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# “Non-axisymmetric oscillations of rotating relativistic stars by conformally flat approximation”

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## RESCEU SYMPOSIUM ON

## GENERAL RELATIVITY AND GRAVITATION

# JGRG 22

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# Non-axisymmetric oscillations of rotating relativistic stars by conformally flat approximation

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# Studies of oscillations of rapidly rotating relativistic stars

astrophysical interests

as sources of gravitational wave $\nu \sim \mathrm{several} \cdot (10^2 - 10^3)\mathrm{Hz}$

ground-based laser-interferometers
resonant detectors

stability of stars

determining maximum masses/rotational period of NS

Aim of the studies:

Compute characteristic(eigen-) frequencies of "realistic" NS models.

Determine the stability boundaries of a parameter space of rotating stars.

# Preceding studies of oscillations of rotating stars in GR

## via traditional eigenvalue problem

*Kojima (1992,1997) slow-rotation*
*Yoshida & Kojima (1997) slow-rotation*

*Cutler & Lindblom (1992) post-Newtonian*

*Yoshida & Eriguchi (1997,1999) Cowling; f-modes*
*Yoshida & Eriguchi (2001) Cowling; axisymmetric modes*
*Yoshida et al. (2002,2005) Cowling; f-modes*

*Lockitch et al. (2001) slow-rotation, relativistic inertial modes*

*Ruoff & Kokkotas (2001,2002) slow-rotation, relativistic r-modes*

*Ferrari et al. (2004) Cowling; rotating proto-NS*
*Passamonti et al. (2006) slow-rotation*
*Ferrari et al. (2007)*

## via direct hydrodynamical simulations

*Font et al. (2001) Cowling; axisymmetric oscillations*
*Dimmelmeier et al. (2006) spatially conformal-flat; axisymmetric oscillations*

*Saijo et al. (2001) post-Newtonian*
*Saijo (2005) spatially conformal-flat*

*Shibata & Sekiguchi (2003) full GR; axisymmetric oscillations*
*Shibata & Karino (2004) post-Newtonian; bar-mode instability*
*Shibata & Sekiguchi (2005) full GR; dynamical instability*

*Stavridis et al. (2007) slow-rotation(differential)*
*Kastaun et al. (2010) Cowling*
*Krueger et al. (2010) Cowling*
*Gaertig & Kokkotas (2009,2011) Cowling (also g-modes)*

*Baiotti et al. (2007) full GR; bar-mode instability*
*Manca et al. (2007) full GR; dynamical instability*
*Takami et al. (2011) full GR; axisymmetric instability*

"Cowling"=*Cowling approximation* where metric perturbation is neglected

* *non-linear hydrodynamic simulation*

* *linear hydrodynamic simulation*

Full GR treatment is rather expensive!

* eigenvalue problem - no formulation exists except for slowly rotating cases!
(messy equations; how do we impose boundary conditions?)

* direct numerical simulations - numerically expensive
identification & extraction of eigenmodes?

For a “fluid mode”, its gravitational damping time is several orders of magnitude longer than the period ⟶ We may neglect the gravitational radiation effect.

But, neglecting all the gravitational perturbation (Cowling approximation) is not satisfactory, especially for the low-order modes.

We want something better than Cowling, but easier to handle than “full GR”.

↓

Conformally flat approximation!!

# General relativistic hydrodynamics with spatially conformally-flat approximation

Isenberg (1978) ; Wilson et al. (1996) : Solving system of elliptic eqs. for metrics
"Isenberg-Wilson-Mathews theory"

Applications

Wilson et al.(1996) - binary NS quasi-equilibrium

Glanclement et al. (2002) - binary BH quasi-equilibrium

Oechslin et al. (2002;2004) - NS merger

Faber et al. (2004) - NS merger

Dimmelmeier et al.(2002) - Core-collapse SN

Saijo (2005) - bar-instability in core collapse

Dimmelmeier et al.(2006) - Non-linear oscillations of NS

# Formulation

assumptions
- a background star - stationary, axisymmetric, isentropic
- barotropic EOS - no buoyancy (no g-modes)
- linear perturbation
- spatially conformally flat approximation (CFA) of gravity
  - both equilibrium & perturbed state

[CFA]

$$ds^2 = -\alpha^2 dt^2 + \psi^4 f_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt); \quad f_{ij}\text{: flat metric}$$

