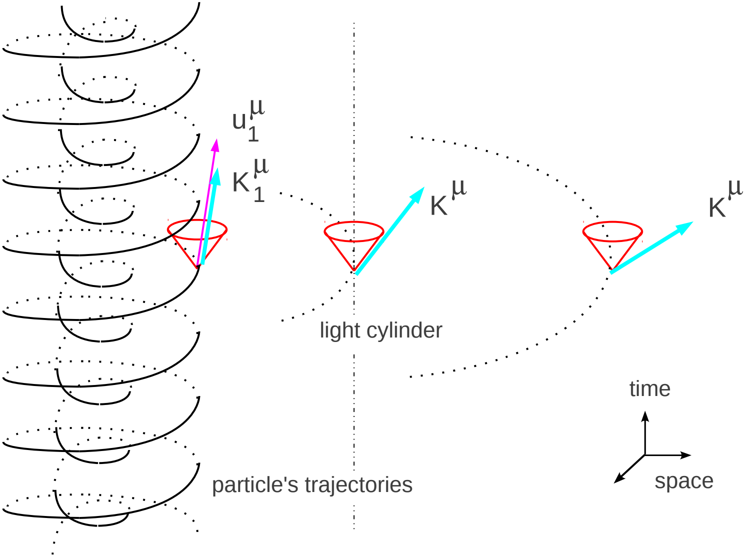


Situation with the HKV

Looking at the Conservative Part of the Dynamics

- One way to deal with the problem is to look at approximate solutions which are asymptotically flat. A possible solution is to suppress radiation degrees of freedom by imposing a condition of conformal flatness for the spatial metric [Isenberg & Nester 1980; Wilson & Mathews 1989]
- Here we follow a different route which is to consider only the conservative part of the dynamics in a post-Newtonian (PN) expansion, neglecting the dissipative effects due to the emission of gravitational radiation
- Thus we derive the First Law for a class of conservative PN space-times admitting a HKV and describing point particles (possibly with spins) moving on an exactly circular orbit

Two Point Particles on an Exactly Circular Orbit



Conservative versus Dissipative Dynamics in PN theory

- Internal acceleration of a matter system is written as a formal PN expansion

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{A}_{\mathrm{N}} + \frac{1}{c^2}\mathbf{A}_{1\mathrm{PN}} + \frac{1}{c^4}\mathbf{A}_{2\mathrm{PN}} + \frac{1}{c^5}\mathbf{A}_{2.5\mathrm{PN}} + \frac{1}{c^6}\mathbf{A}_{3\mathrm{PN}} + \frac{1}{c^7}\mathbf{A}_{3.5\mathrm{PN}} + \frac{1}{c^8}\mathbf{A}_{4\mathrm{PN}} + \mathcal{O}\left(\frac{1}{c^9}\right)$$

- Naive split would be to say that conservative effects are those which carry an even power of $1/c$, while dissipative effects, linked to gravitational radiation reaction, are those which carry an odd power of $1/c$

Conservative versus Dissipative Dynamics in PN theory

- This is correct at leading 2.5PN order where the force derives from a scalar in an appropriate gauge, $\mathbf{A}_{2.5\text{PN}} = \boldsymbol{\nabla} V_{2.5\text{PN}}$ with [Burke & Thorne]

$$V_{2.5\text{PN}}(\boldsymbol{x}, t) = -\frac{1}{5}\, x^i x^j I_{ij}^{(5)}(t)$$

  This term would change sign if we change the prescription of retarded potentials to the advanced potentials

- This is still correct at sub-leading order 3.5PN where the force involves both scalar and vector potentials given by [Blanchet 1997]

$$\begin{aligned} V_{3.5\text{PN}} &= \frac{1}{189} x^i x^j x^k I_{ijk}^{(7)}(t) - \frac{1}{70} \boldsymbol{x}^2 x^i x^j I_{ij}^{(7)}(t) \\ V_{3.5\text{PN}}^i &= \frac{1}{21} x^{\langle i} x^j x^{k\rangle} I_{jk}^{(6)}(t) - \frac{4}{45} \varepsilon_{ijk} x^j x^l J_{kl}^{(5)}(t) \end{aligned}$$

