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Adiabatic evolution of resonant orbits on Kerr spacetime

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Abstract

We discuss the inspiral of a small body in a background Kerr spacetime. In the adiabatic regime, the radiation reaction is solely characterized by the averaged rate of the change of the constants of motion, the energy E, the azimuthal angular momentum L and the Carter constant Q. Despite the lack of the global conservation law for the Carter constant, it has been shown that its long time averaged rate of change can be simply computed via "global conservation law like" formula when there exists a simultaneous turning point of the radial and polar oscillations. However, an inspiralling orbit may cross a *resonance* point, where the frequencies of the radial and polar orbital oscillations are in a rational ratio. At the resonant point, one cannot find a simultaneous turning point in general, thus a direct computation of the self-forces, quite challenging especially in the Kerr background, seems to be necessary. Contrary to this expectation, we here show that we can still compute the averaged rate of change of the Carter constant in a relatively simple manner even at the resonance point.

1 Introduction

A supermassive black hole accompanied by a compact object is one of the potential sources of the low frequency gravitational waves. This system can be considered as a particle with mass μ moving along a bound orbit on a much larger Kerr black hole with the mass $M(\gg \mu)$ and the spin parameter a. Then, the dominant part of the long time orbital evolution due to the radiation reaction of the gravitational wave emission is dictated by the long time averaged rates of the change of three constants of motion: the energy E, the azimuthal angular momentum L and the Carter constant Q as long as the characteristic time scale of the secular orbital evolution due to the radiation reaction is sufficiently longer than the orbital period. This method is called the adiabatic approximation [1, 2].

The averaged rates of change of the energy and the azimuthal angular momentum of the particle can be balanced with the ones carried by the gravitational waves due to their global conservation laws. Although we do not have any balance argument for the Carter constant, based on Mino's work [1], it has shown that the averaged rate of change of the Carter constant can be also computed by a practically simple formula [2, 3]. However, the derivation of above mentioned formula implicitly assumes that there exists a simultaneous turning point of the radial and polar oscillations. The problem is that we cannot find such a turning point for a inspiral that crosses a resonant point, where the inspiral's radial and polar orbital frequencies, Ω_r and Ω_θ, takes $\Omega_r/\Omega_\theta = j_r/j_\theta$ with coprime integers, j_r and j_θ.

The purpose of this article is overviewing our claim that we can still easily compute the adiabatic evolution of the Carter constant in the resonance case, together with the scalar toy model: a point scalar particle with mass μ_0 and charge q coupled to its own scalar field, and moving on the resonant bounded geodesic around the Kerr black hole. Further details are presented in our preparing manuscript [5]. In this paper, we use geometrical units $G = c = 1$ and take the sign convention of the metric is $(-, +, +, +)$. For saving the space of the article, the basic tools required for the adiabatic evolution of the scalar toy model, such as the solution of geodesic equation and the mode-decomposed retarded solution of the scalar field, are borrowed from Drasco *et al.*[6] without derivation.

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2 A bounded resonant geodesic in Kerr spacetime

We label the geodesic motion of a scalar particle with the bare rest mass $\mu = 1$ and a scalar charge q as $z(\lambda) := (t(\lambda),\ r(\lambda),\ \theta(\lambda),\ \phi(\lambda))$ with the "Mino time" λ, which is related to the proper time τ as $d\lambda := d\tau/\Sigma$ with $\Sigma := r^2 + a^2\cos^2\theta$. In terms of the Mino time, the r and θ oscillations are independently periodic for bound orbits:

$$r(\lambda) = r(\lambda + n\Lambda_r), \qquad \theta(\lambda) = \theta(\lambda + k\Lambda_\theta), \tag{1}$$

where n and k are integers, and Λ_r and Λ_θ are the periods with respect to the Mino time λ. The precise meaning of the resonance is that the orbital frequencies $\Upsilon_{r,\theta} := 2\pi/\Lambda_{r,\theta}$ with respect to the Mino time λ, or the frequency $\Omega_{r,\theta} = \Upsilon_{r,\theta}/\Upsilon_t$ with respect to t are related to each other as

$$\frac{\Upsilon_r}{j_r} = \frac{\Upsilon_\theta}{j_\theta} =: \Upsilon\ , \tag{2}$$

where j_r and j_θ are coprime integers. In the resonance case, there is always a difference between the times reaching the minima of r and θ oscillations. We call this difference "offset phase" and denote it as $\Delta\lambda$,and choose $\Delta\lambda$ such that $|\Delta\lambda| \le \Lambda'/2$, where $\Lambda' = (2\pi/\Upsilon)/(j_r + j_\theta)$.