"(3+1)-" form of Einstein's equation in CFA:

Hamiltonian & Momentum constraint, "evolution" of Kij (K=0)

$$\hat{\nabla}^i \hat{\nabla}_i \psi = -2\pi\psi^5 \left( \rho_{_H} + \frac{1}{64\pi\alpha^2} \hat{\Lambda}_{ij} \hat{\Lambda}^{ij} \right)$$

$$\hat{\nabla}^i \hat{\nabla}_i (\alpha\psi) = 2\pi\alpha\psi^5 \left( \rho h (3(\alpha u^t)^2 - 2) + 5p + \frac{7}{64\pi\alpha^2} \hat{\Lambda}_{ij} \hat{\Lambda}^{ij} \right)$$

$$\hat{\nabla}^i \hat{\nabla}_i \beta^j = 16\pi\alpha\psi^4 S^j - \frac{1}{3} \hat{\nabla}^j (\hat{\nabla}_k \beta^k) + \hat{\Lambda}^{jk} \hat{\nabla}_k \ln \left( \frac{\alpha}{\psi^6} \right)$$

$$\rho_H := n_a n_b T^{ab} \quad S^j := -\delta^j_a n_c T^a_c \quad \hat{\Lambda}^{ij} := \hat{\nabla}^i \beta^j + \hat{\nabla}^j \beta^i - \frac{2}{3} f^{ij} \hat{\nabla}_k \beta^k$$

$$T^{ap} := \rho h u^a u^b + p g^{ab}$$

# Formulation - contd.

[equations of hydrodynamics]

rest mass conservation

$$\partial_a \pi^a + \pi^a \partial_a \ln\left(\frac{\rho}{h}\right) + \pi^a \left(\frac{\partial_a \alpha}{\alpha} + \frac{6\partial_a \psi}{\psi} + \frac{2}{r}\delta^r_a + \cot\theta \delta^\theta_a\right) = 0$$

momentum equation

$$\pi^b \partial_b \pi_j - \pi^b \partial_j \pi_b = 0$$

$\pi_a := h u_a$

*Linearizing equations above and supplementing with boundary conditions, we set up an eigenvalue problem.*

% boundary conditions

\+ metric components $\psi, \ \alpha, \ \beta^j$ --- regularity at the origin & at the infinity
\+ fluid variables $h, \ \pi_j$ --- regularity at the origin
stress free surface :
$\Delta p = 0$ (Lagrangian perturbation of pressure vanishes).

# Numerical treatment

(A) metric part: given sources (S), formal solutions are obtained by Green's functions

$$\hat{\nabla}_i \hat{\nabla}^i q = S \qquad \rightarrow \qquad q = \int S(\vec{y}) G(\vec{x}|\vec{y}) dV_y$$

(B) fluid part: equations are descretized

$$\mathbf{U}(\mathbf{f}) = \mathfrak{g}(q)$$

$\mathbf{f}$ : array vector of fluid variables

$\mathbf{U}$ : array of operators describing hydro-equations

$\uparrow$

regarded as source terms when q is given

Equations of fluid part are solved by Newton-Raphson method.

(A) and (B) are iteratively solved. => eigenmode!

# Numerical treatment - contd.

* background star is stationary and axisymmetric

=> perturbation is decomposed into harmonic components

$$\delta f \sim \exp[-i\sigma t + im\varphi] \cdot F(r,\theta)$$

The problem is reduced to 2D eigenvalue problem.

