Kazuharu Bamba, JGRG 22(2012)111414

"Finite-time future singularities and rip cosmology in f(T) gravity"

RESCEU SYMPOSIUM ON

GENERAL RELATIVITY AND GRAVITATION

JGRG 22

November 12-16 2012

Koshiba Hall, The University of Tokyo, Hongo, Tokyo, Japan

Finite-time future singularities and Rip cosmology in $f(T)$ gravity

Kazuharu Bamba[1] (a), Ratbay Myrzakulov[2] (b), Shin'ichi Nojiri[3] (a),(c) and Sergei D. Odintsov[4] (b),(c),(d),(e),(f)

(a) *Kobayashi-Maskawa Institute for the Origin of Particles and the Universe, Nagoya University, Nagoya 464-8602, Japan*

(b) *Eurasian International Center for Theoretical Physics, Eurasian National University, Astana 010008, Kazakhstan*

(c) *Department of Physics, Nagoya University, Nagoya 464-8602, Japan*

(d) *Instituciò Catalana de Recerca i Estudis Avançats (ICREA), Barcelona, Spain*

(e) *Institut de Ciencies de l'Espai (CSIC-IEEC), Campus UAB, Facultat de Ciencies, Torre C5-Par-2a pl, E-08193 Bellaterra (Barcelona), Spain*

(f) *Tomsk State Pedagogical University, Tomsk, Russia*

Abstract

We show that the finite-time future singularities appear in $f(T)$ gravity with T being the torsion scalar. We reconstruct $f(T)$ gravity models with realizing the finite-time future singularities. It is explicitly demonstrated that a power-law type correction term T^{β} $(\beta > 1)$ such as a T^2 term can remove the finite-time future singularities in $f(T)$ gravity. We also investigate $f(T)$ models with realizing inflation in the early universe, the ΛCDM model, Little Rip cosmology and Pseudo-Rip cosmology.

1 Introduction

Cosmological observations such as Type Ia Supernovae (SNe Ia), cosmic microwave background (CMB) radiation, large scale structure (LSS), baryon acoustic oscillations (BAO), and weak lensing have suggested the current accelerated expansion of the universe. There is two representative approaches to understand this phenomenon. The first is to introduce the so-called "dark energy" (for a very recent review, see, e.g., [1]). The second is to extend the gravitational theory, for example, $F(R)$ gravity (for reviews, see, e.g., [2]). There also exists "teleparallelism" constructed by using the Weitzenböck connection. This is an alternative gravity theory to general relativity and it is described by the torsion scalar T and not the scalar curvature R defined with the Levi-Civita connection [3]. It has recently been revealed that the late time cosmic acceleration [4, 5, 6] as well as inflation [7] can be realized by extending this theory to $f(T)$ gravity, similarly to that in $F(R)$ gravity. In this paper, we review our results in Ref. [8]. In particular, we demonstrate the existence of finite-time future singularities in $f(T)$ gravity. We note that the procedure used here has also been extended to Loop quantum cosmology [9]. In addition, we explain the relevant cosmologies to future singularities. We use units of $k_{\mathrm{B}} = c = \hbar = 1$ and denote the gravitational constant $8\pi G$ by $\kappa^2 \equiv 8\pi/{M_{\mathrm{Pl}}}^2$ with the Planck mass of $M_{\mathrm{Pl}} = G^{-1/2} = 1.2 \times 10^{19}$GeV.

2 Finite-time future singularities in $f(T)$ gravity

In the teleparallelism, the metric $g^{\mu\nu}$ is written $g_{\mu\nu} = \eta_{AB} e^A_\mu e^B_\nu$ with $e_A(x^\mu)$ orthonormal tetrad components, where $A = 0,1,2,3$ for the tangent space at each point x^μ of the manifold, μ, $\nu = 0,1,2,3$ are coordinate indices on the manifold, and e^μ_A is the tangent vector of the manifold. The teleparallel Lagrangian density is expressed by using the torsion scalar T, although in general relativity the Lagrangian density is described by the Ricci scalar R. The torsion scalar T is given by $T \equiv {S_\rho}^{\mu\nu} {T^\rho}_{\mu\nu}$

[1] Email address: bamba@kmi.nagoya-u.ac.jp
[2] Email address: rmyrzakulov@gmail.com
[3] Email address: nojiri@phys.nagoya-u.ac.jp
[4] Email address: odintsov@ieec.uab.es

Table 1: Conditions for the finite-time future singularities to occur on q in the expressions of H in Eq. (1), ρ_{DE}, P_{DE}, and the behaviors of H and $\dot{H}$ in the limit $t \to t_{\mathrm{s}}$.

