Kouji Nakamura, JGRG 22(2012)111315

"Problems in nth-order extension of the gauge-invariant perturbation theory"

RESCEU SYMPOSIUM ON

GENERAL RELATIVITY AND GRAVITATION

JGRG 22

November 12-16 2012

Koshiba Hall, The University of Tokyo, Hongo, Tokyo, Japan

Problems in nth-order extension of the gauge-invariant perturbation theory

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References :

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K.N. in progress

I. Introduction

- The higher order perturbation theory in general relativity has very wide physical motivation.

  – Cosmological perturbation theory
    - Expansion law of inhomogeneous universe (ΛCDM v.s. inhomogeneous cosmology)
    - Non-Gaussianity in CMB (beyond WMAP)

  – Black hole perturbations
    - Radiation reaction effects due to the gravitational wave emission.

  – Perturbation of a star (Neutron star)
    - Rotation – pulsation coupling (Kojima 1997)

There are many physical situations to which higher order perturbation theory should be applied.

However, general relativistic perturbation theory requires very delicate treatments of “gauges”.

It is worthwhile to formulate the higher-order gauge-invariant perturbation theory from general point of view.

- According to this motivation, we have been formulating the general relativistic second-order perturbation theory in a gauge-invariant manner.
  - **General formulation :**
    - Framework of higher-order gauge-invariant perturbations :
      - K.N. PTP**110** (2003), 723; *ibid.* **113** (2005), 413.
    - Construction of gauge-invariant variables for the linear order metric perturbation :
      - K.N. CQG**28** (2011), 122001; 1105.4007[gr-qc]; IJMPD**21** (2012), 1242004.
    - **The nth-order extension of the definitions of gauge-invariant variables :**
      - K.N. in progress. (I am trying to resolve this issue.)

  - **Application to cosmological perturbation theory** :
    - Einstein equations : K.N. PRD**74** (2006), 101301R; PTP**117** (2007), 17.
    - Equations of motion for matter fields : K.N. PRD**80** (2009), 124021.
    - Consistency of the 2nd order Einstein equations : K.N. PTP**121** (2009), 1321.
    - Summary of current status of this formulation : K.N. Adv. in Astron. **2010** (2010), 576273.
    - Comparison with a different formulation : A.J. Christopherson, et al., CQG**28** (2011), 225024.

- In this poster presentation,
  - We propose an outline to define the gauge-invariant variables for nth-order general-relativistic perturbations.
  - Some problems to complete this outline are summarized.

II. "Gauge" in general relativity

(R.K. Sachs (1964).)

- There are two kinds of "gauge" in general relativity.
  - The concepts of these two "gauge" are closely related to the general covariance.
  - "General covariance" :

    There is no preferred coordinate system in nature.

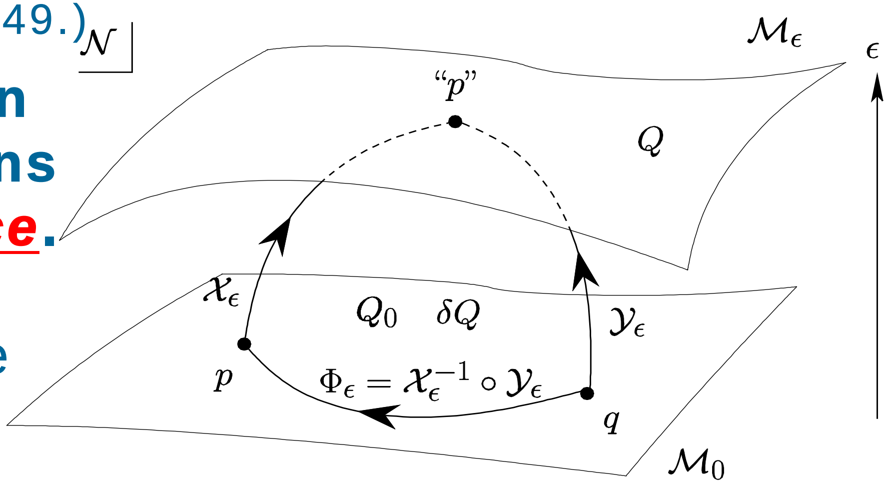
- **The first kind "gauge"** is a coordinate system on a single spacetime manifold.

