Naomasa Fushimi, JGRG 22(2012)111313

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RESCEU SYMPOSIUM ON

GENERAL RELATIVITY AND GRAVITATION

JGRG 22

November 12-16 2012

Koshiba Hall, The University of Tokyo, Hongo, Tokyo, Japan

Criterion for bound (or unbound) orbits in the Kottler spacetime

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JGRG22 in Tokyo November 12 - 16, 2012

Abstract: We discuss criteria for bound (or unbound) orbits of a test particle in the Kottler (Shcwarzschild-de Sitter) spacetime.

1 The equation of motion

The equation of motion for a test particle in Kottler spacetime:

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{\Lambda}{3}r^2 + E + \frac{\Lambda}{3}j^2 + \frac{r_g}{r} - \frac{j^2}{r^2} + \frac{r_g j^2}{r^3} \tag{1}$$

where E , j^2 are constants defined by

$$E = \left(1 - \frac{r_g}{r} - \frac{\Lambda}{3}r^2\right)^2 \dot{t}^2 - 1$$

$$j^2 = r^2\dot{\phi}.$$

They are the energy per unit mass and angular momentum from correspondence with Newtonian mechanics, respectively.
And r_g is Schwarzschild radius, Λ is cosmological constant.

$$\frac{d^2r}{d\tau^2} = \frac{\Lambda}{3}r - \frac{r_g}{2r^2} + \frac{j^2}{4r^3} - \frac{r_g j^2}{6r^4} \tag{2}$$

2 Criterion for bound (or unbound) orbits

2.1 Sch

Well-known criterion for bound orbits near Schwarzschild blackhole.
$(k = \frac{3r_g^2}{j^2} < 1)$

$j^2 < 3r_g^2$

Unbound

$3r_g^2 \leq j^2 < 4r_g^2$

$$\frac{-2(1-k)^{3/2} + 2 - 3k}{9k} < E < \frac{2(1-k)^{3/2} + 2 - 3k}{9k}$$

$4r_g^2 \leq j^2$

$$\frac{-2(1-k)^{3/2} + 2 - 3k}{9k} < E < 0.$$

2.2 Kottler

Since we do not have a quintic formula, we cannot solve equation (1) nor equation (2), either. Then, Strum's theorem is applied.
equation (1) to six inequalities were obtained and equation (2) to three inequalities were obtained. Criterion for bound orbits in Kottler spacetime are found
$(\lambda = r_g^2\Lambda \ll 1)$

$j^2 < 3r_g^2(1-18\lambda)$

Unbound

$3r_g^2(1-18\lambda) < j^2 < 4r_g^2[1-(10\lambda)^{1/3}]$

$$\frac{-2(1-k)^{3/2} + 2 - 3k}{9k} - \lambda A_- < E < \frac{2(1-k)^{3/2} + 2 - 3k}{9k} + \lambda A_+$$

$$\left(A_\pm = 2\frac{16-60k+69k^2-15k^3-12k^4 \pm \sqrt{1-k}(16-52k+45k^2+2k^3-8k^4)}{k\{\pm 2(1-k)^{3/2}+2-3k\}^3}\right)$$

$4r_g^2[1-(10\lambda)^{1/3}] < j^2 < \frac{3}{16}r_g^2\left(\frac{3}{\lambda}\right)^{1/3}$

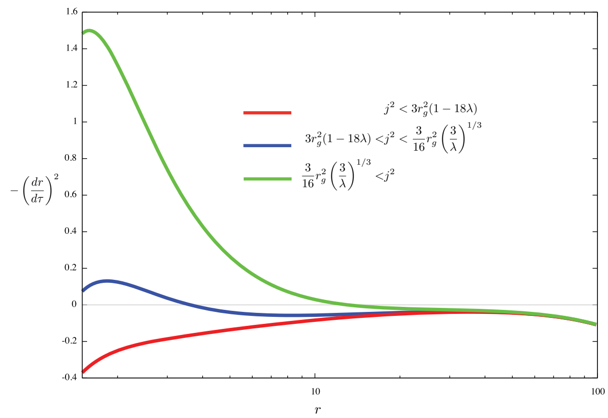
$$\frac{-2(1-k)^{3/2} + 2 - 3k}{9k} - \lambda A_- < E < X^{1/3} - \frac{10\Lambda j^2}{27X^{1/3}} - \frac{\Lambda j^2}{8}$$

$$\left(X = \frac{5\Lambda\sqrt{6561r_g^4 + \frac{2560\Lambda j^6}{3}}}{648} - \frac{5r_g^2\Lambda}{8}\right)$$

$\frac{3}{16}r_g^2\left(\frac{3}{\lambda}\right)^{1/3} < j^2$

Unbound

The upper limit of angular momentum is new criterion.



$(E = 0, \Lambda = 10^{-5}, r_g = 1)$

3 Outermost orbit

In equation (2), the fourth item loses its effect at the distant place. it changes into:

$$\frac{d^2r}{d\tau^2} \simeq \frac{\Lambda}{3}r - \frac{r_g}{2r^2} + \frac{j^2}{4r^3}. \tag{3}$$

On the boundary line at the distant place which changes from bound to unbound, it is required that the solution for r should become a multiple solution. Therefore, angular momentum is obtained as

$$j^2 = \frac{3}{16}r_g^2\left(\frac{3}{\lambda}\right)^{1/3}.$$

and r is obtained as

$$r = \frac{r_g}{2}\left(\frac{3}{\lambda}\right)^{1/3}. \tag{4}$$

This is outermost orbit.

4 Conclusion

We obtained criteria for bound (or unbound) orbits of a test particle.
Then substituting a suitable parameter value into obtained equation (4),

$$r \sim 10^{18}[\mathrm{m}]. \tag{5}$$

References

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