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Einstein equation of state and its cosmological applicationss

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Abstract

We investigate the applications of the Einstein equation of state to cosmological problems. In order to apply the cosmology, we expand the shear term in higher orders and find that the leading part of them are described in terms of the Weyl curvature . Consequently, the Penrose's Weyl curvature conjecture has been formulated from spacetime thermodynamical aspect.

1 Introduction

Recent cosmological observations suggest the cosmological scenario that the Universe has evolved from an almost homogeneous state to the present inhomogeneous state due to formation of the astronomical objects. If the Universe can be regarded as a system of the gravity theory, how can we describe evolution of the system? For this question, R.Penrose has proposed the following conjecture [1, 2], namely, 'The entropy of gravitational filed should be somehow related to the Weyl curvature' and 'The entropy, defined as a functional of the Weyl curvature, should be increasing in time.' The Wely curvature is traceless part of Riemann curvature tensor $R_{\alpha\beta\gamma\delta}$, which defined as

$$C_{\alpha\beta\gamma\delta} := R_{\alpha\beta\gamma\delta} - g_{\alpha[\gamma}R_{\delta]\beta} + g_{\beta[\gamma}R_{\delta]\alpha} + \frac{1}{3}Rg_{\alpha[\gamma}g_{\delta]\beta}, \tag{1}$$

Namly, this Penrose's Weyl curvature hypothesis means that gravitational field itself should carry an entropy, and it may be related to the Weyl tensor In this context, the Universe can be regard as a physical system with the following property, i.e. it has an almost homogeneous initial state that the gravitational entropy is nearly zero, After the formation of very inhomogeneous large scale structure, a final (present) state with large gravitational entropy. Then the gravitational entropy continues increasing globally from a initial equilibrium, Ricci dominant state to a final npn-equilibrium Weyl dominant state.
The entropy function mentioned above has been not clear to be defined by the fundamental gravity theory in spite of previous investigations, e.g. Grøn and Hervik(2001) by minisperspace model[3], Rudjord et. al(2012) introduce the gravitational entropy density $s(x)$ through examining the possibility to explain some kind of black hole entropy[4]. However these are just only phenomenological approach. Thus one of our aim of this report, we obtain more directly the conjecture from the fundamental gravitational theory. In this research, we study the cosmological applications of spacetime thermodynamics. This approach of gravity has recently been remarkably evolved and suggest the alternative of gravitational degree of freedom (detail in the next section). And we point out that the Penrose's Weyl curvature conjecture can be formulated naturally by expanding the shear term.

2 What is the Gravity?

The gravitational theory given by A. Einstein deduced by the principle of 'general relativity' and 'equivalent principle., in which the gravitational interaction is expressed by the Riemannian Manifold. Though

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the cosmological application has been successful, the quantization has difficulties due to the divergence of quantum renormalizations. However, it is widely known that the black hole thermodynamics seems to be related the quantum gravity. One of them so-called, 'Area theorem' has been studied from the quantum gravity aspects, e.g.D-brane, which states that the black hole entropy $S$is propotional to the stational black hole area A

$$S = \frac{1}{4G}A \tag{2}$$

Recently T. Jacobson (1995) give the alternative description to the gravity, namely, he has derive the Einstein equation from the Clausius's low and the area theorem applying to the local Rindler horizon at any point of any spacetime[5]. This gravity of approach seems to reviel the problem what the physical degree of freedom of gravity. Although the approach seems to be successful to the black hole thermodynamics, the cosmological applications have some problems due to just the construction.

3 Einstein equation of state

First we briefly review the Jacobson's derivation of the Einstein equation of state.
#1 **Equivalence Principe**
We can take a local inertial frame near any point in a spacetime manifold.

$$g_{\mu\nu}(X)\mathrm{d}X^{\mu}\mathrm{d}X^{\nu} = \left[\eta_{\mu\nu} + O(\ell^2)\right]\mathrm{d}X^{\mu}\mathrm{d}X^{\nu}, \tag{3}$$

We take a pacetime area $\Delta A(P)$ including P that is tangent to the light (here X_1 at P.
#2 **Local Rindler frame**
We consider the local Rindler flame (χ, η, X_2, X_3) from the local inertial flame $X_i (i = 0, 1, 2, 3)$ by the following transformation as

$$X^0 = \chi \sinh(\kappa\eta), \tag{4}$$
$$X^1 = -\chi \cosh(\kappa\eta), \tag{5}$$

where κ is a constant. Then the line element is given by

$$\mathrm{d}s^2 = -\kappa^2\chi^2\mathrm{d}\eta^2 + \mathrm{d}\chi^2 + \mathrm{d}\ell_\perp^2, \tag{6}$$

