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# Gauss-Bonnet braneworld redux: A novel scenario for the bouncing universe

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## Abstract

We propose a new scenario for the bouncing universe in a simple five-dimensional braneworld model in the framework of Einstein-Gauss-Bonnet gravity, which works even with ordinary matter on the brane. In this scenario, the so-called branch singularity located at a finite physical radius in the bulk spacetime plays an essential role. We show that a three-brane moving in the bulk may reach and pass through it in spite of the fact that it is a curvature singularity. The bulk spacetime is extended beyond the branch singularity in the $C^0$ sense and then the branch singularity is identified as a massive thin shell. From the bulk point of view, this process is the collision of the three-brane with the shell of branch singularity. From the point of view on the brane, this process is a sudden transition from the collapsing phase to the expanding phase of the universe. This manuscript is based on [1].

# 1 Preliminaries

We consider the following five-dimensional action for the bulk spacetime:

$$I = \frac{1}{2\kappa_5^2}\int d^5x\sqrt{-g}\Big(R-2\Lambda+\alpha L_{\rm GB}\Big), \tag{1}$$

where the Gauss-Bonnet term $L_{\rm GB}$ is defined by

$$L_{\rm GB} := R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, \tag{2}$$

which does not give any higher-derivative term in the field equations. The constant $\alpha$ allows the GR limit $\alpha \to 0$. The gravitational equation given from the action (1) is

$$G^\mu{}_\nu + \alpha H^\mu{}_\nu + \Lambda\delta^\mu{}_\nu = 0, \tag{3}$$

where

$$H_{\mu\nu} := 2\Big(RR_{\mu\nu} - 2R_{\mu\alpha}R^\alpha{}_\nu - 2R^{\alpha\beta}R_{\mu\alpha\nu\beta} + R_\mu{}^{\alpha\beta\gamma}R_{\nu\alpha\beta\gamma}\Big) - \frac{1}{2}g_{\mu\nu}L_{\rm GB}. \tag{4}$$

The Gauss-Bonnet term in the action is obtained in the low-energy limit of heterotic superstring theory together with a dilaton in ten dimensions, in which case $\alpha$ is regarded as the inverse string tension and positive definite. We therefore assume $\alpha > 0$ throughout this paper. We also assume $\Lambda < 0$ and $1+4\alpha\Lambda/3 > 0$ in addition, the latter of which ensures the existence of nondegenerate maximally symmetric vacuum solutions.

## 1.1 Bulk solution

In this system, a vacuum solution is obtained as a warped product manifold $\mathcal{M}^5 \approx M^2 \times K^3$, where $K^3$ is a three-dimensional space of constant curvature. In the equations which follow, $k$ denotes the curvature

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of $K^3$ and takes the values 1 (positive curvature), 0 (zero curvature), and $-1$ (negative curvature). The metric of the vacuum solution is given by

$$ds_5^2 = g_{\mu\nu}dx^\mu dx^\nu = -h(r)dt^2 + \frac{dr^2}{h(r)} + r^2\left[d\chi^2 + f_k(\chi)^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{5}$$

$$h(r) := k + \frac{r^2}{4\alpha}\left(1 \mp \sqrt{1 + \frac{\alpha\mu}{r^4} + \frac{4}{3}\alpha\Lambda}\right), \tag{6}$$

where $\mu$ is constant, $f_0(\chi) := \chi$, $f_1(\chi) := \sin\chi$, and $f_{-1}(\chi) := \sinh\chi$ [2, 3].

It is seen that the solution has two branches corresponding to the sign in the metric function $h(r)$. We call the family with the minus (plus) sign the GR branch (non-GR branch). Only the GR branch solution has the general relativistic limit. The maximally symmetric vacuum in the non-GR branch was shown to be unstable [2] and so we only consider the solution in the GR branch in the present paper.