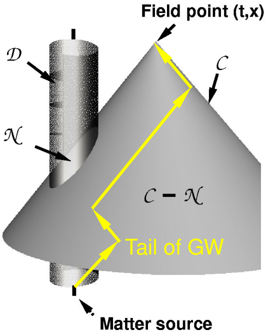
  which also change sign from retarded to advanced potentials

Dissipative Tail Effect in the PN Dynamics

However the naive split fails starting at 4PN order because of the appearance of tails in the radiation reaction force [Blanchet & Damour 1988]

$$V_{\mathrm{4PN}} = -\frac{4M}{5}\,x^i x^j \int_{-\infty}^{t} \mathrm{d}t'\, I^{(7)}_{ij}(t')\ln\left(\frac{t-t'}{2r}\right)$$

This term is not invariant when we go from retarded to advanced potentials



Logarithms at 4PN order in the Conservative Dynamics

With the HKV we have at our disposal the binary's orbital period $P = 2\pi/\Omega$. We split

$$\ln\left(\frac{t-t'}{2r}\right) = -\ln\left(\frac{r}{P}\right) + \ln\left(\frac{t-t'}{2P}\right)$$

Tails produce a conservative 4PN logarithmic term

$$V_{4\text{PN}} = -\frac{4M^2}{5}\,x^i x^j \Bigg[\underbrace{-I^{(6)}_{ij}(t)\ln\left(\frac{r}{P}\right)}_{\text{conservative 4PN log term}} + \underbrace{\int_{-\infty}^{t} \mathrm{d}t'\, I^{(7)}_{ij}(t')\ln\left(\frac{t-t'}{2P}\right)}_{\text{dissipative term (neglected)}} \Bigg]$$

We shall see appearing at 4PN and higher orders like 5PN some logarithmic contributions in the conservative part of the dynamics of binary black holes

Short History of the PN Approximation

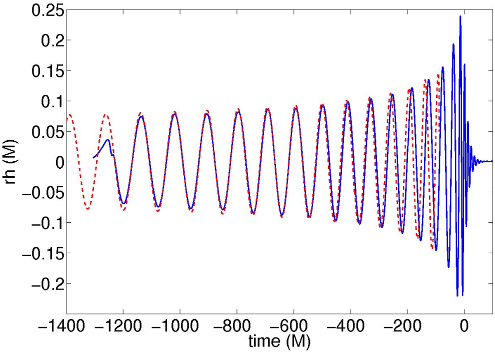
Equations of motion

- 1PN equations of motion [Lorentz & Droste 1917; Einstein, Infeld & Hoffmann 1938]
- Radiation-reaction controvercy [Ehlers *et al* 1979; Walker & Will 1982]
- 2.5PN equations of motion and GR prediction for the binary pulsar [Damour & Deruelle 1982; Damour 1983]
- The "3mn" Caltech paper [Cutler, Flanagan, Poisson & Thorne 1993]
- 3.5PN equations of motion [Jaranowski & Schäfer 1999; BF 2001; ABF 2002; BI 2003; Itoh & Futamase 2003, Foffa & Sturani 2011]
- Ambiguity parameters resolved [DJS 2001; BDE 2003]

Radiation field

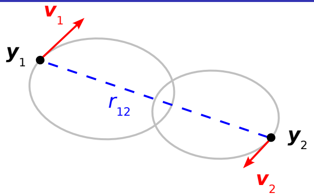
- 1918 Einstein quadrupole formula
- 1940 Landau-Lifchitz formula
- 1960 Peters-Mathews formula
- EW moments [Thorne 1980]
- BD moments and wave generation formalism [BD 1989; B 1995, 1998]
- 1PN phasing [Wagoner & Will 1976; BS 1989]
- Test-particle limit using BH perturbations [Tagoshi & Sasaki 1994]
- 2PN waveform [BDIWW 1995]
- 3.5PN phasing and 3PN waveform [BFIJ 2003, BFIS 2007]
- Ambiguity parameters resolved [BI 2004; BDEI 2004, 2005]