- **The second kind "gauge"** appears in the perturbation theory.

  This is a point identification between the physical spacetime and the background spacetime.
  - To explain this second kind "gauge", we have to remind what we are doing in perturbation theory.

III. The second kind gauge in GR.

(Stewart and Walker, PRSL **A341** (1974), 49.)



- **"*Gauge degree of freedom*" in general relativistic perturbations arises due to *general covariance*.**

- In any perturbation theories, we always treat two spacetimes :
  - *Physical Spacetime (PS);*
  - *Background Spacetime (BGS).*

- In perturbation theories, we always write equations like

$$Q(\text{“}p\text{”}) = Q_0(p) + \delta Q(p)$$

Through this equation, we always identify the points on these two spacetimes and this identification is called "gauge choice" in perturbation theory.

Gauge transformation rules to second order

- **Expansion of gauge choices :**

We assume that each gauge choice is an exponential map.

$$Q_{\mathcal{X}} = \mathcal{X}_\epsilon^* Q = Q + \epsilon \pounds_u Q + \frac{1}{2}\epsilon^2 \pounds_u^2 Q + O(\epsilon^3)$$

$$Q_{\mathcal{Y}} = \mathcal{Y}_\epsilon^* Q = Q + \epsilon \pounds_v Q + \frac{1}{2}\epsilon^2 \pounds_v^2 Q + O(\epsilon^3)$$

$$\Phi_\epsilon^* Q = \left(\mathcal{X}_\epsilon^{-1} \circ \mathcal{Y}_\epsilon\right)^* Q = Q + \epsilon \pounds_{\xi_1} Q + \frac{1}{2}\epsilon^2 \left(\pounds_{\xi_2} + \pounds_{\xi_1}^2\right) Q + O(\epsilon^3)$$

(Sonego and Bruni, CMP, **193** (1998), 209.)

-------> $\xi_1 = u - v, \quad \xi_2 = [u, v]$

- **Expansion of the variable :**

$$Q = Q_0 + \epsilon Q_1 + \frac{1}{2} Q_2 + O(\epsilon^3)$$

- **Order by order gauge transformation rules :**

$$ {}_{\mathcal{Y}}Q_1 - {}_{\mathcal{X}}Q_1 = \pounds_{\xi_1} Q_0$$

$$ {}_{\mathcal{Y}}Q_2 - {}_{\mathcal{X}}Q_2 = 2\pounds_{\xi_1} {}_{\mathcal{X}}Q_1 + \left(\pounds_{\xi_2} + \pounds_{\xi_1}^2\right) Q_0$$

Gauge transformation rules for nth-order perturbations (Sonego and Bruni, CMP, **193** (1998), 209.)

- **Representation of general diffeomorphism :**

$$\Phi^*_\epsilon Q = \phi_{\frac{\epsilon^n}{(n)!}\xi_{(n)}} \circ \phi_{\frac{\epsilon^{n-1}}{(n-1)!}\xi_{(n-1)}} \circ \cdots \phi_{\frac{\epsilon^2}{(2)!}\xi_{(2)}} \circ \phi_{\frac{\epsilon}{(1)!}\xi_{(1)}} Q + O(\epsilon^{n+1})$$

$$= \sum_{l=0}^{n} \frac{\epsilon^l}{l!} \sum_{J_l} \frac{l!}{(1!)^{j_1}(2!)^{j_2}\cdots(l!)^{j_l} j_1! j_2! \cdots j_l!} \pounds^{j_1}_{\xi_{(1)}} \pounds^{j_2}_{\xi_{(2)}} \cdots \pounds^{j_l}_{\xi_{(l)}} Q + O(\epsilon^{n+1})$$

where $J_l := \left\{ (j_1, ..., j_l) \in \mathbb{N}^l \middle| \sum_{i=1}^{l} i j_i = l \right\}$, $\phi_{\frac{\epsilon^l}{(l!)}\xi_{(l)}}$: the exponential map generated by $\frac{\epsilon^l}{(l!)}\xi^a_{(l)}$.