The global structures of this spacetime depending on the parameters have been clarified [4]. In this spacetime, there are two classes of curvature singularity for $\mu \neq 0$. One is the central singularity at $r = 0$ and the other is the branch singularity at $r = r_{\rm b}(> 0)$, where the term inside the square-root in the metric function (6) vanishes. $r_{\rm b}$ is explicitly given by

$$r_{\rm b} := \left(-\frac{3\alpha\mu}{3 + 4\alpha\Lambda}\right)^{1/4}. \tag{7}$$

The branch singularity exists if $\mu$ is negative. The metric and its inverse are finite at $r = r_{\rm b}$ (but their derivatives blow up) and the metric becomes complex and hence unphysical at $r < r_{\rm b}$.

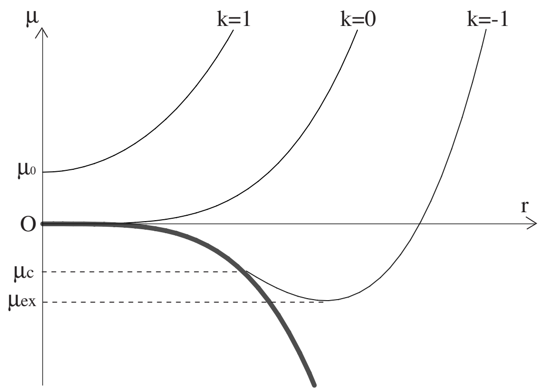


Figure 1: Thin curves show the relations between the mass parameter and the horizon radius for each $k$. A thick curve shows to the relation between the mass parameter and the branch-singularity radius. The metric becomes complex and unphysical below the thick curve.

Near the branch singularity, the metric function behaves as

$$h(r) \simeq \left(k + \frac{r_{\rm b}^2}{4\alpha}\right) - \frac{r_{\rm b}^{3/2}}{2\alpha}\sqrt{1 + \frac{4}{3}\alpha\Lambda}(r - r_{\rm b})^{1/2}. \tag{8}$$

Therefore, the branch singularity is timelike and spacelike for $k + r_{\rm b}^2/(4\alpha) > 0$ and $k + r_{\rm b}^2/(4\alpha) \leq 0$, respectively.

## 1.2 Friedmann equation on the brane

We consider a three-brane in the bulk spacetime (5), which is a timlike hypersurface described by $r = a(\tau)$ and $t = T(\tau)$, where the parameter $\tau$ is the proper time on the brane. The induced metric of the three-brane $\bar{g}_{ab}$ is given by

$$ds_4^2 = \bar{g}_{ab}dy^a dy^b = -d\tau^2 + a(\tau)^2\left[d\chi^2 + f_k(\chi)^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{9}$$

This is the FRW metric with the spatial curvature $k$.

The dynamics of the three-brane, namely the behavior of the scale factor $a(\tau)$ on the brane, is determined by the junction condition. Here we simply assume the $Z_2$-symmetry of reflection with respect to the brane; we take two copies of the bulk spacetime with $r < a(\tau)$ and paste them at $r = a(\tau)$. Using the junction condition in Einstein-Gauss-Bonnet gravity [5, 6] with the following form of the energy-momentum tensor $S^a{}_b$ on the brane;

$$S^a{}_b = \mathrm{diag}(-\rho, p, p, p) + \mathrm{diag}(-\sigma, -\sigma, -\sigma, -\sigma, ), \tag{10}$$

we obtain the modified Friedmann equation for the brane universe as

$$\frac{\kappa_5^4}{36}(\rho+\sigma)^2 = \left(\frac{h(a)}{a^2} + H^2\right)\left[1 + \frac{4\alpha}{3}\left(\frac{3k - h(a)}{a^2} + 2H^2\right)\right]^2, \tag{11}$$

where $H := \dot{a}/a$ and a dot denotes the differentiation with respect to $\tau$ [5, 7]. Here $\rho$ and $p$ are the energy density and pressure of a perfect fluid on the brane, and the constant $\sigma$ is the brane tension.