The motion in the t and ϕ directions is also decoupled into r- and θ-dependent parts. Indeed, we have

$$t(\lambda) = \Upsilon_t\lambda + \Delta t_r(\lambda - \Delta\lambda) + \Delta t_\theta(\lambda), \qquad \phi(\lambda) = \Upsilon_\phi\lambda + \Delta\phi_r(\lambda - \Delta\lambda) + \Delta\phi_\theta(\lambda), \tag{3}$$

where Υ_t and Υ_ϕ are the averaged orbital frequency, and Δt_r and $\Delta\phi_r$ are the oscillating function with the period Λ_r. The meaning of the functions Δt_θ and $\Delta\phi_\theta$ can be understood in the same manner.

For detailed derivations, see Draso *et al.*[6].

3 Adiabatic evolution of the Carter constant with resonance

Due to the lack of the global conservation law for the Carter constant, we go back to the definition of the Carter constant, and start from the bare formula of its long time averaged rate of the change, which is written in terms of the scalar self-force acting on the particle. The Carter constant in the Kerr spacetime is defined as (*e.g.*. see Ref. [3])

$$Q := \left(\frac{L}{\sin\theta} - a\sin\theta E\right)^2 + a^2\cos^2\theta + \Sigma^2(u^\theta)^2, \tag{4}$$

where $u^\alpha := dz^\alpha/d\tau$, and E and L are the conserved energy and the azimuthal angular momentum of the particle, respectively. With the aid of the geodesic equations, the expression of the long time averaged value of the Mino-time derivative of the Carter constants takes

$$\left\langle \frac{dQ}{d\lambda} \right\rangle_\lambda = -2q \left\langle \left\{ \left(V_{tr}(r)\partial_t + V_{\phi r}(r)\partial_\phi + \frac{dr(\lambda)}{d\lambda}\partial_r \right) \left[\Sigma(x)\Phi^{(R)}(x) \right] \right\}_{x=z(\lambda)} \right\rangle_\lambda , \tag{5}$$

where $V_{tr}(r)$ and $V_{\phi r}(r)$ are the r-dependent potentials of the t and ϕ components of the geodesic equations [6]. In Eq. (5), the symbol $\langle \dots \rangle$ denotes the λ average: $\langle F(\lambda)\rangle_\lambda := \lim_{T\to+\infty}(1/(2T))\int_{-T}^{+T} d\lambda' F(\lambda')$ for a function $F(\lambda)$. The field $\Phi^{(R)}(x)$ is the R-part of the retarded scalar field, which is regular even at the particle's location [7]. With the aid of the "radiative (anti-symmetric)" field defined by the half-retarded *minus* half-advanced fields, and the "symmetric" field also defined by the half-retarded *plus* half-advanced field, we have $\Phi^{(R)}(x) := \Phi^{(\mathrm{rad})}(x) + \left\{\Phi^{(\mathrm{sym})}(x) - \Phi^{(S)}(x)\right\}$, where $\Phi^{(S)}(x)$ is the S-part of the retarded field, which is the symmetric for the argument of the Green function, satisfies the same inhomogeneous Klein-Gordon equation as retarded solution, and shares the same singular structure as the one of retarded/advanced field. To ensure the regularity of $\Phi^{(R)}(x)$, the subtraction of $\Phi^{(S)}(x)$ is essential.

Noting that either radiative, symmetric and S-part field, which we put the symbol $\Phi^{(\sharp)}$ to schematically denote them, is defined through the Green function as $\Phi^{(\sharp)} := -q\int d\lambda\Sigma[z(\lambda)]G^{(\sharp)}[x, z(\lambda)]$, Eq. (5) can be rewritten with these Green functions. After integral by parts of Eq. (5) and rather involved algebra, we arrive at expression for the radiative part:

$$\left\langle \frac{dQ}{d\lambda} \right\rangle_\lambda^{(\mathrm{rad})} = 2q^2 \int_{-\infty}^{+\infty} d\lambda' \, \Sigma[z(\lambda')] \left\langle \left[\Sigma(x) \left(\langle V_{tr} \rangle \partial_t - \langle V_{\phi r} \rangle \partial_\phi \right) G^{(\mathrm{rad})}(x, z(\lambda')) \right]_{x=z(\lambda)} \right\rangle_\lambda \\ -2q^2 \int_{-\infty}^{+\infty} d\lambda' \left[\Sigma(x') \frac{d}{d(\Delta\lambda)} \left\langle \Sigma[z(\lambda)] G^{(\mathrm{rad})}(z(\lambda), x') \right\rangle_\lambda \right]_{x'=z(\lambda')} . \quad (6)$$

Note that $z(\lambda)$ implicitly depends on the offset phase $\Delta\lambda$.