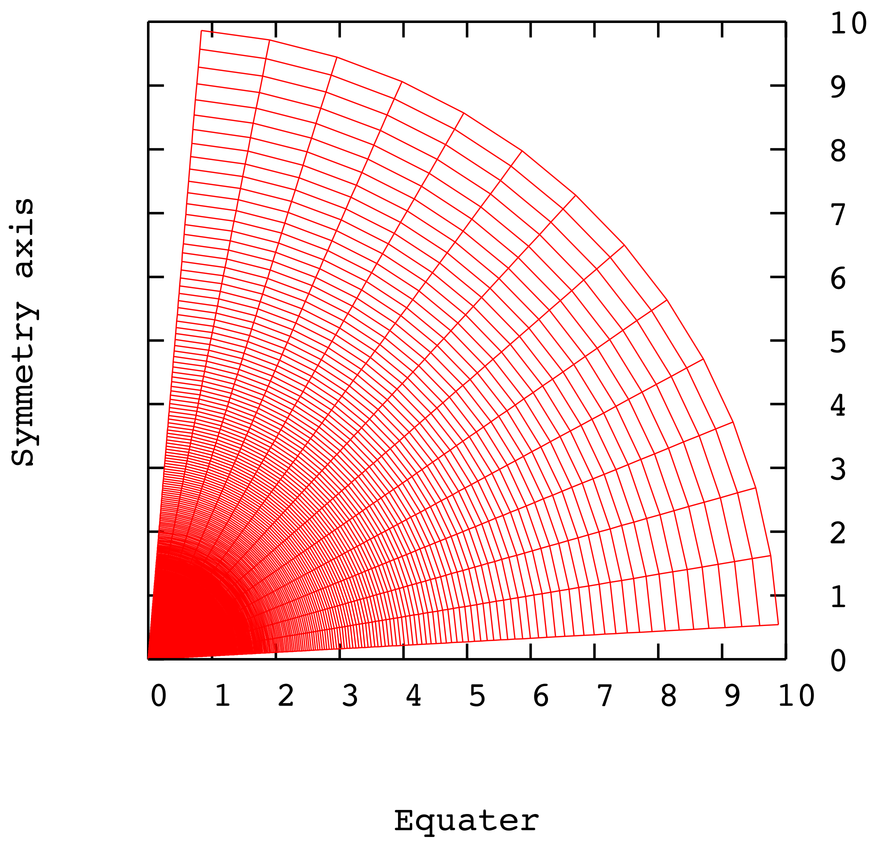
* Surface-fitted coordinate: $r = R_s(\theta^*)\ r^*$ $\quad p(R_s,\theta) = 0$ (equilibrium surface)

$$\theta = \theta^*$$

Equilibrium stellar surface is always mapped to r*=1.

==> Boundary condition is easy to impose.

computational grids



* radial grid points: non-uniform distribution (r*=1 on the equatorial surface)

* angular grid points: zeroes of Legendre function

-- For stellar interior, "fluid" and "metric" grids are staggered.

# Results - quadrupole modes of slowly rotating star

$$\delta f \sim \exp(-i\sigma t + i m \varphi)$$

$$\sigma = \underset{\substack{\nearrow \\ \text{non-rotating}}}{\sigma^0} + m\, \underset{\substack{\nwarrow \\ \text{rotational correction}}}{\sigma'}\ \Omega$$

| Mode | M / R | $\sigma^0_{CFA}/\sigma^0_R$ | $\sigma^0_{Cw}/\sigma^0_R$ | $\sigma'_{CFA}/\sigma'_R$ | $\sigma'_{Cw}/\sigma'_R$ |
|---|---|---|---|---|---|
| f | 0.100 | 0.998 | 1.26 | 0.998 | 1.02 |
|  | 0.200 | 0.997 | 1.15 | 1.00 | 1.01 |
| $P_1$ | 0.100 | 1.00 | 1.10 | 0.994 | 1.00 |
|  | 0.200 | 0.997 | 1.11 | 0.991 | 1.01 |
| $P_2$ | 0.100 | 1.00 | 1.05 | 0.997 | 1.00 |
|  | 0.200 | 0.999 | 1.06 | 1.00 | 1.00 |

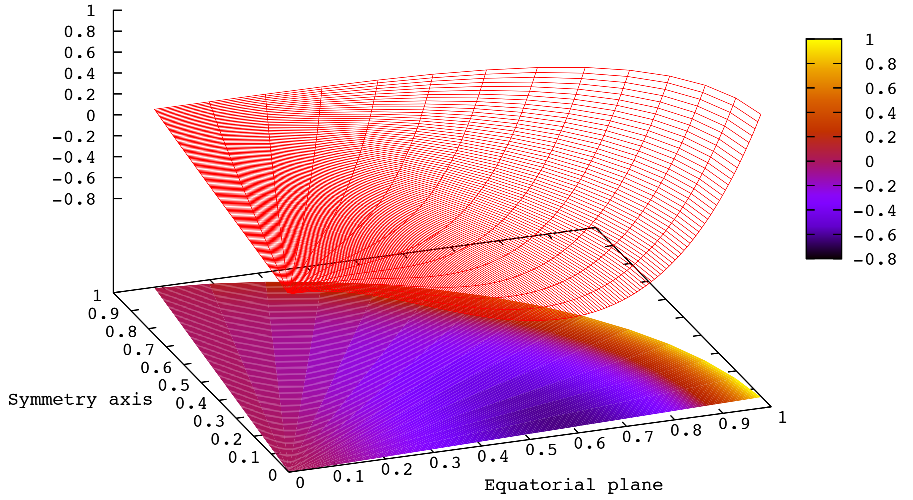
EOS --- $p = K\epsilon^2$

Cw: Cowling approximation (Yoshida & Kojima 1997)