We assume the following linear equation of state $p = (\gamma - 1)\rho$ and then the energy-conservation equation on the brane is integrated to give

$$\rho = \frac{\rho_0}{a^{3\gamma}}, \tag{12}$$

where a constant $\rho_0$ is assumed to be positive. Because $\gamma = 0$ is equivalent to the cosmological constant, we assume $0 < \gamma \leq 2$ which satisfies the dominant energy condition.

For $\alpha > 0$, Eq. (11) is solved to give the following modified Friedmann equation on the brane:

$$H^2 = V(a), \tag{13}$$

$$V(a) := \frac{1}{8\alpha}\left[-\frac{8k\alpha}{a^2} - 2 + \left\{A^{3/2} + 256\alpha^3P^2 + 16\sqrt{2\alpha^3P^2\left(128\alpha^3P^2 + A^{3/2}\right)}\right\}^{1/3} + A\left\{A^{3/2} + 256\alpha^3P^2 + 16\sqrt{2\alpha^3P^2\left(128\alpha^3P^2 + A^{3/2}\right)}\right\}^{-1/3}\right], \tag{14}$$

where

$$A := 1 + \frac{\alpha\mu}{a^4} + \frac{4}{3}\alpha\Lambda, \qquad P^2 := \frac{\kappa_5^4}{256\alpha^2}\left(\frac{\rho_0}{a^{3\gamma}} + \sigma\right)^2. \tag{15}$$

Here $V(a)$ denotes the effective potential and the region with $V(a) > 0$ is the allowed region for dynamics. Now we have a one-dimensional potential problem with one dynamical degree of freedom $a(\tau)$ and the qualitative behavior of $a$ is completely understood by the form of the potential $V$.

## 2 Bouncing Gauss-Bonnet braneworld

The bouncing solution is characterized by the transition from the contracting phase ($\dot{a} < 0$) to the expanding one ($\dot{a} > 0$) of the universe. The bounce in a conventional sense occurs at the lower bound of this domain $a = a_{\mathrm{B}}$ satisfying $V(a_{\mathrm{B}}) = 0$. The evolution of the contracting universe in the domain $a > a_{\mathrm{B}}$ momentarily stops at $a = a_{\mathrm{B}}$ and then starts to expand in the domain $a > a_{\mathrm{B}}$. Therefore, the sufficient condition for this conventional bounce is $V(0) < 0$.

It is shown that the conventional bounce may occur in some cases in the Gauss-Bonnet braneworld with $\mu > 0$. In contrast, the situation in the Gauss-Bonnet braneworld with $\mu < 0$ is very different. The primary reason is that there is a branch singularity in the bulk and $a < r_{\mathrm{b}}$ is not in the physical domain of $a$. We will see what happens when the brane hits the branch singularity in the bulk.

## 2.1 Novel bounce for $\mu < 0$

For $\mu < 0$, $a < r_{\rm b}$ is not in the physical domain of $a$ and so we focus on the behavior of the potential around $a = r_{\rm b}$. If the potential is non-negative, the brane reaches $a = r_{\rm b}$. If the potential is negative near $a = r_{\rm b}$, the bounce (in the conventional sense) occurs or a singularity appears at some $a > r_{\rm b}$.

First we present the condition that the brane hits the branch singularity in the bulk spacetime. The behavior of $V(a)$ around $a = r_{\rm b}$ is given by

$$V(a) \simeq V(r_{\rm b}) + O(a - r_{\rm b}), \tag{16}$$

where

$$V(r_{\rm b}) = -\frac{k}{r_{\rm b}^2} - \frac{1}{4\alpha} + P(r_{\rm b})^{2/3}. \tag{17}$$

Therefore, the brane reaches the branch singularity if $V(r_{\rm b}) \geq 0$. This condition is satisfied if

$$h(r_{\rm b}) = k + \frac{r_{\rm b}^2}{4\alpha} \leq 0. \tag{18}$$

Therefore, if the branch singularity is spacelike, which is realized only for $k = -1$, the brane reaches there. If the branch singularity is timelike, the condition $V(r_{\rm b}) \geq 0$ gives

$$\sigma \geq \frac{16\alpha}{\kappa_5^2}\left(\frac{k}{r_{\rm b}^2} + \frac{1}{4\alpha}\right)^{3/2} - \frac{\rho_0}{r_{\rm b}^{3\gamma}}. \tag{19}$$