The Gravitational Chirp of Compact Binaries



The waveform is obtained by matching a high-order post-Newtonian waveform describing the long inspiralling phase and a highly accurate numerical waveform describing the final merger and ringdown phases

3.5PN Equations of Motion of Compact Binary Systems



Explicit EOM for non-spinning compact binaries

$$\frac{\mathrm{d}v_1^i}{\mathrm{d}t} = -\frac{Gm_2}{r_{12}^2} n_{12}^i$$
$$+ \frac{1}{c^2} \overbrace{\left\{ \left[\frac{5G^2 m_1 m_2}{r_{12}^3} + \frac{4G^2 m_2^2}{r_{12}^3} + \frac{Gm_2}{r_{12}^2} \left(\frac{3}{2}(n_{12}v_2)^2 - v_1^2 - 2v_2^2 \right) \right] n_{12}^i + \cdots \right\}}^{\text{1PN}}$$
$$+ \underbrace{\frac{1}{c^4}[\cdots]}_{\text{2PN}} + \underbrace{\frac{1}{c^5}[\cdots]}_{\text{2.5PN}} + \underbrace{\frac{1}{c^6}[\cdots]}_{\text{3PN}} + \underbrace{\frac{1}{c^7}[\cdots]}_{\text{3.5PN}} + \mathcal{O}\left(\frac{1}{c^8}\right)$$

Spin effects arise at orders 1.5PN for the spin-orbit and 2PN for the spin-spin.

Mass and Angular Momentum of Compact Binaries

It is convenient tu use the gauge invariant PN parameter

$$x = \left(\frac{Gm\Omega}{c^3}\right)^{3/2}$$

with the mass parameters $m = m_1 + m_2$ and $\nu = m_1 m_2 / m^2$.

Conservative PN energy for circular orbits

$$E = -\frac{1}{2} m\nu \Big\{ 1 + \overbrace{\left(-\frac{3}{4} - \frac{\nu}{12}\right) x}^{\text{1PN}} + \overbrace{[\cdots]\, x^2}^{\text{2PN}} + \overbrace{[\cdots]\, x^3}^{\text{3PN}}$$
$$+ \overbrace{\left(\cdots + \frac{448}{15} \nu \ln x\right) x^4}^{\text{4PN}} + \overbrace{\left(\cdots + \left[-\frac{4988}{35} - 6565\nu\right] \nu \ln x\right) x^5}^{\text{5PN}} + \mathcal{O}\left(x^6\right) \Big\}$$

The 4PN and 5PN conservative logarithmic terms have been computed recently [Blanchet, Detweiler, Le Tiec & Whiting 2010]

Mass and Angular Momentum of Compact Binaries

The angular momentum J is checked to satisfy for all the terms up to 3PN order, and also for the 4PN and 5PN log terms, the

Thermodynamic relation valid for circular orbits

$$\frac{\partial M}{\partial \Omega} = \Omega \, \frac{\partial J}{\partial \Omega}$$

which constitutes the first ingredient in the First Law of binary black holes.

- The thermodynamic relation states that the flux of energy emitted in the form of gravitational waves is proportional to the flux of angular momentum
- It is used in numerical computations of the binary evolution based on a sequence of quasi-equilibrium configurations [Gourgoulhon *et al* 2002]

The Redshift Observable [Detweiler 2008]

- The geometry has a Helical Killing Vector (HKV) asymptotically given by

$$K^\mu \partial_\mu = \partial_t + \Omega\, \partial_\varphi$$

- The four-velocity u_1^μ of the particle must be proportional to the HKV at the location of the particle

$$K_1^\mu = z_1\, u_1^\mu$$

- In suitable coordinate systems z_1 reduces to the inverse of the zeroth component of the particle's velocity,

$$z_1 = \frac{1}{u_1^t} = \sqrt{-(g_{\mu\nu})_1\, v_1^\mu v_1^\nu}$$

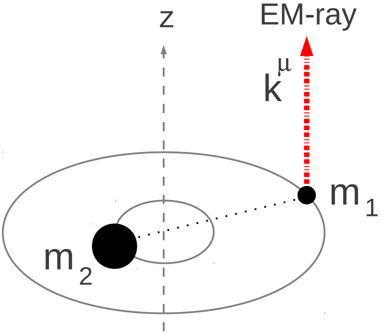
- The relation $z_1(\Omega)$ is a well-defined observable which can be computed to high precision in PN theory