- **Expansion of the variable :** $Q = \sum_{l=0}^{n} \frac{\epsilon^l}{l!} Q_{(l)} + O(\epsilon^{n+1})$

- **Order by order gauge transformation rules :**

$${}_{\mathcal{Y}}Q_{(k)} = \sum_{l=0}^{k} \frac{k!}{(k-l)!} \sum_{J_l} \frac{1}{(1!)^{j_1}(2!)^{j_2}\cdots(l!)^{j_l} j_1! j_2! \cdots j_l!} \pounds^{j_1}_{\xi_{(1)}} \pounds^{j_2}_{\xi_{(2)}} \cdots \pounds^{j_l}_{\xi_{(l)}} {}_{\mathcal{X}}Q_{(k-l)}$$

To develop nth-order gauge-invariant perturbation theory, we have to construct gauge-invariant variables for each order perturbation through this gauge-transformation rule.

- **Problem 1 :**
  General diffeomorphism should form a group. How to prove?

III. Construction of gauge invariant variables in higher order perturbations

- metric perturbation : metric on PS : $\bar{g}_{ab}$, metric on BGS : g_{ab}

  metric expansion : $\bar{g}_{ab} = g_{ab} + \epsilon h_{ab} + \frac{1}{2}\epsilon^2 l_{ab} + O(\epsilon^3)$

  Our general framework of the second-order gauge invariant perturbation theory is based on a single assumption.

- **linear order (decomposition conjecture) :**

Suppose that the linear order perturbation h_{ab} is decomposed as

$$h_{ab} = \mathcal{H}_{ab} + \pounds_X g_{ab}$$

so that the variable $\mathcal{H}_{ab}$ and X^a are the gauge invariant and the gauge variant parts of h_{ab} , respectively.

These variables are transformed as

$${}_{\mathcal{Y}}\mathcal{H}_{ab} - {}_{\mathcal{X}}\mathcal{H}_{ab} = 0 \qquad {}_{\mathcal{Y}}X^a - {}_{\mathcal{X}}X^a = \xi_1^a$$

under the gauge transformation $\Phi_\epsilon = \mathcal{X}_\epsilon^{-1} \circ \mathcal{Y}_\epsilon$.

This conjecture is almost proved but is still a conjecture due to the "zero-mode problem" !! (Problem 2)

[See K.N. CQG **28** (2011), 122001; arXiv:1105.4007[gr-qc]; IJMPD **21** (2012), 1242004.]

Second order :

Once we accept the above assumption for the linear order metric perturbation h_{ab} , we can always decompose the second order metric perturbations l_{ab} as follows :

$$l_{ab} =: \mathcal{L}_{ab} + 2\pounds_X h_{ab} + \left(\pounds_Z - \pounds_X^2\right) g_{ab}$$

where $\mathcal{L}_{ab}$ is gauge invariant part and Z_a is gauge variant part.

Under the gauge transformation $\Phi_\epsilon = \mathcal{X}_\epsilon^{-1} \circ \mathcal{Y}_\epsilon$, the vector field Z_a is transformed as $_{\mathcal{Y}}Z^a - {}_{\mathcal{X}}Z^a = \xi_2^a + [\xi_1, X]^a$

Perturbations of an arbitrary matter field Q :

Using gauge variant part of the metric perturbation of each order, gauge invariant variables for an arbitrary tensor fields Q other than metric are defined by

- ▲ First order perturbation of Q : $\mathcal{Q}_1 := Q_1 - \pounds_X Q_0$
- ▲ Second order perturbation of Q :

$$\mathcal{Q}_2 := Q_2 - 2\pounds_X Q_1 - \left\{\pounds_Z - \pounds_X^2\right\} Q_0$$

The essence of the above construction of gauge-invariant variables is to find gauge-variant variables for each-order metric perturbations.

---> The definitions of gauge-invariant variables for an arbitrary matter field are automatically given through the gauge-variant variables for each-order metric perturbation.

We may concentrate on each-order metric perturbation.