The symmetric and S-part contribution to $\langle dQ/dt \rangle^{(\sharp)}$ is more tricky since the each of $G^{(\mathrm{sym})}[z(\lambda), z(\lambda')]$ and $G^S[z(\lambda), z(\lambda')]$ diverges in the coincidence limit $z(\lambda) \to z(\lambda')$, while their difference is regular in this limit. The problem here is that we can just compute symmetric and S-part field separately since the S-part Green function is only defined near the particle location[7]. To remedy this situation, we here introduce the point splitting regularization by displacing the orbits $z(\lambda)$ and $z(\lambda')$ as $z(\lambda) \to z_+(\lambda) := z(\lambda) + (\epsilon/2)\xi$, and $z(\lambda') \to z_-(\lambda') := z(\lambda') - (\epsilon/2)\xi$ with a small parameter $\epsilon \ll 1$, and the Killing field $\xi := \cos\zeta \, \xi^{(t)\mu} + (\Omega_\phi \cos\zeta - \Omega \sin\zeta) \, \xi^{(\phi)\mu}$ as well as the Killing vectors associated with the stationarity and axisymmetry of the Kerr spacetime: $\xi^{(t)\mu}$ and $\xi^{(\phi)\mu}$. Here Ω_ϕ and Ω are the orbital frequency with respect to t, not to the Mino-time, and we introduce the parameter ξ to specify the Killing field.

Taking care of this point splitting regularization, the symmetric contribution to $\langle dQ/dt \rangle$, after subtracting the S-part contribution and making it regularized, can be written as

$$\left\langle \frac{dQ}{d\lambda} \right\rangle_\lambda^{(\mathrm{sym}-S)} = -\lim_{\epsilon \to 0} q^2 \frac{d}{d(\Delta\lambda)} \left[\Psi^{(\mathrm{sym})}(\Delta\lambda, \epsilon) - \Psi^{(S)}(\Delta\lambda, \epsilon) \right] . \quad (7)$$

where we introduce the symmetric and S-part potential $\Psi^{(\mathrm{sym})/(S)}(\Delta\lambda, \epsilon)$ defined by

$$\Psi^{(\mathrm{sym})/(S)}(\Delta\lambda, \epsilon) := \int_{-\infty}^{+\infty} d\lambda' \Sigma[z_-(\lambda')] \left\langle \Sigma[z_+(\lambda)] G^{(\mathrm{sym})/\mathrm{S}}[z_+(\lambda), z_-(\lambda')] \right\rangle_\lambda . \quad (8)$$

4 Simplified formula for the regularized symmetric part

The expressions Eqs. (6) and (7) are rewritten as more practical expressions if we take care of that the variables of the scalar Klein-Gordon equation is separable in the Kerr spacetime. In fact, its homogeneous solution, namely mode functions is written by $\pi^\flat_{\omega\ell m}(t, r, \theta, \phi) := (2/\sqrt{r^2 + a^2}) e^{-i\omega t} e^{im\phi} \theta_{\omega\ell m}(\theta) u^\flat_{\omega\ell m}(r^*)$ where m is the integer, r^* is the tortoise coordinate defined by $dr^* := ((r^2 + a^2)/(r^2 - 2Mr + a^2))dr$, and $\theta_{\omega\ell m}(\theta) e^{im\phi} := S_{\omega\ell m}(\theta, \phi)$ is the spheroidal harmonics normalized as $\int d\theta d\phi \sin\theta S_{\omega\ell m}(\theta, \phi)^* S_{\omega\ell' m'}(\theta, \phi) = \delta_{\ell\ell'} \delta_{mm'}$.. The symbol $\flat$ represents one of the four distinct boundary conditions: "up", "down", "in" and "out". The corresponding radial functions $u^\flat_{\omega\ell m}(r^*)$ are respectively defined in *e.g.* Ref.[6]. The mode functions enable us to write down the retarded Green function as the factorized form:

$$G^{(\mathrm{ret})}(x, \, x') = \frac{1}{16\pi i} \int_{-\infty}^{+\infty} d\omega \sum_{\ell=0}^{+\infty} \sum_{m=-\ell}^{\ell} \frac{\omega}{|\omega|} \frac{1}{\alpha_{\omega\ell m} \beta_{\omega\ell m}} \\ \times \left[\pi^{\mathrm{up}}_{\omega\ell m}(x) \pi^{\mathrm{out}\,*}_{\omega\ell m}(x') H(r - r') + \pi^{\mathrm{in}}_{\omega\ell m}(x) \pi^{\mathrm{down}\,*}_{\omega\ell m}(x') H(r' - r) \right] \quad (9)$$