The Redshift Observable [Detweiler 2008]

- The redshift obervable was introduced in self-force computations of the motion of a particle around the black hole in the limit $m_1/m_2 \ll 1$
- It represents the redshift of light rays emitted by the particle and received at infinity along the symmetry axis

$$z_1 = \frac{(k_\mu u^\mu)_{\text{rec}}}{(k_\mu u^\mu)_{\text{em}}} = \frac{1}{u_1^t}$$

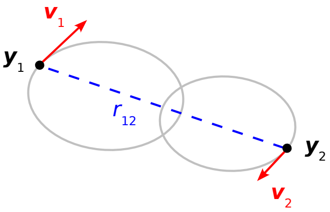
- This is also the Killing energy of the particle associated with the HKV



Post-Newtonian Computation of the Redshift Observable

The PN metric is to be evaluated at the location of one of the particles

$$z_1 = \Big[- \underbrace{(g_{\mu\nu})_1}_{\text{regularized metric}} v_1^\mu v_1^\nu \Big]^{1/2}$$



A self-field regularization is required

- Hadamard's regularization [Hadamard 1932; Schwartz 1978] is convenient but has been shown to yield ambiguities at the 3PN order
- Dimensional regularization ['t Hooft & Veltman 1972] is extremely powerful and is free of any ambiguity at 3PN order

High-order PN result for the Redshift Observable

[Blanchet, Detweiler, Le Tiec & Whiting 2010ab]

- Posing $X_1 = m_1/m$ and still $x = (Gm\Omega/c^3)^{3/2}$, the redshift observable of particle 1 through 3PN order and augmented by 4PN and 5PN logarithmic contributions is

$$\begin{aligned} z_1 &= 1 + \left(-\frac{3}{2}X_1 + \frac{\nu}{2}\right) x + \overbrace{[\cdots]\, x^2}^{\text{1PN}} + \overbrace{[\cdots]\, x^3}^{\text{2PN}} + \overbrace{[\cdots]\, x^4}^{\text{3PN}} \\ &\quad + \Big(\cdots \underbrace{+ [\cdots]\, \nu \ln x}_{\text{4PN log}}\Big) x^5 + \Big(\cdots \underbrace{+ [\cdots]\, \nu \ln x}_{\text{5PN log}}\Big) x^6 + \mathcal{O}\left(x^7\right) \end{aligned}$$

- We can re-expand in the small mass-ratio limit $\nu = m_1 m_2/m^2 \ll 1$ so that

$$z_1 = z_{\text{Schw}} + \underbrace{\nu\, z_{\text{SF}}}_{\text{self-force}} + \underbrace{\nu^2\, z_{\text{PSF}}}_{\text{post-self-force}} + \mathcal{O}(\nu^3)$$

High-order PN fit to the Numerical Self Force

- The 3PN prediction agrees with the SF value with 7 significant digits

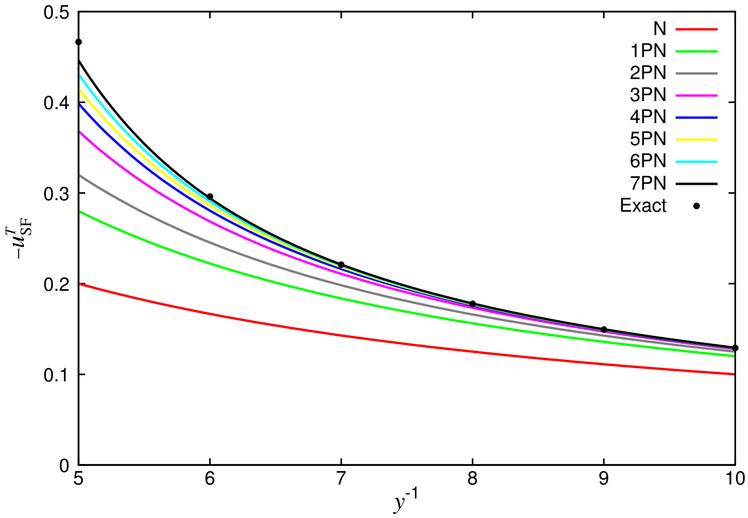
3PN value	SF fit
$a_3 = -\frac{121}{3} + \frac{41}{32}\pi^2 = -27.6879026\cdots$	$-27.6879034 \pm 0.0000004$