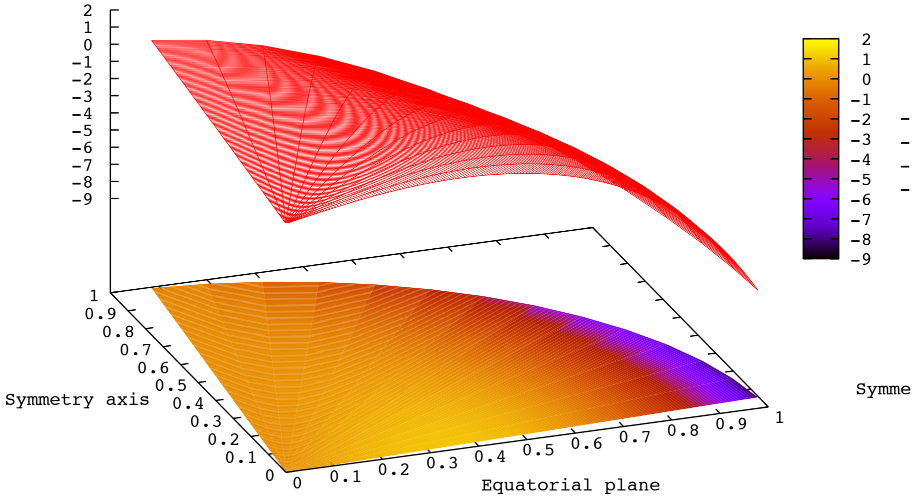
# Results - quadrupole modes of slowly rotating star

