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Curvature perturbation in conformally related frames

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Abstract

We consider the curvature perturbation resulting from multi-field inflation models in which potentially all fields are non-minimally coupled to the Ricci scalar. We confirm that, unlike in the single field case, the curvature perturbations associated with the Jordan and Einstein frames are not the same, and that the difference is a direct consequence of the isocurvature perturbations inherent to multi-field models. Using the δN formalism, and focusing on analytically soluble examples, we see how the statistical properties of the two curvature perturbations are also not necessarily the same.

1 Introduction

Despite the agreement of predictions from single-field inflation models with current observational data, in the context of unifying and higher-dimensional theories it is natural to consider the presence of multiple fields during inflation. Moreover, as a consequence of the compactification of higher dimensions, or following renormalisation arguments, it is expected that these multiple fields be non-minimally coupled to the Ricci scalar [1]. It is therefore very important to determine any possible observational signatures that such multi-field models with non-minimal coupling might give rise to. The type of model that we are considering takes an action of the form

$$S = \int d^4x\sqrt{-g}\left\{f(\boldsymbol{\phi})R - \frac{1}{2}h_{ab}(\boldsymbol{\phi})g^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b - V(\boldsymbol{\phi})\right\}, \tag{1}$$

In determining the observational predictions of different inflation models, we are largely interested in the statistical properties of the primordial fluctuations they produce. In particular, we are interested in determining the curvature perturbation on constant density hypersurfaces, ζ, and its statistical properties. In order to make calculations more tractable for an action of the form (1) it is common practise to make the conformal transformation $g_{\mu\nu} = \Omega\tilde{g}_{\mu\nu}$ in order to bring the gravity part of the action into the canonical Einstein-Hilbert form, namely

$$S = \int d^4x\sqrt{-\tilde{g}}\left\{\frac{\tilde{R}}{2} - \frac{1}{2}S_{ab}(\boldsymbol{\phi})\tilde{g}^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b - \tilde{V}(\boldsymbol{\phi})\right\}, \tag{2}$$

with

$$S_{ab} = \frac{1}{2f}\left(h_{ab} + \frac{3f_af_b}{f}\right) \qquad \text{and} \qquad \tilde{V}(\boldsymbol{\phi}) = \frac{V}{4f^2}. \tag{3}$$

The action in its original form is referred to as being in the Jordan frame, and the transformed form as being in the Einstein frame. However, we see that we need to be careful to consider which metric matter is minimally coupled to, and if it is minimally to the original metric then we must be careful to relate $\tilde{\zeta}$ calculated in the Einstein frame back to the original ζ of the Jordan frame. In the case of a single non-minimally coupled field it is known that the two curvature perturbations are the same to all orders, and thus their statistical properties too, but this is no longer the case when multiple fields are considered [2]. The difference is a consequence of the isocurvature modes inherent to multi-field models

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that are absent in the single-field case. Despite the difference in the curvature perturbations, however, it is important to note that any *observable* quantities will be independent of the frame in which they are calculated [3].

In this work we try to clarify the relation between the two curvature perturbations in the case of multiple non-minimally coupled fields. Using linear perturbation theory we first find that the notion of adiabaticity is not common to both frames. Namely, even if the curvature perturbation is conserved in one frame, it is not necessarily conserved in the other. We also find that the two curvature perturbations are of different magnitude. We then use the δN formalism, applied to analytically soluble examples, to confirm that predictions for the power spectrum, its tilt and non-gaussianity are in general different for ζ and $\tilde{\zeta}$ if one does not assume the vanishing of isocurvature modes before the end of inflation. If we expect that the isocurvature modes do eventually decay before the radiation domination era, where $\zeta = \tilde{\zeta}$, then this begs the question how is the $\zeta = \tilde{\zeta}$ limit reached, and how is the final curvature perturbation related to that at the end of inflation. The details of reheating therefore have to be considered more carefully.

2 Linear perturbations

In each frame we consider perturbations around a FLRW background of the form

$$ds^2 = -(1+2AY)dt^2 - 2aBY_i dt dx^i + a^2 \left[(1+2\mathcal{R})\,\delta_{ij} + 2H_T \frac{1}{k^2} Y_{,ij} \right] dx^i dx^j, \tag{4}$$

The curvature perturbation on constant density hypersurfaces in the Jordan frame is given as

$$\zeta = \mathcal{R} - \frac{H}{\dot{\rho}}\delta\rho \tag{5}$$

and similarly in the Einstein frame.