$q(\neq 0,\, -1)$	H $(t \to t_{\mathrm{s}})$	$\dot{H}$ $(t \to t_{\mathrm{s}})$	ρ_{DE}	P_{DE}
$q \geq 1$ [Type I ("Big Rip") singularity]	$H \to \infty$	$\dot{H} \to \infty$	$J_1 \neq 0$	$J_1 \neq 0$ or $J_2 \neq 0$
$0 < q < 1$ [Type III singularity]	$H \to \infty$	$\dot{H} \to \infty$	$J_1 \neq 0$	$J_1 \neq 0$
$-1 < q < 0$ [Type II ("sudden") singularity]	$H \to H_{\mathrm{s}}$	$\dot{H} \to \infty$		$J_2 \neq 0$
$q < -1$, but q is not any integer [Type IV singularity]	$H \to H_{\mathrm{s}}$	$\dot{H} \to 0$ (Higher derivatives of H diverge.)		

with $T^{\rho}{}_{\mu\nu} \equiv e^{\rho}_{A}\left(\partial_{\mu} e^{A}_{\nu} - \partial_{\nu} e^{A}_{\mu}\right)$ the torsion tensor and $S_{\rho}{}^{\mu\nu} \equiv (1/2)\left(K^{\mu\nu}{}_{\rho} + \delta^{\mu}_{\rho}\, T^{\alpha\nu}{}_{\alpha} - \delta^{\nu}_{\rho}\, T^{\alpha\mu}{}_{\alpha}\right)$, where $K^{\mu\nu}{}_{\rho} \equiv -(1/2)\left(T^{\mu\nu}{}_{\rho} - T^{\nu\mu}{}_{\rho} - T_{\rho}{}^{\mu\nu}\right)$ is the contorsion tensor. The modified teleparallel action is given by [5]$I = \int d^4x |e| \left[f(T)/\left(2\kappa^2\right) + \mathcal{L}_{\mathrm{M}}\right]$, where $|e| = \det\left(e^{A}_{\mu}\right) = \sqrt{-g}$ and $\mathcal{L}_{\mathrm{M}}$ is the Lagrangian of matter.

We take the four-dimensional flat Friedmann-Lemaître-Robertson-Walker (FLRW) space-time with the metric $ds^2 = dt^2 - a^2(t) \sum_{i=1,2,3} \left(dx^i\right)^2$, where $a(t)$ is the scale factor. In this background, we have $g_{\mu\nu} = \mathrm{diag}(1, -a^2, -a^2, -a^2)$ and the tetrad components $e^{A}_{\mu} = (1, a, a, a)$. The torsion scalar $T = -6H^2$ with $H = \dot{a}/a$ the Hubble parameter, where the dot denotes the time derivative, $\partial/\partial t$. In the flat FLRW background, the gravitational field equations read $H^2 = \left(\kappa^2/3\right)\left(\rho_{\mathrm{M}} + \rho_{\mathrm{DE}}\right)$ and $\dot{H} = -\left(\kappa^2/2\right)\left(\rho_{\mathrm{M}} + P_{\mathrm{M}} + \rho_{\mathrm{DE}} + P_{\mathrm{DE}}\right)$ with where ρ_{M} and P_{M} the energy density and pressure of all perfect fluids of generic matter, respectively. The perfect fluid satisfies the continuity equation $\dot{\rho}_{\mathrm{M}} + 3H\left(\rho_{\mathrm{M}} + P_{\mathrm{M}}\right) = 0$. In addition, the energy density and pressure of dark components become $\rho_{\mathrm{DE}} = \left[1/\left(2\kappa^2\right)\right] J_1$ and $P_{\mathrm{DE}} = -\left[1/\left(2\kappa^2\right)\right]\left(4J_2 + J_1\right)$, respectively, where $J_1 \equiv -T - f + 2TF$ and $J_2 \equiv \left(1 - F - 2TF'\right)\dot{H}$ with $F \equiv df/dT$ and $F' = dF/dT$. The standard continuity equation is also satisfied as $\dot{\rho}_{\mathrm{DE}} + 3H\left(\rho_{\mathrm{DE}} + P_{\mathrm{DE}}\right) = 0$.