with the Heaviside step function $H(x) := \int_{-\infty}^{x} \delta(y) dy$. Thinking of the definition of the radiative and the symmetric field, at least, the Green functions associated with these two field are also written in the factorized form with the relation given by $G^{(\mathrm{rad})}(x, x') := (1/2)[G^{(\mathrm{ret})}(x, x') - G^{(\mathrm{ret})}(x', x)]$ and $G^{(\mathrm{sym})}(x, x') := (1/2)[G^{(\mathrm{ret})}(x, x') + G^{(\mathrm{ret})}(x', x)]$ with $G^{(\mathrm{ret})}(x, x') = G^{(\mathrm{adv})}(x', x)$. Their explicit forms will be shown our manuscript [5].

The simplified formula is essentially derived by substituting the radiative and the symmetric Green function in the factorized form into Eqs. (6) and (7). Considering $\langle dQ/d\lambda \rangle_\lambda^{(\mathrm{rad})}$ has already computed by both Flanagan *et.al.* [8] and us [5], we bravely skip to show its explicit form here. See above two references for technical issues and its final form.

Compared to $\langle dQ/d\lambda\rangle_\lambda^{(\mathrm{rad})}$, the simplification of $\langle dQ/d\lambda\rangle_\lambda^{(\mathrm{sym})}$ and $\langle dQ/d\lambda\rangle_\lambda^{(S)}$ is much involved. The main obstacle is that the S-part Green function does not admit the mode decomposition in terms of the spheroidal harmonics, while we can easily decompose the symmetric Green function with the aid of Eq. (9). Here, the key observation here is what we need is the mode decomposition of the potentials defined in Eq. (8), rater than that of the Green functions themselves. We note that the bound geodesic allows to discretized the frequency ω as $\omega_{mN} := m\Omega_\phi + N\Omega$ with the integer N. With the aid of this information and the retarded Green fucntion in mode decomposed form given in Eq. (9), it is straightforward to obtain the regularized symmetric potential in the (N,m)-decomposed form:

$$\Psi^{(\mathrm{sym})}(\Delta\lambda;\epsilon) = \sum_{Nm} e^{i(\epsilon_1 N+\epsilon_2 m)\Omega}\Psi^{(\mathrm{sym})}_{Nm}(\Delta\lambda), \tag{10}$$

where $\epsilon_1 = \epsilon\cos\zeta$ and $\epsilon_2 = \epsilon\sin\zeta$. remarkably, Eq. (10) is nothing but a two-dimensional Fourier series expansion of $\Psi^{(\mathrm{sym})}(\Delta\lambda;\epsilon)$. In other words, the (N,m)-mode decomposition is simply achieved via its inverse Fourier transformation. This is clearly good news for the S-part potential since its local expansion near the particle can be computed as

$$\Psi^{(S)}(\Delta\lambda;\epsilon) = -\frac{\psi(\zeta)}{\epsilon} + O(\epsilon), \tag{11}$$

with a ζ-dependent function. Thus, the (N,m)-mode decomposition of Eq. (11) is just derived as

$$\Psi^{(S)}_{Nm}(\Delta\lambda) = \frac{\Omega^2}{4\pi^2}\int_{-\frac{\pi}{\Omega}}^{\frac{\pi}{\Omega}} d\epsilon_1 \int_{-\frac{\pi}{\Omega}}^{\frac{\pi}{\Omega}} d\epsilon_2 e^{-i(\epsilon_1 N+\epsilon_2 m)\Omega}\Psi^{(S)}(\Delta\lambda;\epsilon) \tag{12}$$

All in all, the expression given in Eq. (7) can be computed as by mode-by-mode subtraction:

$$\left\langle \frac{dQ}{dt}\right\rangle_t^{(\mathrm{sym})} = -\frac{q^2}{\Upsilon_t}\sum_{Nm} e^{i(\epsilon_1 N+\epsilon_2 m)\Omega}\left[\frac{d}{d(\Delta\lambda)}\left\{\Psi^{(\mathrm{sym})}_{Nm}(\Delta\lambda) - \Psi^{(S)}_{Nm}(\Delta\lambda)\right\}\right]. \tag{13}$$

The merit of Eq. (13) is that both (N,m)-modes of the symmetric and S-part potentials are easily handled in the numerical implementation without bothering the divergence. In addition, the S-part has already subtracted at the level of mode decomposed form, we do not see any divergence even after summing up with respect to the N and m modes of the right hand side of Eq. (13). Again, further technical details will be explained our manuscript [5].

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