The non-conservtion of ζ on super horizon scales is given as

$$\dot{\zeta} \simeq -\frac{H}{\rho + p}\delta p_{\rm nad} \tag{6}$$

where $\delta p_{\rm nad} = \delta p - \frac{\dot{p}}{\dot{\rho}}\delta\rho$. The explicit expressions for δp, $\delta\rho$ etc are given in terms of perturbations of the fields. It is useful to decompose these field perturbations into components along and perpendicular to the background trajectory - adiabatic and isocurvature modes respectively. On making this decomposition in the canonical two-field case, one finds that the isocurvature perturbations source the curvature perturbation when there is a turn in the background trajectory, thus leading to its non-conservation [4]. In the single-field case there is no such isocurvature component (on super-horizon scales) and thus the curvature perturbation is conserved. In our case we have N fields, meaning that there are $N-1$ isocurvature modes. These can be expressed in terms of the combination

$$\mathcal{K}^{ab} = \delta\phi^a\dot{\phi}^b - \delta\phi^b\dot{\phi}^a, \tag{7}$$

which we see will vanish for an adiabatic mode where $\delta\phi^a \propto \dot{\phi}^a$.

Making the metric decomposition (4) in the Jordan and Einstein frames we find the relation

$$\tilde{\mathcal{R}} = \mathcal{R} - \frac{\delta\Omega}{2\Omega}. \tag{8}$$

We thus see that the two curvature perturbations would be the same in a gauge corresponding to $\delta\Omega = 0$. In the single field case, as $\Omega = \Omega(\phi)$, $\delta\Omega = 0 \Rightarrow \delta\phi = 0$, which coincides with the constant energy and comoving gauge conditions in both frames. As such, one finds that the curvature perturbation on constant density hypersurfaces is the same in the two frames. More generally, however, the gauge $\delta\Omega = 0$ does not coincide with the comoving or constant energy density gauges of either frame. Moreover, if Ω is only a function of isocurvature modes, then there is no gauge for which $\delta\Omega = 0$, as isocurvature perturbations are gauge invariant.

Explicitly, on super horizon scales we find that the difference in the curvature perturbations on constant density hypersurfaces as defined in the two frames is of the form [5]

$$\zeta - \tilde{\zeta} \approx \mathcal{A}_{ab}\mathcal{K}^{ab} + \mathcal{B}_{ab}\dot{\mathcal{K}}^{ab}, \tag{9}$$

where $\mathcal{A}_{ab}$ and $\mathcal{B}_{ab}$ take some complex form in terms of background quantities. The important thing to note is that the difference is given in terms of the isocurvature modes $\mathcal{K}^{ab}$ and their derivatives. As such, in the absence of isocurvature modes, i.e. $\delta\phi^a \propto \dot{\phi}^a$, the two curvature perturbations do coincide.

With a non-zero difference between the two curvature perturbations we might also expect that their evolutions be different. In particular, it may be possible that despite being conserved in one frame, the curvature perturbation continues to evolve in the other.

In the Einstein frame we find that

$$\delta\tilde{p}_{nad} = -\frac{2S_{ab}\phi'^a}{(2\kappa^2 f)^2(\tilde{\rho}+\tilde{p})}S_{cd}\frac{D^{(S)}\phi'^d}{d\tilde{t}}\tilde{\mathcal{K}}^{bc}, \tag{10}$$

where a prime denotes a derivative with respect to $\tilde{t}$ ($d\tilde{t} = \sqrt{2f}dt$), $\tilde{\mathcal{K}}^{ab}$ is defined as in (7) but with dots replaced by primes and $D^{(S)}/d\tilde{t}$ is the covariant derivative with respect to the field-space metric S_{ab}. The curvature perturbation is thus sourced by the isocurvature modes when the background trajectory deviates from a geodesic of the field-space. This is the intuitive generalisation of the canonical two-field example given above.

Unfortunately, in the Jordan frame there seems to be no such intuitive interpretation as to when the curvature perturbations is conserved. However, with the difference between the two frames (9) given in terms of $\mathcal{K}^{ab}$, it is clear that the non-conservation in the Jordan frame will also be a result of sourcing from the isocurvature modes.

One interesting scenario would arise if the curvature perturbations in both frames were conserved but with a non-zero difference in their magnitudes. The first property might naively lead us to believe the evolution to be effectively single-field, but this would then be in disagreement with the second property. Using a two-field example we show that such a scenario is indeed possible [5].