It is noted that this condition is not so sensitive about $\gamma$ and realized even with positive pressure ($\gamma > 1$). On the other hand, if

$$\sigma < \frac{16\alpha}{\kappa_5^2}\left(\frac{k}{r_{\rm b}^2} + \frac{1}{4\alpha}\right)^{3/2} - \frac{\rho_0}{r_{\rm b}^{3\gamma}} \tag{20}$$

is satisfied, $a = r_{\rm b}$ is not in the physical domains of $a$ and hence the conventional bounce occurs or a singularity appears at some $a > r_{\rm b}$ instead.

We have seen the condition under which the brane hits the branch singularity in the bulk. Let us see what happens then. Now the important fact is that the curvature invariants on the brane do *not* blow up even when the brane approaches the branch singularity in the bulk. In fact, the behavior of the scale factor $a(\tau)$ near the branch singularity is obtained as

$$a(\tau) \simeq r_{\rm b} + a_1(\tau - \tau_{\rm b}) + O((\tau - \tau_{\rm b})^2), \tag{21}$$

$$a_1^2 := r_{\rm b}^2 V(r_{\rm b}), \tag{22}$$

where $\tau_{\rm b}$ is the cosmological time on the brane to reach the branch singularity. This is the Taylor series around $\tau = \tau_{\rm b}$ and hence the curvature invariants are all finite around there. However, because the allowed domain of $a$ is $a \geq r_{\rm b}$, we must take the minus and plus signs of $a_1$ for $\tau < \tau_{\rm b}$ and $\tau > \tau_{\rm b}$, respectively. As a result, the evolution near $\tau = \tau_{\rm b}$ represents the transition from the collapsing phase ($\tau < \tau_{\rm b}$) to the expanding phase ($\tau > \tau_{\rm b}$). Thus, the bouncing universe is realized on the brane.

From the bulk point of view, this process is that the brane reaches the branch singularity and passes across it. It is emphasized that the spacetime on the brane is not free of singularities then, but there appears just a shell-type instantaneous singularity. In the generic case, the derivative of $a(\tau)$ (velocity) is not continuous and the metric on the brane is $C^0$ at $\tau = \tau_{\rm b}$. The junction condition on the brane then shows that there is a matter distribution on the spacelike hypersurface $\tau = \tau_{\rm b}$ on the brane. This means that a shell-type singularity appears instantaneously on the brane at $\tau = \tau_{\rm b}$ but it is rather harmless since it stems from the thin-shell approximation of the brane as well as the branch singularity. With a fine-tuning giving $a_1 \equiv 0$, in contrast, the metric on the brane becomes analytic around $\tau = \tau_{\rm b}$ and there is no shell-type singularity.

In summary, our claim is that the brane can reach the branch singularity and pass through it safely. In order to support this claim, we will show in the next subsections that the branch singularity is indeed harmless for a finite body moving radially and can be considered as a massive thin shell. From the bulk point of view, this novel bouncing process is the collision of the three-brane with the shell of branch singularity.

### 2.2 Branch singularity is weak

In this subsection, we show that the branch singularity is a weak singularity. The physical consequence of this property is that a finite body may reach there safely.