#3 **The observer and thermal system**
Consider an uniformally accelerating observer running away from the point near P in spatial direction of the light wavefront (X_1), whose trajectory is $\chi = const.$. Define χ^μ as a unit vector of this trajectory.
#4 **The local Rindler horizon** Then, for this observer, the null-geodesics $X^0 - X^1 = 0$ is a causal horizon which is parametrized by a affine parameter λ, X^2 and X^3.
When a unit vector k^μ of the null-geodesics with $X^2 = const.$ and $X^3 = const.$, then χ^μ i always approximate near the causal horizon as

$$\chi^\mu \approx -\kappa\lambda k^\mu \tag{7}$$

Thus we can regard the inside of the causal horizon as the thermal system for the observers.
#5 **The Clausius's low and construction of Einstein equation of state**
The Clausius's low is expressed in terms of the temperature T, entropy change δS and heat flow δQ of the thermodynamical system by

$$T\delta S \geq \delta Q \tag{8}$$

We can obtain the Einstein equation using the Area theorem $\delta S = \alpha\delta A$ for the entropy change, the Unruh temperature $T_u\sqrt{-g} = \frac{\hbar\kappa}{2\pi}$ [7] from the local Rindler horizon as the temperature$T$ and the energy flow through the horizon into the system as δQ.The energy flow is

$$\delta Q = \int_0^{\Delta\eta}\int_{\Delta A(\eta)} \lambda T_{\mu\nu}\chi^\mu k^\nu d\lambda dA = -\kappa\int_{\Delta\lambda} \lambda T_{\mu\nu}k^\mu k^\nu d\lambda dA \tag{9}$$

In the last step, we need to the change of the horizon area, i.e.

$$\delta A = \int_S \theta \, \mathrm{d}\lambda \, \mathrm{d}\ell_\perp^2, \tag{10}$$

where the expansion θ of null congruence defined as $\theta = -\frac{1}{\Delta A}\frac{d\Delta A}{d\lambda}$ and ΔA is infinitesimal cross section to the null congruence at P. The change of expansion θ obeys at P with respect to λ with the Rauchaudhri's equation given by

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma^2 - R_{\mu\nu}k^\mu k^\nu \tag{11}$$

and of shear σ[8]:

$$\frac{d\sigma_{\mu\nu}}{d\lambda} = -\theta\sigma_{\mu\nu} - C_{\alpha\nu\mu\beta}k^\alpha k^\beta. \tag{12}$$

Now, under the preparation above, if we make a assumption $\theta\big|_P = 0, \sigma_{\mu\nu}\big|_P = 0,$, then

$$\begin{aligned} & T\delta S - \delta Q = 0 \\ & = -\kappa \int \lambda \, \mathrm{d}\lambda \int_{S(\lambda)} \mathrm{d}\ell_\perp^2 \Big\{\alpha\frac{\hbar}{2\pi}\Big[\theta/\lambda - \Big(\frac{1}{2}\theta^2 + \sigma_{\mu\nu}\sigma^{\mu\nu} + R_{\mu\nu}k^\mu k^\nu\Big)\Big]_P - T_{\mu\nu}k^\mu k^\nu\Big\}. \end{aligned} \tag{13}$$

up to $O(\lambda^2)$. Since this derivation is not depend on the point P, Eq.(**??**) yields the Einstein equation with cosmological constant by using the Bianchi identity and setting $\alpha = \frac{1}{4G\hbar}$. This result means that thermal equilibrium condition of the system, i.e. $T\delta S = \delta Q$ with $\theta\big|_P = 0, \sigma_{\mu\nu}\big|_P = 0,$,is equivalent to the Einstein equation.

4 Expansions of the shear term and the cosmological applications

Can the spacetime thermodynamics be applied to cosmological problems? How can we do it? The derivation of the Einstein equation of state is supposed that the evolution of the expansion θ and the shear $\sigma_{\mu\nu}$ depend on the point P in the spite of the independence of choice of P in the manifold. In order to apply the discussion above to any point on any geodesics parametrized by λ, we expand the θ and $\sigma_{\mu\nu}$ as

$$\theta = \theta\big|_P + \lambda\frac{d\theta}{d\lambda}\big|_P + \frac{1}{2!}\lambda^2\frac{d^2\theta}{d\lambda^2}\big|_P + \frac{1}{3!}\lambda^3\frac{d^3\theta}{d\lambda^3}\big|_P + \cdots \tag{14}$$

$$\sigma_{\mu\nu} = \sigma_{\mu\nu}\big|_P + \lambda\frac{d\sigma_{\mu\nu}}{d\lambda}\big|_P + \frac{1}{2!}\lambda^2\frac{d^2\sigma_{\mu\nu}}{d\lambda^2}\big|_P + \frac{1}{3!}\lambda^3\frac{d^3\sigma_{\mu\nu}}{d\lambda^3}\big|_P + \cdots \tag{15}$$