3 $f(T)$ models with realizing cosmologies

In the FLRW background, the effective equation of state (EoS) for the universe is written as [2] $w_{\mathrm{eff}} \equiv P_{\mathrm{eff}}/\rho_{\mathrm{eff}} = -1 - 2\dot{H}/\left(3H^2\right)$ with $\rho_{\mathrm{eff}} \equiv 3H^2/\kappa^2$ and $P_{\mathrm{eff}} \equiv -\left(2\dot{H} + 3H^2\right)/\kappa^2$ the total energy density and pressure of the universe, respectively. At the dark energy dominated stage, we have $w_{\mathrm{DE}} \approx w_{\mathrm{eff}}$. In the following, we consider this case that dark energy is completely dominant over matter because we explore the evolution of the universe around the time when the finite-time future singularities appear. For $\dot{H} < 0$ (> 0), we find $w_{\mathrm{eff}} > -1$ (< -1), which describes the non-phantom [namely, quintessence] (phantom) phase. The case that $w_{\mathrm{eff}} = -1$ for $\dot{H} = 0$ corresponds to the cosmological constant.

It is known that there exist four types of the finite-time future singularities [10]. Type I ("Big Rip" [11]): For the limit $t \to t_{\mathrm{s}}$, $a \to \infty$, $\rho_{\mathrm{eff}} \to \infty$ and $|P_{\mathrm{eff}}| \to \infty$. The case that ρ_{eff} and P_{eff} are finite values at $t = t_{\mathrm{s}}$ [12] is included. (ii) Type II ("sudden" [13]): For the limit $t \to t_{\mathrm{s}}$, $a \to a_{\mathrm{s}}$, $\rho_{\mathrm{eff}} \to \rho_{\mathrm{s}}$ and $|P_{\mathrm{eff}}| \to \infty$. (iii) Type III: For the limit $t \to t_{\mathrm{s}}$, $a \to a_{\mathrm{s}}$, $\rho_{\mathrm{eff}} \to \infty$ and $|P_{\mathrm{eff}}| \to \infty$. (iv) Type IV: For the limit $t \to t_{\mathrm{s}}$, $a \to a_{\mathrm{s}}$, $\rho_{\mathrm{eff}} \to 0$, $|P_{\mathrm{eff}}| \to 0$, and higher derivatives of H diverge. Also, the case that ρ_{eff} and/or $|P_{\mathrm{eff}}|$ become finite values at $t = t_{\mathrm{s}}$ is included. Here, t_{s} is the time when the finite-time future singularities occur, and $a_{\mathrm{s}}(\neq 0)$ and ρ_{s} are constants. We examine the case that H is expressed as [14]

$$H \sim h_{\mathrm{s}}\left(t_{\mathrm{s}} - t\right)^{-q} \;\text{ for }\; q > 0\,, \quad H \sim H_{\mathrm{s}} + h_{\mathrm{s}}\left(t_{\mathrm{s}} - t\right)^{-q} \;\text{ for }\; q < -1,\;\; -1 < q < 0\,, \tag{1}$$

where $h_{\mathrm{s}}(> 0)$ and $H_{\mathrm{s}}(> 0)$ are positive constants, and $q(\neq 0,\, -1)$ is a non-zero constant. Only the period $0 < t < t_{\mathrm{s}}$ is studied because H should be a real number. In Table 1, we show the conditions for the finite-time future singularities to exist on q in the expression of H in Eq. (1), ρ_{DE}, P_{DE}, and the behaviors of H and $\dot{H}$ in the limit $t \to t_{\mathrm{s}}$.

Table 2: Necessary conditions on the model parameters of a power-law model of $f(T)$ in order to realize the finite-time future singularities, what types of the finite-time future singularities finally emerge, and those of the correction term $f_{\rm c}(T) = BT^{\beta}$ with removing the finite-time future singularities.