There are several definitions of the strength of a singularity. We first present the definition by Tipler [8]. Let $\bar{\gamma} : [\lambda_0, \lambda_s) \to M$ be an affinely parametrized causal geodesic which approaches a singularity as $\lambda \to \lambda_s^-$, where $\lambda$ is an affine parameter. Define $J_{\lambda_1}(\bar{\gamma})$ for $\lambda_1 \in [\lambda_0, \lambda_s)$ to be a set of maps $Z_{(I)}$ : $[\lambda_0, \lambda_s) \to TM$ ($TM$ means the tangent bundle and $I = 1, 2, 3, 4$ ($I = 1, 2, 3$) for timelike (null) $\bar{\gamma}$) satisfying the following four:

$$Z^\mu_{(I)}(\lambda) \in T_{\bar{\gamma}(\lambda)}M, \tag{23}$$

$$Z^\mu_{(I)}(\lambda_1) = 0, \tag{24}$$

$$\ddot{Z}^\mu_{(I)} = -R^\mu{}_{\nu\rho\sigma} Z^\rho_{(I)} k^\nu k^\sigma, \tag{25}$$

$$Z^\mu_{(I)} k_\mu = 0. \tag{26}$$

where $k^\mu$ is the tangent of $\bar{\gamma}$. Equation (25) is called the Jacobi equation (or geodesic deviation equation). Along a timelike geodesic, four independent Jacobi fields define a volume element $V(\lambda)$ along $\bar{\gamma}$ by the exterior product. Along a null geodesic, three such fields define an area element which we also denote $V(\lambda)$. A singularity is called *Tipler strong* if

$$\lim_{\lambda \to \lambda_s^-} \inf V(\lambda) = 0 \tag{27}$$

is satisfied for all $\lambda_1 \in [\lambda_0, \lambda_s)$ and all four (three) linearly independent Jacobi fields $Z \in J_{\lambda_1}(\bar{\gamma})$ [8]. The singularity is called Tipler weak if it is not Tipler strong. This definition of the Tipler strong singularity intuitively says that any object that hits a strong singularity is crushed to zero volume (area).

The above definition ignores the case where $V(\lambda)$ blows up in the approach to the singularity. Also, $V(\lambda)$ may remain finite overall when some of the elements of $J_{\lambda_1}(\bar{\gamma})$ blow up but some others converge to zero. In order to include such situations, Ori defined deformationally strong singularity [9]. A singularity is called *deformationally strong* if it is either (i) Tipler strong, or (ii) if there exists an element of $J_{\lambda_1}(\bar{\gamma})$ that has infinite norm for $\lambda \to \lambda_s^-$ for all $\lambda_1 \in [\lambda_0, \lambda_s)$ [9]. A singularity is called deformationally weak if it is not deformationally strong. A singularity is Tipler weak if it is deformationally weak. Actually, it is shown by direct calculations that the branch singularity is deformationally weak along radial causal geodesics [1].

### 2.3 $C^0$ extension of the bulk beyond the branch singularity

In the previous subsection, we have seen that the branch singularity is harmless for radially moving finite bodies, which strongly supports our main claim. Then, the subsequent natural question is what the bulk spacetime on the other side of the branch singularity is. In order to answer to this question, we consider the extension of the bulk spacetime beyond $r = r_b$.

It is seen in (5) that the metric and its inverse are finite at the branch singularity. This implies that the spacetime can be extended beyond $r = r_b$ in the $C^0$ sense. The $C^0$ extension is not unique in general; however, thanks to the Birkhoff's theorem in the system [7], there are only two candidates for the extended spacetime, namely the GR and non-GR branches of the vacuum solution (5). Among these two, the GR branch should be chosen because of its dynamical stability. By attaching another the vacuum solution (5) in the GR-branch with the same mass parameter at the position of the branch singularity, we construct the $C^0$-extended bulk spacetime.

Now let us study the branch singularity in this extended spacetime in more detail. The induced metric on $r = r_b$ is given by

$$ds_4^2 = \bar{g}_{ab} dy^a dy^b = -h(r_b) dt^2 + r_b^2 \left[ d\chi^2 + f_k(\chi)^2 (d\theta^2 + \sin^2\theta d\phi^2) \right]. \tag{28}$$

We know that the derivatives of $h$ are not finite at $r = r_b$. The component of the extrinsic curvature $K_{ab}$ of the hypersurface $r$ =constant diverges in the limit $r \to r_b$. Nevertheless, the junction condition provides a finite value of the energy-momentum tensor on $r = r_{\rm b}$. This implies that the branch singularity can be identified as a massive shell in the extended bulk spacetime. We consider the cases where $r = r_{\rm b}$ is timelike ($h(r_{\rm b}) > 0$) and spacelike ($h(r_{\rm b}) < 0$), separately. (A more careful treatment is required in the case of $h(r_{\rm b}) = 0$.)