- Post-Newtonian coefficients are fitted up to 7PN order

PN coefficient	SF value
a_4	$-114.34747(5)$
a_5	$-245.53(1)$
a_6	$-695(2)$
b_6	$+339.3(5)$
a_7	$-5837(16)$

Comparison with the Self-Force Prediction

[Blanchet, Detweiler, Le Tiec & Whiting 2010]



First Law of Binary Point Particle Mechanics

[Le Tiec, Blanchet & Whiting 2011]

1. We find by direct computation that the redshift observables z_1 and z_2 are related to the ADM mass and angular momentum by

$$\frac{\partial M}{\partial m_1} - \Omega \frac{\partial J}{\partial m_1} = z_1 \quad \text{and} \quad (1 \leftrightarrow 2)$$

2. Finally those relations can be summarized into the

First law of binary point-particles mechanics

$$\delta M - \Omega\, \delta J = z_1\, \delta m_1 + z_2\, \delta m_2$$

The first law tells how the ADM quantities change when the individual masses m_1 and m_2 of the particles vary (keeping the frequency Ω fixed)

3. An interesting consequence is the remarkably simple relation

First integral of the first law

$$M - 2\Omega J = z_1 m_1 + z_2 m_2$$

Agreement with the Generalized First Law of Mechanics

[Friedman, Uryū & Shibata 2002]

- Space-time generated by black holes and perfect fluid matter distributions
- Globally defined HKV field
- Asymptotic flatness

Generalized law of perfect fluid and black hole mechanics

$$\delta M - \Omega \delta J = \int_\Sigma \left[\bar{\mu}\, \Delta(\mathrm{d}m) + \bar{T}\, \Delta(\mathrm{d}S) + w^\mu \Delta(\mathrm{d}C_\mu) \right] + \sum_n \frac{\kappa_n}{8\pi}\, \delta A_n$$

where Δ denotes the Lagrangian variation of the matter fluid, where $\mathrm{d}m$ is the conserved baryonic mass element, and where $\overline{T} = zT$ and $\overline{\mu} = z(h - Ts)$ are the redshifted temperature and chemical potential

In the point-particle limit for the fluid bodies (without BHs) one recovers formally the PN result

First law of mechanics for binary point particles with spins

[Blanchet, Buonanno & Le Tiec 2012]

The spins must be aligned or anti-aligned with the orbital angular momentum.

First law of binary point particles with spins

$$\delta M - \Omega\,\delta J = \sum_{n=1}^{2} \Big[z_n\,\delta m_n + (\Omega_n - \Omega)\,\delta S_n \Big]$$

- The precession frequency Ω_n of the spins obeys

$$\frac{\mathrm{d}\boldsymbol{S}_n}{\mathrm{d}t} = \boldsymbol{\Omega}_n \times \boldsymbol{S}_n$$

- The total angular momentum is related to the orbital angular momentum by

$$J = L + S_1 + S_2$$

Analogies with single and binary black holes

1. Single black hole [Bardeen et al 1972]

$$\delta M - \omega_{\mathcal{H}}\, \delta J = \frac{\kappa}{8\pi} \delta \mathcal{A}$$

2. Two black holes [Friedman, Uryū & Shibata 2002]

$$\delta M - \Omega\, \delta J = \sum_{n=1}^{2} \frac{\kappa_n}{8\pi} \delta \mathcal{A}_n$$