We consider two fields ϕ and χ and require that $\dot{\chi} = 0$ and $f = f(\chi)$. This means that the χ field corresponds to the isocurvature mode and that the non-minimal coupling function only depends on this component. On imposing conservation of the curvature perturbation in the Einstein frame we find that the curvature perturbation in the Jordan frame is not necessarily conserved. However, if we further impose that the effective mass of the χ field be $\mathcal{O}(\epsilon)$ then we find

$$\frac{1}{H}\frac{\mathrm{d}}{\mathrm{dt}}\ln(\zeta - \tilde{\zeta}) \sim \mathcal{O}(\epsilon) \qquad \text{and} \qquad \zeta - \tilde{\zeta} = \frac{f_\chi}{2f\epsilon}\frac{H}{\sqrt{2k^3}}(1 + 3f_\chi^2/f)^{-1/2}, \tag{11}$$

so that to zeroth order in slow roll we also have conservation of ζ whilst maintaining a non-zero $\zeta - \tilde{\zeta}$. Comparing this expression with standard slow-roll expressions, we see that in order for ζ and $\tilde{\zeta}$ to be of the correct order of magnitude we require $f_\chi/\sqrt{f} \sim \mathcal{O}(\epsilon^{1/2})$.

The above two-field example indicates that we have a scenario such as that sketched in Figure 1. During inflation the two curvature perturbations are not equivalent, but both are conserved. The fact that we eventually require them to agree (under the assumption that an adiabatic limit is reached) then means that there must be an intermediate phase where the evolution of the curvature perturbation in the two frames is very different.

3 δN formalism

In applying the δN formalism we expand the total number of e-foldings between an initial flat hypersurface shortly after horizon crossing and a final constant energy hypersurface as a function of the initial field values on the flat hypersurface [6]. In the Jordan and Einstein frames we have

$$\zeta = \delta N = N_{,a}\delta\phi^a_f + \frac{1}{2}N_{,ab}\delta\phi^a_f\delta\phi^b_f + ... \qquad \text{and} \qquad \tilde{\zeta} = \delta\tilde{N} = \tilde{N}_a\delta\phi^a_{\tilde{f}} + \frac{1}{2}\tilde{N}_{ab}\delta\phi^a_{\tilde{f}}\delta\phi^b_{\tilde{f}} + ... \tag{12}$$

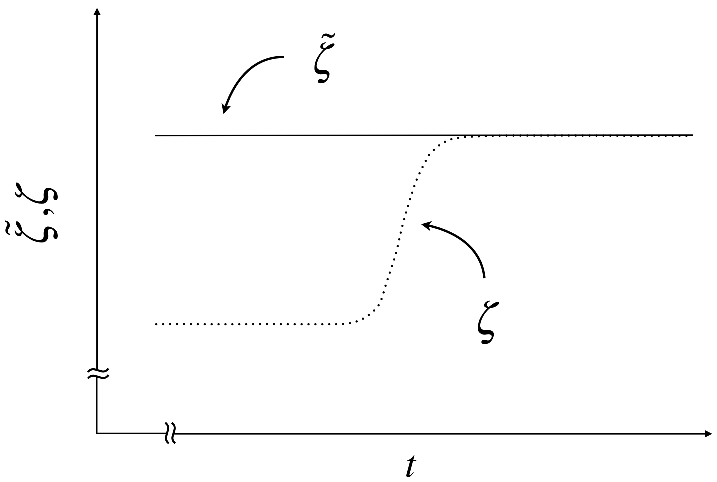

Figure 1:

respectively. Comparing these expansions we see that any difference between the curvature perturbations in the two frames comes from differences in the derivatives of N and $\tilde{N}$ with respect to the initial conditions and differences in the definition of the flat-gauge field perturbations $\delta\phi^a_f$ and $\delta\phi^a_{\tilde{f}}$.

By considering the definition of $\delta\phi^a_f$ and $\delta\phi^a_{\tilde{f}}$ at first order we see that

$$\delta\phi^a_f \equiv \delta\phi^a - \frac{\phi'^a}{\mathcal{H}}\mathcal{R} = (\delta^a_b + \frac{\phi'^a f_{,b}}{2f\mathcal{H}})\delta\phi^b_{\tilde{f}}, \tag{13}$$

where we have used $\tilde{\mathcal{R}} = \mathcal{R} + \frac{f_{,a}\delta\phi^a}{2f}$, $\tilde{\mathcal{H}} = \mathcal{H} + \frac{f'}{2f}$ and here a prime denotes a derivative with respect to conformal time. A similar relation can be found at second order [7][8].