First we consider the case where $r = r_{\rm b}$ is timelike. Then, we have

$$h(r_{\rm b}) = k + \frac{r_{\rm b}^2}{4\alpha}(> 0) \tag{29}$$

and the junction condition shows that $S^a{}_b$ on the hypersurface remains finite for $r \to r_{\rm b}$ as

$$\lim_{r\to r_{\rm b}} S^a{}_b = \mathrm{diag}(-\rho_{\rm b}, p_{\rm b}, p_{\rm b}, p_{\rm b}), \tag{30}$$

where

$$\rho_{\rm b} = -16\alpha\frac{h(r_{\rm b})^{3/2}}{r_{\rm b}^3}, \quad p_{\rm b} = \frac{8h(r_{\rm b})r_{\rm b}^4 - \mu}{2\sqrt{h(r_{\rm b})}r_{\rm b}^5}. \tag{31}$$

Thus, the branch singularity can be considered as a massive thin shell. Since $\rho_{\rm b} < 0$, the matter on $r = r_{\rm b}$ violates the weak energy condition.

Next we consider the case where $r = r_{\rm b}$ is spacelike. Then the junction condition gives the energy-momentum tensor $S^a{}_b$ on the spacelike hypersurface as

$$\lim_{r\to r_{\rm b}} S^a{}_b = \mathrm{diag}(P_{\rm b(r)}, P_{\rm b(t)}, P_{\rm b(t)}, P_{\rm b(t)}), \tag{32}$$

where

$$P_{\rm b(r)} = 16\alpha\frac{(-h(r_{\rm b}))^{3/2}}{r_{\rm b}^3}, \quad P_{\rm b(t)} = \frac{8h(r_{\rm b})r_{\rm b}^4 + \mu}{2\sqrt{-h(r_{\rm b})}r_{\rm b}^5}. \tag{33}$$

They are finite and hence the branch singularity may be considered as a massive spacelike thin shell. Since this is a spacelike shell, it is difficult to discuss the energy condition for the matter field there.

# 3 Summary and discussions

In this paper, we have presented a novel scenario for the bouncing universe in the five-dimensional braneworld in the framework of Einstein-Gauss-Bonnet gravity. In this scenario, the branch singularity located at the finite physical radius in the bulk, which appears for the negative mass parameter, plays an essential role. We have shown that the bulk spacetime is extended beyond the branch singularity in the $C^0$ sense and the branch singularity is identified as a massive thin shell. A three-brane may pass through the shell of branch singularity and then the bouncing universe is realized on the brane. This claim is strongly supported by the fact that the branch singularity is radially deformationally weak, which implies that the singularity is harmless for a finite body moving radially.

The present result opens a completely new possibility to achieve the bouncing brane universe as an effect of the higher-curvature terms. Our scenario is not sensitive about the equation of state for the matter on the brane and does work even with ordinary matter. Since the existence of the branch singularity stems from the quadratic nature of the theory, such a singularity is characteristic and must be quite generic in higher-curvature theories. In Lovelock higher-curvature gravity [10], which contains general relativity and Einstein-Gauss-Bonnet gravity as special cases, such a singularity appears in this class of vacuum solutions rather generically [11]. Interestingly, the central singularity is totally absent and the branch singularity is generic independent of the mass paraeter $\mu$ if there is a U(1) gauge field in the bulk spacetime in Einstein-Gauss-Bonnet gravity [12]. Undoubtedly, the effect of such singularities in cosmology is an interesting problem and should be investigated further.

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