3. Two point particles [Le Tiec, LB & Whiting 2012]

$$\delta M - \Omega\, \delta J = \sum_{n=1}^{2} z_n \delta m_n$$

4. Two spinning point particles [LB, Buonanno & Le Tiec 2012]

$$\delta M - \Omega\, \delta J = \sum_{n=1}^{2} \Big[z_n\, \delta m_n + (\Omega_n - \Omega)\, \delta S_n \Big]$$

Analogies with single and binary black holes

1. Single black hole [Smarr 1973]

$$M - 2\omega_{\mathcal{H}}\, J = \frac{\kappa}{4\pi}\mathcal{A}$$

2. Two black holes [Friedman, Uryū & Shibata 2002]

$$M - 2\Omega\, J = \sum_{n=1}^{2} \frac{\kappa_n}{4\pi}\mathcal{A}_n$$

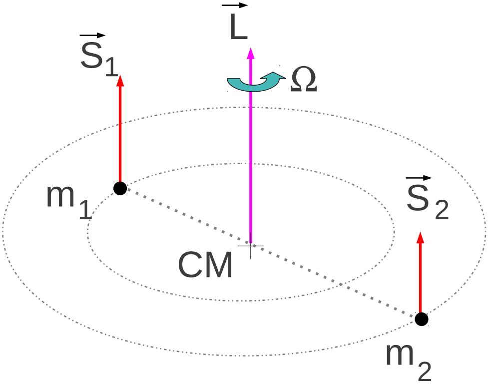
3. Two point particles [Le Tiec, LB & Whiting 2012]

$$M - 2\Omega\, J = \sum_{n=1}^{2} z_n m_n$$

4. Two spinning point particles [LB, Buonanno & Le Tiec 2012]

$$M - 2\Omega\, J = \sum_{n=1}^{2} \Big[z_n\, m_n + 2\,(\Omega_n - \Omega)\, S_n \Big]$$

Analogies with single and binary black holes



For point particles which have no finite extension the notion of rotation frequency of the body is meaningless. Thus the First Law is valid for arbitrary aligned or anti-aligned spins

The first law for Binary Corotating Black Holes

1. To describe extended bodies such as black holes one must suplement the point particles with some internal constitutive relation of the type

$$m_n = m_n\left(m_n^{\text{irr}}, S_n\right)$$

where S_n is the spin and m_n^{irr} is some "irreducible" constant mass

2. We define the response coefficients associated with the internal structure

$$c_n = \left(\frac{\partial m_n}{\partial m_n^{\text{irr}}}\right)_{S_n}, \qquad \omega_n = \left(\frac{\partial m_n}{\partial S_n}\right)_{m_n^{\text{irr}}}$$

where in particular ω_n is the rotation frequency of the body

3. The First Law becomes

$$\delta M - \Omega\,\delta J = \sum_{n=1}^{2}\left[z_n\,c_n\,\delta m_n^{\text{irr}} + \left(z_n\,\omega_n + \Omega_n - \Omega\right)\delta S_n\right]$$

The First Law for Binary Corotating Black Holes

Corotation condition for extended particles [LB, Buonanno & Le Tiec 2012]

$$z_n \, \omega_n = \Omega - \Omega_n$$

The First Law is then in agreement with the first law of two black holes [Friedman, Uryū & Shibata 2002]

$$\delta M - \Omega \, \delta J = \sum_{n=1}^{2} \frac{\kappa_n}{8\pi} \delta \mathcal{A}_n$$

provided that we make the identifications

$$m_n^{\text{irr}} \quad \longleftrightarrow \quad \sqrt{\frac{\mathcal{A}_n}{16\pi}}$$

$$z_n \, c_n \quad \longleftrightarrow \quad 4 m_n^{\text{irr}} \kappa_n$$

Conclusions

1. Compact binary star systems are the most important source for gravitational wave detectors LIGO/VIRGO and LISA
2. Post-Newtonian theory has proved to be the appropriate tool for describing the inspiral phase of compact binaries up to the ISCO
3. For massive BH binaries the PN templates should be matched to the results of numerical relativity for the merger and ringdown phases
4. The PN approximation is now tested against different approaches such as the SF and performs extremely well
5. The conservative part of the dynamics of compact binaries exhibits a First Law which is the analogue of the First Law of black hole mechanics