Turning to the derivatives of N and $\tilde{N}$, the integral expressions for N and $\tilde{N}$ are given as

$$N = \int_E^* \mathcal{H} d\eta \qquad \text{and} \qquad \tilde{N} = \int_{\tilde{E}}^* \tilde{\mathcal{H}} d\eta = \int_{\tilde{E}}^* \mathcal{H} d\eta + \frac{1}{2}\ln\left(\frac{f_*}{f_{\tilde{E}}}\right) \tag{14}$$

respectively, where $*$ denotes the initial conditions, $E =$ const. specifies the final constant energy surface in the Jordan frame and $\tilde{E} =$ const. the final constant energy surface in the Einstein frame. We see that the differences thus come from the additional log term in the Einstein frame and the difference in the definition of the final constant energy surface.

Combining the results up to second order we find [8]

$$\begin{aligned}\zeta - \tilde{\zeta} &= (N_a - N_a^{\tilde{E}})\delta\phi^a_{\tilde{f}} + \frac{1}{2}(N_{ab} - N_{ab}^{\tilde{E}})\delta\phi^a_{\tilde{f}(1)}\delta\phi^b_{\tilde{f}(1)} \\ &+ \frac{f_b}{2f}\bigg|_{\tilde{E}} \frac{\partial\phi^b_{\tilde{E}}}{\partial\phi^a_*}\delta\phi^a_{\tilde{f}} + \left(\frac{f_{cd}}{2f}\bigg|_{\tilde{E}} \frac{\partial\phi^c_{\tilde{E}}}{\partial\phi^a_*}\frac{\partial\phi^d_{\tilde{E}}}{\partial\phi^b_*} - \frac{f_c f_d}{2f^2}\bigg|_{\tilde{E}} \frac{\partial\phi^c_{\tilde{E}}}{\partial\phi^a_*}\frac{\partial\phi^d_{\tilde{E}}}{\partial\phi^b_*} + \frac{f_c}{2f}\bigg|_{\tilde{E}} \frac{\partial^2\phi^c_{\tilde{E}}}{\partial\phi^a_*\partial\phi^b_*} \right)\delta\phi^a_{\tilde{f}(1)}\delta\phi^b_{\tilde{f}(1)},\end{aligned} \tag{15}$$

where

$$N_a^{\tilde{E}} = \frac{\partial}{\partial\phi^a_*}\left(\int_{\tilde{E}}^* \mathcal{H} d\eta\right). \tag{16}$$

It turns out that contributions to this difference coming from the relation between $\delta\phi^a_f$ and $\delta\phi^a_{\tilde{f}}$ cancel with those from the $\ln(f_*)$ term in (14), meaning that it is the difference in definition of the final constant energy surface that is important. If we consider an adiabatic limit, where final field values are independent of the initial conditions ($\partial\phi^a_{\tilde{E}}/\partial\phi^b_* = 0$) and the surfaces $E =$ const. and $\tilde{E} =$ const. coincide, then we see that $\zeta = \tilde{\zeta}$ is recovered.

To determine the power-spectrum, its tilt and non-gaussianity in the Jordan frame we can exploit relation (13) between $\delta\phi^a_f$ and $\delta\phi^a_{\tilde f}$ and our knowledge of the correlation functions of $\delta\phi^a_{\tilde f}$

$$\langle\delta\phi^a_{\tilde f \boldsymbol{k}}\delta\phi^b_{\tilde f \boldsymbol{k}'}\rangle = (2\pi)^3\delta^3(\boldsymbol{k}+\boldsymbol{k}')\frac{\tilde H^2}{2k^3}S^{ab}. \tag{17}$$

Defining

$$\mathcal{N}_a = N_a + \frac{f_a}{2f} \qquad \text{and} \qquad \mathcal{N}_{ab} = \nabla_a\nabla_b N + \frac{\nabla_a\nabla_b f}{2f} - \frac{f_a f_b}{2f^2}, \tag{18}$$

we find

$$\mathcal{P}_\zeta(k) = \mathcal{N}_a\mathcal{N}_b S^{ab}\left(\frac{\tilde H}{2\pi}\right)^2 \qquad \text{and} \qquad f_{NL} = \frac{\mathcal{N}_a\mathcal{N}_b\mathcal{N}_{cd}S^{ac}S^{bd}}{[\mathcal{N}_e\mathcal{N}_f S^{ef}]^2}. \tag{19}$$