## eigenfunctions of $P_l$ mode

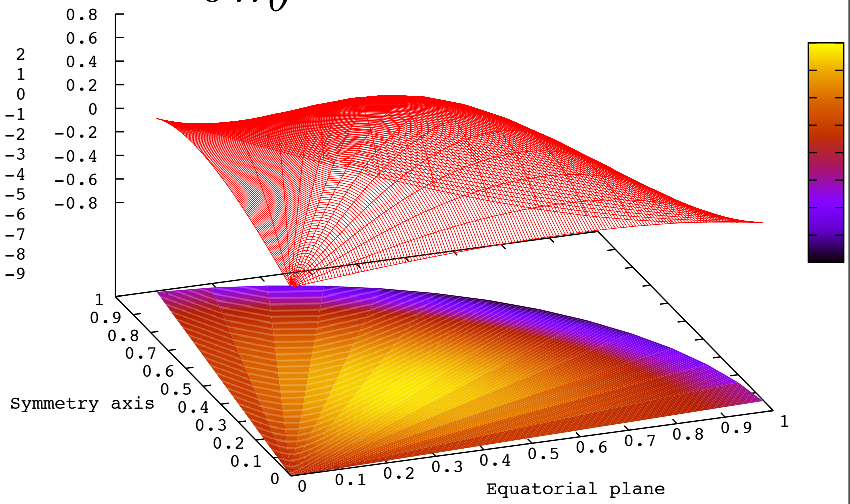
$\delta h$



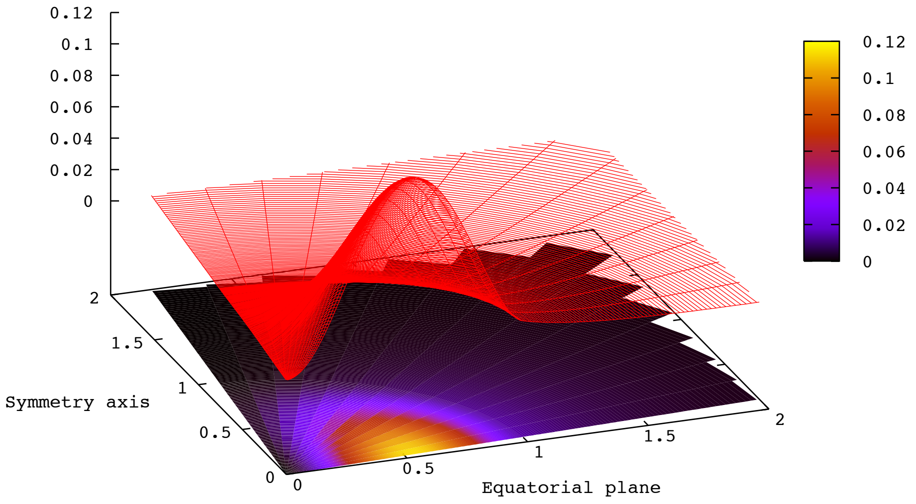
$\delta \pi_r$



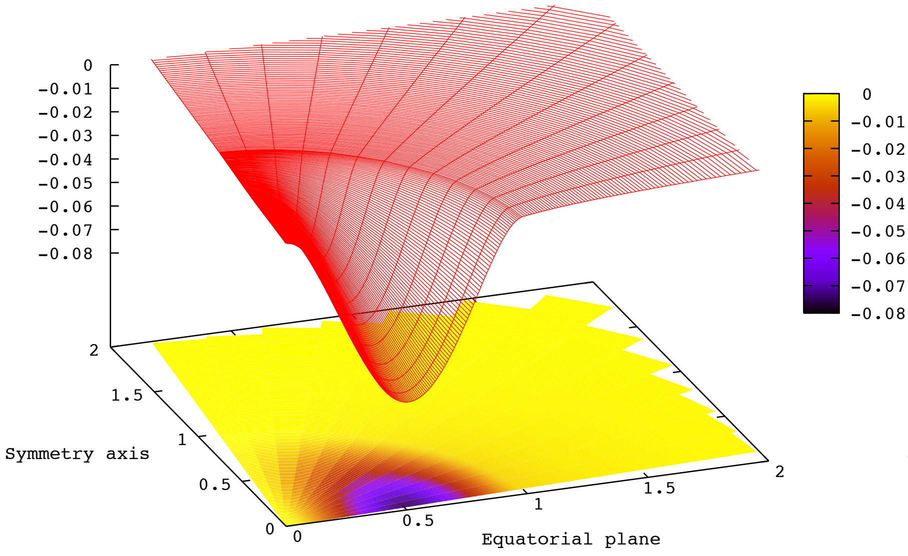
$\delta \pi_\theta$



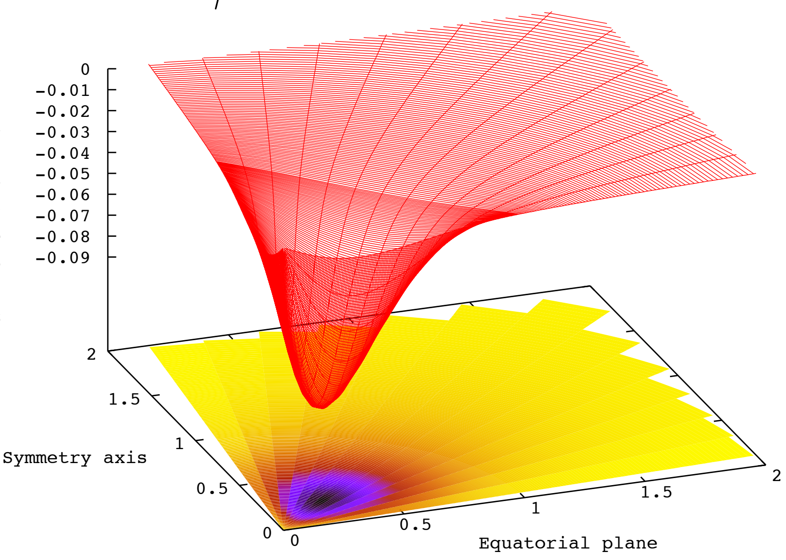
$\delta \alpha$



$\delta \psi$



$\delta \beta^\theta$



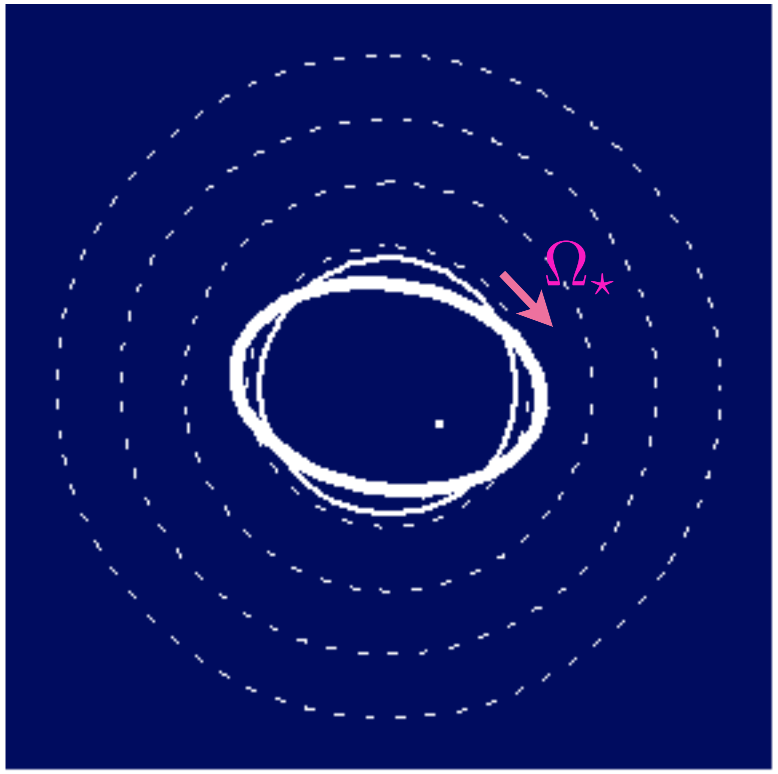
# CFS instability

Chandrasekhar (1970), Friedman & Schutz (1985)

instability in pulsations of rotating stars that couple to (gravitational) radiation field --- f-mode, r-mode instability