The metric expansion : $\bar{g}_{ab} = \sum_{l=0}^{n} \frac{\epsilon^l}{l!} {}^{(l)}g_{ab} + O(\epsilon^{n+1})$

We denote the gauge-variant parts of the first- and the second-order metric perturbation as

$$ {}^{(1)}X^a := X^a, \quad {}^{(2)}X^a := Z^a. $$

Third-order metric perturbation [K.N. (2003)]

Gauge transformation rule :

$$ {}^{(3)}_{\mathcal{Y}}g_{ab} = {}^{(3)}_{\mathcal{X}}g_{ab} + 3\pounds_{\xi_{(1)}} {}^{(2)}_{\mathcal{X}}g_{ab} + 3\left(\pounds^2_{\xi_{(1)}} + \pounds_{\xi_{(2)}}\right) {}^{(1)}_{\mathcal{X}}g_{ab} + \left(\pounds^3_{\xi_{(1)}} + 3\pounds_{\xi_{(1)}}\pounds_{\xi_{(2)}} + \pounds_{\xi_{(3)}}\right) g_{ab} $$

Inspecting this gauge-transformation rule, we define the variable ${}^{(3)}\hat{H}_{ab}$ by

$$ {}^{(3)}\hat{H}_{ab} := {}^{(3)}g_{ab} + 3\pounds_{-{}^{(1)}X} {}^{(2)}g_{ab} + 3\left(\pounds^2_{-{}^{(1)}X} + \pounds_{-{}^{(2)}X}\right) {}^{(1)}g_{ab} + \left(\pounds^3_{-{}^{(1)}X} + 3\pounds_{-{}^{(1)}X}\pounds_{-{}^{(2)}X}\right) g_{ab} $$

We can check that the gauge-transformation rule for ${}^{(3)}\hat{H}_{ab}$ is given by

$$ {}^{(3)}_{\mathcal{Y}}\hat{H}_{ab} - {}^{(3)}_{\mathcal{X}}\hat{H}_{ab} = \pounds_{\sigma_{(3)}} {}^{(0)}g_{ab} $$

$$ \sigma^a_{(3)} := \xi^a_{(3)} + 3[\xi_{(1)}, {}^{(2)}_{\mathcal{X}}X]^a + 3[\xi_{(1)}, \xi_{(2)}]^a + 3[{}^{(1)}_{\mathcal{X}}X, \xi_{(2)}]^a + 2[\xi_{(1)}, [\xi_{(1)}, {}^{(1)}_{\mathcal{X}}X]]^a + [{}^{(1)}_{\mathcal{X}}X, [\xi_{(1)}, {}^{(1)}_{\mathcal{X}}X]]^a $$

Applying the above decomposition conjecture, we can decompose the variable ${}^{(3)}\hat{H}_{ab}$ into its gauge-invariant and gauge-variant parts as

$$ {}^{(3)}\hat{H}_{ab} =: {}^{(3)}\mathcal{H}_{ab} + \pounds_{{}^{(3)}X} {}^{(0)}g_{ab}, \quad {}^{(3)}_{\mathcal{Y}}\mathcal{H}_{ab} - {}^{(3)}_{\mathcal{X}}\mathcal{H}_{ab} = 0, \quad {}^{(3)}_{\mathcal{Y}}X^a - {}^{(3)}_{\mathcal{X}}X^a = \sigma^a_{(3)}. $$

This implies that we have decomposed ${}^{(3)}g_{ab}$ as

$$ {}^{(3)}g_{ab} =: {}^{(3)}\mathcal{H}_{ab} + \pounds_{{}^{(3)}X} g_{ab} - 3\pounds_{-{}^{(1)}X} {}^{(2)}g_{ab} - 3\left(\pounds^2_{-{}^{(1)}X} + \pounds_{-{}^{(2)}X}\right) {}^{(1)}g_{ab} - \left(\pounds^3_{-{}^{(1)}X} + 3\pounds_{-{}^{(1)}X}\pounds_{-{}^{(2)}X}\right) g_{ab} $$

(cyan) : gauge-invariant part, (yellow) : gauge-variant part.

○ Fourth-order metric perturbation :

Gauge transformation rule :

$$
{}^{(4)}_{\mathcal{Y}}g_{ab} = {}^{(4)}_{\mathcal{X}}g_{ab} + 4\pounds_{\xi_{(1)}}{}^{(3)}_{\mathcal{X}}g_{ab} + 6\left(\pounds_{\xi_{(2)}} + \pounds^2_{\xi_{(1)}}\right){}^{(2)}_{\mathcal{X}}g_{ab} + 4\left(\pounds_{\xi_{(3)}} + 3\pounds_{\xi_{(1)}}\pounds_{\xi_{(2)}} + \pounds^3_{\xi_{(1)}}\right){}^{(1)}_{\mathcal{X}}g_{ab}
$$
$$
+\left(\pounds_{\xi_{(4)}} + 3\pounds^2_{\xi_{(2)}} + 4\pounds_{\xi_{(1)}}\pounds_{\xi_{(3)}} + 6\pounds^2_{\xi_{(1)}}\pounds_{\xi_{(2)}} + \pounds^4_{\xi_{(1)}}\right)g_{ab}
$$