$q(\neq 0, -1)$	Final behavior	$f(T) = AT^{\alpha}$ $(A \neq 0,\ \alpha \neq 0)$	$f_{\rm c}(T) = BT^{\beta}$ $(B \neq 0,\ \beta \neq 0)$
$q \geq 1$ [Type I ("Big Rip") singularity]	emerges	$\alpha < 0$	$\beta > 1$
$0 < q < 1$ [Type III singularity]	—	$\alpha < 0$	$\beta > 1$
$-1 < q < 0$ [Type II ("sudden") singularity]	—	$\alpha = 1/2$	$\beta \neq 1/2$
$q < -1$, but q is not any integer [Type IV singularity]	emerges	$\alpha = 1/2$	$\beta \neq 1/2$

For H in Eq. (1), we reconstruct models of $f(T)$ gravity where the finite-time future singularities occur. We note that there have been proposed reconstruction method for modified gravity such as an $F(R)$ gravity in Ref. [15]. As a result, we find that for a power-law form of $f(T)$ as $f(T) = AT^{\alpha}$ with $A(\neq 0)$ and $\alpha(\neq 0)$ non-zero constants, in the flat FLRW background both the two gravitational filed equations can be satisfied. In addition, we investigate a correction term to remove the finite-time future singularities. As an example of a correction term $f_{\rm c}(T)$, we take $f_{\rm c}(T) = BT^{\beta}$ where $B(\neq 0)$ and $\beta(\neq 0)$ are non-zero constants. In $F(R)$ gravity, the correction term with $\beta = 2$, namely, a T^2 term can cure the finite-time future singularities [2]. Consequently, we see that for the combined form as $f(T) = AT^{\alpha}+BT^{\beta}$, the two gravitational filed equations cannot be met simultaneously. This implies that such a power-low correction term can remove the finite-time future singularities. In Table 2, we depict necessary conditions for the model parameters of a power-law form of $f(T)$ where the finite-time future singularities appear, those of a power-low the correction term which can remove the finite-time future singularities. We also describe what types of the finite-time future singularities eventually appear. The finite-time future singularities with the large absolute value of q can emerge. In addition, we explicitly derive $f(T)$ models in which (a) inflation in the early universe, (b) the ΛCDM model, (c) Little Rip cosmology [16], and (d) Pseudo-Rip cosmology [17, 18] can be realized. In Table 3, we represent forms of H and $f(T)$ with realizing these cosmological scenarios.

Furthermore, we study Little Rip cosmology which corresponds to a mild phantom scenario. The Little Rip scenario has been proposed to avoid the finite-time future singularities, in particular a Big Rip singularity. In this scenario, the energy density of dark energy increases in time with $w_{\rm DE}$ being less than -1 and then $w_{\rm DE}$ asymptotically approaches $w_{\rm DE} = -1$. However, such a scenario eventually leads to the dissolution of bound structures at some time in the future via the increase of an inertial force between objects. This process is called the "Little Rip". We also investigate Pseudo-Rip cosmology. The above four cosmological models can be classified by using the behavior of the Hubble parameter as follows [17]. (a) power-law inflation: $H(t) \to \infty,\quad t \to 0$. (b) the ΛCDM model or exponential inflation: $H(t) = H(t_0) =$ constant with t_0 the present time. (c) Little Rip cosmology: $H(t) \to \infty,\quad t \to \infty$. (d) Pseudo-Rip cosmology, which is also phantom asymptotically de Sitter universe: $H(t) \to H_{\infty} < \infty$, $t \to \infty$. Here, $t \geq t_0$, and $H_{\infty}(> 0)$ is a positive constant. We also note that for a Big Rip singularity, $H(t) \to \infty,\quad t \to t_{\rm s}$, as shown in Table 1.

4 Summary

We have illustrated that there appear finite-time future singularities (Type I and IV) in $f(T)$ gravity and reconstructed an $f(T)$ gravity model with realizing the finite-time future singularities. Furthermore, it has been verified that a power-law type correction term T^{β} $(\beta > 1)$ such as a T^2 term can remove the finite-time future singularities in $f(T)$ gravity. This is the same feature as in $F(R)$ gravity. In addition, we have derived the expressions of $f(T)$ gravity models in which (a) Power-law inflation, (b) CDM model, (c) Little Rip cosmology, and (d) Pseudo Rip Cosmology can be realized.

Table 3: Forms of H and $f(T)$ which can realize (a) inflation in the early universe, (b) the ΛCDM model, (c) Little Rip cosmology and (d) Pseudo-Rip cosmology.