These are to be compared to the Einstein frame results

$$\mathcal{P}_{\tilde\zeta}(k) = \tilde N_a\tilde N_b S^{ab}\left(\frac{\tilde H}{2\pi}\right)^2 \qquad \text{and} \qquad \tilde f_{NL} = \frac{\tilde N_a\tilde N_b\nabla_c\nabla_d\tilde N S^{ac}S^{bd}}{\left[\tilde N_e\tilde N_f S^{ef}\right]^2}. \tag{20}$$

In general the derivatives of the total number of e-foldings with respect to the initial conditions must be calculated numerically. However, even in the case of non-minimal coupling there are cases where it can be done analytically. In the slow-roll approximation the equations of motion in the Jordan frame are of the form

$$\frac{d\phi^a}{dN} = 2fh^{ab}\frac{W_b}{W}, \tag{21}$$

where $W = V/f^2$. These can be solved analytically if $h^{ab} = \delta^{ab}$ and W is either product or sum separable. Taking the product separable case, in the Jordan frame $E = fW/2$ and we find [8]

$$N_c = \frac{1}{2f}\bigg|_E \frac{g^c_E}{g^c_*}\frac{\left(\frac{f_c}{f}\Big|_E + g^c_E\right)}{\sum_a\left(\frac{f_a}{f}\Big|_E + g^a_E\right)g^a_E} - \int_{\phi^a_E}^{\phi^a_*}\frac{f_c}{2f^2}\frac{g^c}{g^c_*}\frac{d\phi^a}{g^a} \tag{22}$$

whilst in the Einstein frame we have $\tilde E = W/4$, which gives

$$\tilde N_c = \frac{1}{2f}\bigg|_{\tilde E}\frac{g^c_{\tilde E}}{g^c_*}\frac{g^c_{\tilde E}}{\sum_a g^a_{\tilde E}g^a_{\tilde E}} - \int_{\phi^a_{\tilde E}}^{\phi^a_*}\frac{f_c}{2f^2}\frac{g^c}{g^c_*}\frac{d\phi^a}{g^a}, \tag{23}$$

where $g^a = W_a/W$.

Returning to the two-field example considered at linear order, but no longer imposing the conservation of $\tilde\zeta$, at linear level we find

$$\delta N - \delta\tilde N \simeq \frac{f_\chi}{2f\epsilon_E}\delta\chi_{\tilde f}, \tag{24}$$

which is in agreement with the previous result. Making the same argument as before that $f_\chi/\sqrt{f}$ should be $\mathcal{O}(\sqrt{\epsilon})$, and making some order of magnitude estimates, we find that $\tilde f_{NL} \sim \mathcal{O}(1)\times f_{\chi\chi}$ and that $(f_{NL} - \tilde f_{NL})/\tilde f_{NL} \sim \mathcal{O}(1)$. As such, the difference in non-gaussianity of ζ and $\tilde\zeta$ is potentially significant.

On a more general note, we see that at second order $f_{\chi\chi}$ contributes to possible discrepancies between ζ and $\tilde\zeta$. This means that even if f_χ gives a negligible difference between the results at linear order, a large $f_{\chi\chi}$ could still potentially give a significant difference at the level of non-gaussianity. Indeed, a well motivated form for the non-minimal coupling would be $f = 1 + \xi\chi^2$. In such a case, if the background trajectory is along $\chi = 0$, we would have $f_\chi = 0$ but $f_{\chi\chi} = 2\xi$. The consideration of specific forms for f and V and the resulting spectrum parameters is currently under way.

4 Conclusions

We have explored the relation between the Jordan and Einstein frame curvature perturbations on constant energy density hypersurfaces in the presence of multiple fields and non-minimal coupling. At linear order we saw explicitly that they are not the same and that the difference is a direct consequence of the isocurvature perturbations inherent to multi-field models. We also saw that conservation of the curvature perturbation in one frame does not imply conservation in the other, but that it is also possible that both are conserved with a non-zero difference in their magnitudes. At non-linear order we were able to implement the δN formalism in the Jordan and Einstein frames and saw explicitly that the non-gaussianity of the two curvature perturbations may also differ in the case that an adiabatic limit has not been reached. It was clear that the source of the difference is the difference in definition of the final constant energy surface in the Jordan and Einstein frames. These results highlight the non-observable nature of ζ itself, and the importance of knowing how both curvature perturbations evolve through reheating and are related to the temperature anisotropies of the Cosmic Microwave Background. Work in this direction is currently under way.

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