Inspecting this gauge-transformation rule, we define the variable ${}^{(4)}\hat{H}_{ab}$ by

$$
\begin{aligned}
{}^{(4)}\hat{H}_{ab} := & {}^{(4)}g_{ab} + 4\pounds_{-{}^{(1)}X}{}^{(3)}g_{ab} + 6\left(\pounds_{-{}^{(2)}X} + \pounds^2_{-{}^{(1)}X}\right){}^{(2)}g_{ab} \\
& +4\left(\pounds_{-{}^{(3)}X} + 3\pounds_{-{}^{(1)}X}\pounds_{-{}^{(2)}X} + \pounds^3_{-{}^{(1)}X}\right){}^{(1)}g_{ab} \\
& +\left(3\pounds^2_{-{}^{(2)}X} + 4\pounds_{-{}^{(1)}X}\pounds_{-{}^{(3)}X} + 6\pounds^2_{-{}^{(1)}X}\pounds_{-{}^{(2)}X} + \pounds^4_{-{}^{(1)}X}\right)g_{ab}
\end{aligned}
$$

We can check that the gauge-transformation rule for ${}^{(4)}\hat{H}_{ab}$ is given by

$$
{}^{(4)}_{\mathcal{Y}}\hat{H}_{ab} - {}^{(4)}_{\mathcal{X}}\hat{H}_{ab} = \pounds_{\sigma_{(4)}}{}^{(0)}g_{ab}
$$

$$
\begin{aligned}
\sigma^a_{(4)} := & \xi^a_{(4)} + 4[\xi_{(1)},\xi_{(3)}]^a + 6[\xi_{(1)},[\xi_{(1)},\xi_{(2)}]]^a + 4[\xi_{(1)},{}^{(3)}X]^a + 3[\xi_{(2)},{}^{(2)}X]^a + 6[\xi_{(1)},[\xi_{(1)},{}^{(2)}X]]^a + 3[\xi_{(2)},[\xi_{(1)},{}^{(1)}X]]^a \\
& +3[{}^{(2)}X,[\xi_{(1)},{}^{(1)}X]]^a + 3[\xi_{(1)},[\xi_{(1)},[\xi_{(1)},{}^{(1)}X]]]^a + 3[\xi_{(1)},[{}^{(1)}X,[\xi_{(1)},{}^{(1)}X]]]^a + [{}^{(1)}X,[{}^{(1)}X,[\xi_{(1)},{}^{(1)}X]]]^a
\end{aligned}
$$

Applying the above decomposition conjecture, we can decompose the variable ${}^{(4)}\hat{H}_{ab}$ into its gauge-invariant and gauge-variant parts as

$$
{}^{(4)}\hat{H}_{ab} =: {}^{(4)}\mathcal{H}_{ab} + \pounds_{{}^{(4)}X}{}^{(0)}g_{ab}, \quad {}^{(4)}_{\mathcal{Y}}\mathcal{H}_{ab} - {}^{(4)}_{\mathcal{X}}\mathcal{H}_{ab} = 0, \quad {}^{(4)}_{\mathcal{Y}}X^a - {}^{(4)}_{\mathcal{X}}X^a = \sigma^a_{(4)}.
$$

This implies that we have decomposed ${}^{(4)}g_{ab}$ as

$$
\begin{aligned}
{}^{(4)}g_{ab} =: & {}^{(4)}\mathcal{H}_{ab} + \pounds_{{}^{(4)}X}g_{ab} - 4\pounds_{-{}^{(1)}X}{}^{(3)}g_{ab} - 6\left(\pounds_{-{}^{(2)}X} + \pounds^2_{-{}^{(1)}X}\right){}^{(2)}g_{ab} \\
& -4\left(\pounds_{-{}^{(3)}X} + 3\pounds_{-{}^{(1)}X}\pounds_{-{}^{(2)}X} + \pounds^3_{-{}^{(1)}X}\right){}^{(1)}g_{ab} \\
& -\left(3\pounds^2_{-{}^{(2)}X} + 4\pounds_{-{}^{(1)}X}\pounds_{-{}^{(3)}X} + 6\pounds^2_{-{}^{(1)}X}\pounds_{-{}^{(2)}X} + \pounds^4_{-{}^{(1)}X}\right)g_{ab}
\end{aligned}
$$

: gauge-invariant part, : gauge-variant part.