Cosmology	H	$f(T)$
(a) Power-law inflation	$H = h_{\rm inf}/t\,,$	$f(T) = AT^{\alpha}\,,$
(In the limit $t \to 0$)	$h_{\rm inf}(> 1)$	$\alpha < 0$ or $\alpha = 1/2$
(b) ΛCDM model	$H = \sqrt{\Lambda/3} = \text{constant}\,,$	$f(T) = T - 2\Lambda\,,$
or exponential inflation	$\Lambda > 0$	$\Lambda > 0$
(c) Little Rip cosmology	$H = H_{\rm LR}\exp(\xi t)\,,$	$f(T) = AT^{\alpha}\,,$
(In the limit $t \to \infty$)	$H_{\rm LR} > 0$ and $\xi > 0$	$\alpha < 0$ or $\alpha = 1/2$
(d) Pseudo-Rip cosmology	$H = H_{\rm PR}\tanh(t/t_0)\,, \quad H_{\rm PR} > 0$	$f(T) = A\sqrt{T}$

References

[1] K. Bamba, S. Capozziello, S. Nojiri and S. D. Odintsov, Astrophys. Space Sci. **342**, 155 (2012).

[2] S. Nojiri and S. D. Odintsov, Phys. Rept. **505**, 59 (2011); eConf **C0602061**, 06 (2006) [Int. J. Geom. Meth. Mod. Phys. **4**, 115 (2007)] [arXiv:hep-th/0601213].

[3] F. W. Hehl, P. Von Der Heyde, G. D. Kerlick and J. M. Nester, Rev. Mod. Phys. **48**, 393 (1976); K. Hayashi and T. Shirafuji, Phys. Rev. D **19**, 3524 (1979) [Addendum-ibid. D **24**, 3312 (1981)]; E. E. Flanagan and E. Rosenthal, *ibid.* **75**, 124016 (2007).

[4] G. R. Bengochea and R. Ferraro, Phys. Rev. D **79**, 124019 (2009).

[5] E. V. Linder, Phys. Rev. D **81**, 127301 (2010) [Erratum-ibid. D **82**, 109902 (2010)].

[6] K. Bamba, C. Q. Geng and C. C. Lee, arXiv:1008.4036 [astro-ph.CO]; K. Bamba, C. Q. Geng, C. C. Lee and L. W. Luo, JCAP **1101**, 021 (2011).

[7] R. Ferraro and F. Fiorini, Phys. Rev. D **75**, 084031 (2007); *ibid.* **78**, 124019 (2008).

[8] K. Bamba, R. Myrzakulov, S. Nojiri and S. D. Odintsov, Phys. Rev. D **85**, 104036 (2012).

[9] K. Bamba, J. de Haro and S. D. Odintsov, arXiv:1211.2968 [gr-qc], to appear in JCAP.

[10] S. Nojiri, S. D. Odintsov and S. Tsujikawa, Phys. Rev. D **71**, 063004 (2005).

[11] R. R. Caldwell, M. Kamionkowski and N. N. Weinberg, Phys. Rev. Lett. **91**, 071301 (2003); B. McInnes, JHEP **0208** (2002) 029.

[12] Y. Shtanov and V. Sahni, Class. Quant. Grav. **19**, L101 (2002).

[13] J. D. Barrow, Class. Quant. Grav. **21**, L79 (2004); S. Nojiri and S. D. Odintsov, Phys. Lett. B **595**, 1 (2004).

[14] S. Nojiri and S. D. Odintsov, Phys. Rev. D **78**, 046006 (2008).

[15] S. Nojiri and S. D. Odintsov, Phys. Rev. D **74**, 086005 (2006); J. Phys. Conf. Ser. **66**, 012005 (2007) [arXiv:hep-th/0611071].

[16] P. H. Frampton, K. J. Ludwick and R. J. Scherrer, Phys. Rev. D **84**, 063003 (2011); I. Brevik, E. Elizalde, S. Nojiri and S. D. Odintsov, *ibid.* **84**, 103508 (2011); P. H. Frampton, K. J. Ludwick, S. Nojiri, S. D. Odintsov and R. J. Scherrer, Phys. Lett. B **708**, 204 (2012);

[17] P. H. Frampton, K. J. Ludwick and R. J. Scherrer, Phys. Rev. D **85**, 083001 (2012).

[18] A. V. Astashenok, S. Nojiri, S. D. Odintsov and A. V. Yurov, Phys. Lett. B **709**, 396 (2012).