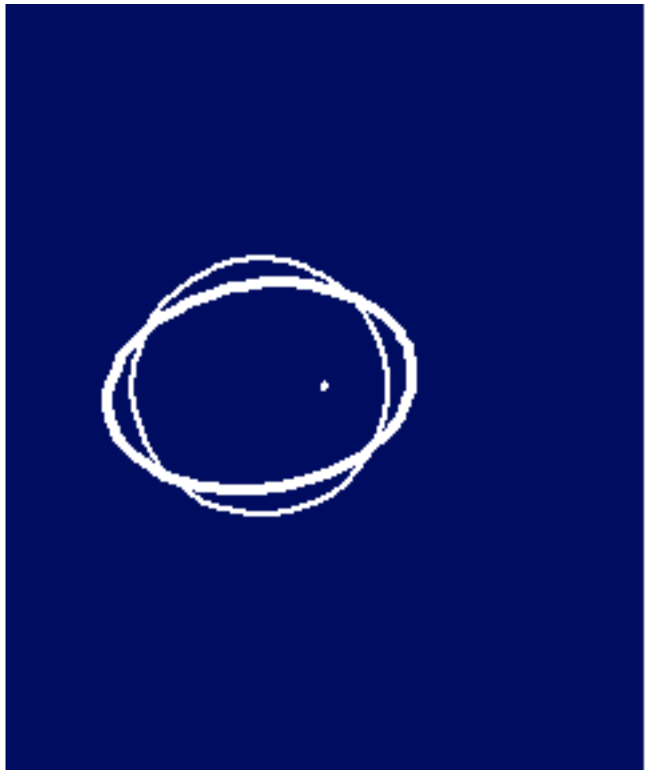
From the inertial frame, the star (with a marker dot) and the mode pattern is rotating in clockwise direction.

Angular momentum carried away by gravitational wave has the same sign as that of the star.



**inertial frame view**

From the corotating frame of star (in which the star is at rest), the mode pattern is rotating in counter-clockwise direction. This mode has an opposite sign of canonical angular momentum to that of star (and that of gravitational wave).