Problem 3 : nth-order metric perturbation :

Gauge transformation rule :

$$ {}^{(n)}_{\mathcal{Y}}g_{ab} - {}^{(n)}_{\mathcal{X}}g_{ab} = \sum_{l=1}^{n} \frac{n!}{(n-l)!} \sum_{J_l} \frac{1}{(1!)^{j_1}(2!)^{j_2}\cdots(l!)^{j_l} j_1! j_2! \cdots j_l!} \pounds^{j_1}_{\xi_{(1)}} \pounds^{j_2}_{\xi_{(2)}} \cdots \pounds^{j_l}_{\xi_{(l)}} {}^{(n-l)}_{\mathcal{X}}g_{ab} $$

$$ =: F\left[\xi^a_{(1)}, \cdots, \xi^a_{(n-1)}, \xi^a_{(n)}; {}^{(0)}_{\mathcal{X}}g_{ab}, {}^{(1)}_{\mathcal{X}}g_{ab}, \cdots, {}^{(n-1)}_{\mathcal{X}}g_{ab}\right] $$

Inspecting this gauge-transformation rule, we define the variable ${}^{(n)}\hat{H}_{ab}$ by

$$ {}^{(n)}\hat{H}_{ab} := {}^{(n)}g_{ab} + F\left[-{}^{(1)}X^a, \cdots, -{}^{(n-1)}X^a, 0; {}^{(0)}g_{ab}, {}^{(1)}g_{ab}, \cdots, {}^{(n-1)}g_{ab}\right] $$

We have to prove the following statement :
There exists a vector field $\sigma^a_{(n)}$ such that the gauge-transformation rule for the variable ${}^{(n)}\hat{H}_{ab}$ is given by ${}^{(n)}_{\mathcal{Y}}\hat{H}_{ab} - {}^{(n)}_{\mathcal{X}}\hat{H}_{ab} = \pounds_{\sigma_{(n)}} {}^{(0)}g_{ab}$

When the proof of the above statement is accomplished, we may apply the decomposition conjecture and we can decompose the variable ${}^{(n)}\hat{H}_{ab}$ into its gauge-invariant and gauge-variant parts as

$$ {}^{(n)}\hat{H}_{ab} =: {}^{(n)}\mathcal{H}_{ab} + \pounds_{{}^{(n)}X} {}^{(0)}g_{ab}, \quad {}^{(n)}_{\mathcal{Y}}\mathcal{H}_{ab} - {}^{(n)}_{\mathcal{X}}\mathcal{H}_{ab} = 0, \quad {}^{(n)}_{\mathcal{Y}}X^a - {}^{(n)}_{\mathcal{X}}X^a = \sigma^a_{(n)}. $$

This implies that we have decomposed ${}^{(n)}g_{ab}$ as

$$ {}^{(n)}g_{ab} := {}^{(n)}\mathcal{H}_{ab} + \pounds_{{}^{(n)}X} g_{ab} - F\left[-{}^{(1)}X^a, \cdots, -{}^{(n-1)}X^a, 0; {}^{(0)}g_{ab}, {}^{(1)}g_{ab}, \cdots, {}^{(n-1)}g_{ab}\right] $$

(cyan) : gauge-invariant part, (yellow) : gauge-variant part.

IV. Summary

We have summarized three problems in the nth-order extension of our general relativistic gauge-invariant perturbation theory.

Problems that we pointed out are as follows:

1. **Group properties of general diffeomorphisms;**
2. **Decomposition conjecture for linear metric perturbations;**
3. **Construction of nth-order gauge-invariant variables.**

Of course, the higher-order perturbation theory requires tough calculations to develop. However, we mainly focus on the essential problems in our formulation, i.e., the construction of gauge-invariant variables.

I am now trying to show the extendibility of our construction of gauge-invariant variables to nth-order perturbations by induction.