With conditions.(ref), the expansion θ at P ($\lambda = 0$) is expressed as

$$\begin{aligned} \theta &= \theta\big|_P + \lambda\frac{d\theta}{d\lambda}\big|_P + \lambda^2\frac{d^2\theta}{d\lambda^2}\big|_P + \cdots + \lambda^n\frac{d^n\theta}{d\lambda^n}\big|_P + \cdots && (16) \\ &= 0 && 0th \ \ order \\ &\quad + \Big(-\lambda R_{\mu\nu}k^\mu k^\nu\Big)\Big|_P && 1st \ \ order \\ &\quad + \lambda^2\frac{1}{2!}\Big(-\frac{d}{d\lambda}R_{\mu\nu}k^\mu k^\nu\Big)\Big|_P && 2nd \ \ order \\ &\quad + \lambda^3\frac{1}{3!}\Big(-\frac{d^2}{d\lambda^2}R_{\mu\nu}k^\mu k^\nu\Big)\Big|_P + \lambda^3\frac{1}{3!}\Big(-E_{\alpha\gamma\delta\beta}k^\gamma k^\delta k^\alpha k^\beta\Big)\Big|_P && 3rd \ \ order \\ &\quad + \cdots \\ &\quad + \lambda^n\frac{1}{n!}\Big(-\frac{d^{n-1}}{d\lambda^{n-1}}R_{\mu\nu}k^\mu k^\nu\Big)\Big|_P + \lambda^n\frac{1}{n!}\Big(\frac{d^{n-3}}{d\lambda^{n-3}}(-E_{\alpha\gamma\delta\beta}k^\gamma k^\delta k^\alpha k^\beta\Big)\Big|_P + Q_n(R_{\mu\nu}, C_{\alpha\nu\mu\beta}) && \mathrm{n}-th \ \ order \\ &\quad + \cdots && (17) \end{aligned}$$

where

$$E_{\alpha\mu\nu\beta} := R_{\mu\nu}R_{\alpha\beta} + 2C_{\alpha\gamma\delta\beta}C_\mu{}^{\gamma\delta}{}_\nu. \tag{18}$$

In Eq.(17), we focus on the 1st and 2nd terms of each order $O(\lambda^n)$ and sum up these 1st terms and similarly the 2nd terms, The remaining terms $Q_n(R_{\mu\nu}, C_{\alpha\nu\mu\beta})$ have the leading term of higher order than those two terms. The summation of the two terms can be considered as the Taylor expansions of a functional given by integrating $-R_{\mu\nu}k^\mu k^\nu$ and $E_{\alpha\mu\nu\beta}k^\gamma k^\delta k^\alpha k^\beta$ with respect to λ.Then it is available at any point $\lambda = \lambda_1 \neq 0$ on the null geodesics by the analytical continuation. Therefore it is expected that spacetime thermodynamics can apply the discussions on a certain cosmological area. Thus we obtain the following inequality:

$$\begin{aligned} &T\delta S - \delta Q \\ &= \kappa \int \lambda \, \mathrm{d}\lambda \int_{\mathcal{S}(\lambda)} \mathrm{d}\ell_\perp^2 \Big[\alpha \frac{\hbar}{2\pi}(R_{\mu\nu}k^\mu k^\nu + E_{\alpha\gamma\delta\beta}k^\gamma k^\delta k^\alpha k^\beta) - T_{\mu\nu}k^\mu k^\nu\Big)\Big]_P + O(\lambda^5)\Big\} \geq 0. \end{aligned} \tag{19}$$

The The 2nd term in the integrands is contributed from the shear term. We should note that this leading term is $O(\lambda^4)$. Comparing the derivation of the Einstein equation of state, this result can be interpreted as the case with non-equilibrium condition $T(\delta S + \delta S_i) = \delta Q$ instead of the Clausius's low [6]and then S_i is regarded as the internal entropy production term:

$$\delta S_i = \int \lambda \, \mathrm{d}\lambda \int_{\mathcal{S}(\lambda)} \mathrm{d}\ell_\perp^2 \Big\{(E_{\alpha\gamma\delta\beta})k^\gamma k^\delta k^\alpha k^\beta\Big\} \geq 0 \tag{20}$$

Consequently, we formulate the Penrose's Weyl curvature conjecture from spacetime thermodynamical point of view.

5 Summary and Outlook

·An attempt to define a gravitational entropy has been made elaborating Jacobson ' s spacetime thermodynamics.
· We have taken into account the higher orders of θ and $\sigma_{\mu\nu}$ with respect to an affine parameter λ, and then we obtained the non-equilibrium thermodynamical inequality which is satisfied at any λ on a light curve by the analytic continuation.
·The Weyl curvature naturally arises in the inequality of entropy variation associated with matter, and the intrinsic one. Consequently, we formulate the Penrose Conjecture from spacetime thermodynamical aspect.
·Now trying to extend the formula, defined only locally, to global one in order to apply to cosmological situations.

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