**corotating frame view**

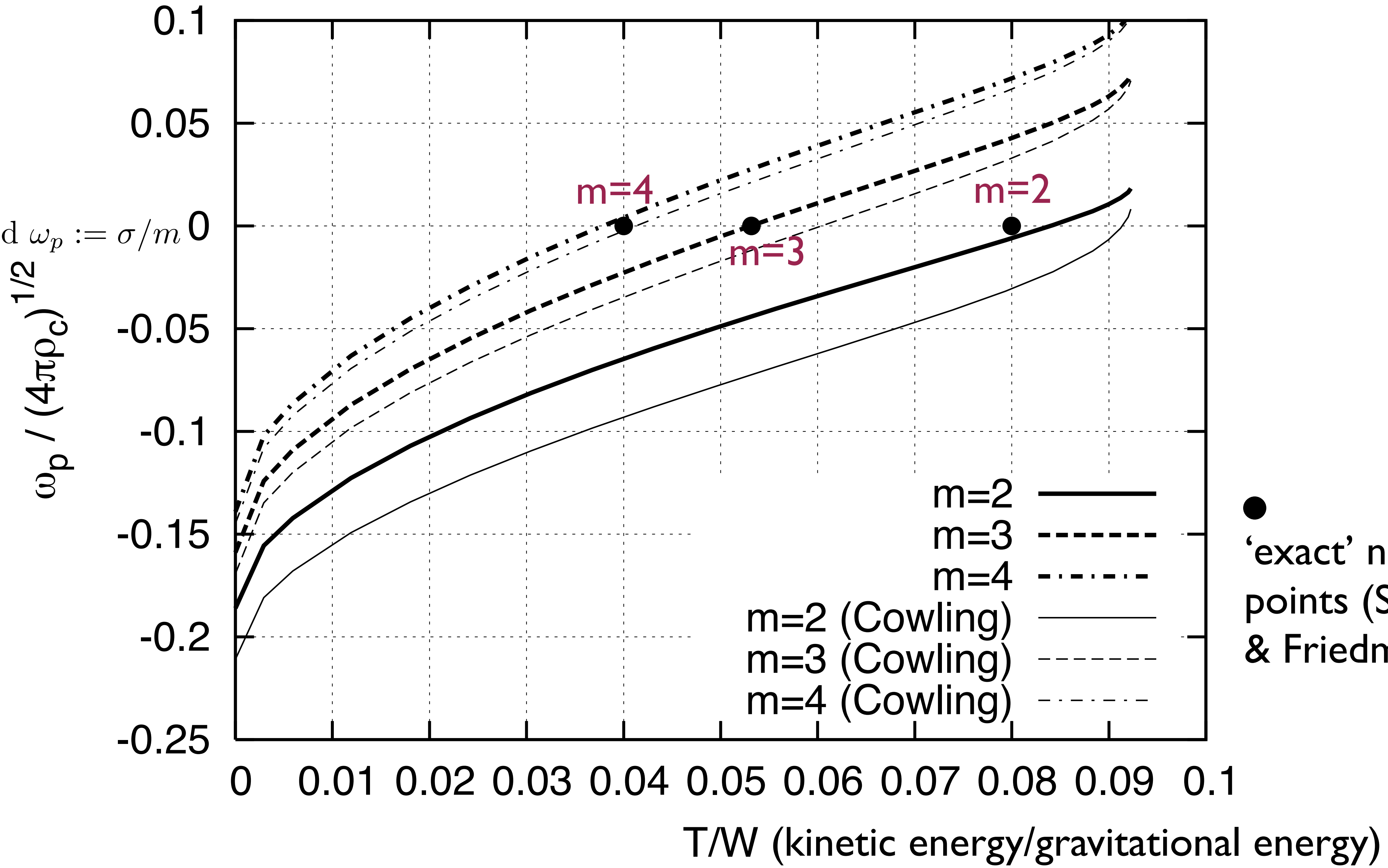
zeroes of frequency (seen from the inertial frame) marks the onsets of the instability.

# Results - sequences of counter-rotating f-modes

Equilibria: computed by COCAL code (Uryu & Tsokaros 2012)

EOS: $p = K\rho^2$

sequences f-modes (l=m=2) with compactness M/R=0.20



pattern speed $\omega_p := \sigma/m$

● ‘exact’ neutral points (Stergioulas & Friedman 1998)

# Summary

We developed a new code to solve eigenmode problem of rotating star with arbitrary rotation.

spatially conformal flatness approximation
=> suppressing gravitational radiation

Compared to conventional Cowling approximation, the eigenmodes obtained are highly improved.

What' s coming next?

* differential rotation --- proto-NS ; dynamical instability of rotating stars
* nuclear EOS --- asteroseismology of NS
* stably stratified star --- g-modes of rotating stars in GR