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Global properties of solutions to the Einstein-matter equations

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Abstract

We investigate global properties of magnetic Gowdy spacetimes. First, a global in time existence theorem for the Einstein-Maxwell equations with Gowdy symmetry. Second, for polarized magnetic Gowdy case, asymptotically velocity terms dominated singular solutions are constructed by using the Fuchsian technique.

1 Introduction

1.1 Singularity theorems and two conjectures

Singularity theorems are the most fundamental and important theorems in classical general relativity, which are proved by Penrose and Hawking in 1960's [6].

Theorem 1 (Penrose) *Suppose the following conditions hold: (1) a Cauchy surface Σ is noncompact, (2) the null convergence condition, (3) Σ contains a closed trapped surface. Then the corresponding maxmal future development $D^+(\Sigma)$ is incomplete.*

Theorem 2 (Hawking) *Suppose the following conditions hold: (1) a Cauchy surface Σ is compact, (2) the timelike convergence condition, (3) the generic condition. Then the corresponding maxmal Cauchy development $D(\Sigma)$ is incomplete.*

These theorems say physically reasonable spacetimes have singularities in general. However, (1) the theorems do not say us *nature of singularity* and (2) *predictability* is breakdown if singularity can be seen. For problem (1), a conjecture has been proposed:

Conjecture 1 (Belinskii-Khalatnikov-Lifshitz (BKL) conjecture) *Solutions to the Einstein(-matter) equations should be Kasner-like ones near spacetime singularity.*

For (2), there should not exist naked singularity physically, that is,

Conjecture 2 (Strong cosmic censorship (SCC) conjecture) *Generic Cauchy data sets have maximal Cauchy developments which are locally inextendible as Lorentzian manifolds.*

To prove these conjectures, we need to show

- *global existence theorems* of solutions to the Einstein(-matter) equations in suitable coordinates,
- existence of Kasner-like solutions near spacetime singularity *in generic*,
- and *inextendibility* of spacetimes.

There are some results for these problems (see [2, 5, 10, 11]). To solve global problems for the Einstein(-matter) equations, some assumptions will be needed, because the equations are very complicated nonlinear PDEs and then we have less mathematical tools to analyze such equations. In this work, we assume

- Existence of some spacelike Killing vectors will be assumed (so called *Gowdy spacetimes*, which are the most simplest inhomogeneous ones including dynamical degree of freedom of gravity) and
- Maxwell field as matter will be assumed.

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2 Einstein-Maxwell system in Gowdy symmetric spacetimes

2.1 Einstein equations in the areal coordinate

The action we will consider is

$$S = \int d^4x\sqrt{-g}\left[-R + F^2\right]. \tag{1}$$

Varying this action, we have the Einstein-Maxwell equations as follows:

$$R_{\mu\nu} = 2F_{\mu\rho}F^{\rho}_{\nu} - \frac{1}{2}g_{\mu\nu}F^2, \tag{2}$$

$$\nabla_\mu(\tilde{F}^{\mu\nu}) = 0, \tag{3}$$

$$\nabla_\mu F^{\mu\nu} = 0, \tag{4}$$

where $F_{\mu\nu} := 2\nabla_{[\mu}A_{\nu]}$, $\tilde{F}_{\mu\nu} := \frac{1}{2}\sqrt{-g}\epsilon_{\mu\nu\delta\rho}F^{\delta\rho}$. The metric for Gowdy symmetric spacetimes, whose spacial topology is T^3 is given by

$$g = -e^{2(\eta-U)}\alpha dt^2 + e^{2(\eta-U)}d\theta^2 + e^{2U}(dx + Ady)^2 + e^{-2U}t^2dy^2, \tag{5}$$

where η, α, U and A are functions of $t \in (0,\infty)$ and $\theta \in T^1$. We call this *areal coordinate*. We assume that the field strength of the Maxwell field is

$$F = Bdx \wedge dy.$$

From the Maxwell equations, B should be a constant. Under these conditions, Einstein equations are as follows:
Constraint equations

$$\frac{\dot{\eta}}{t} = \dot{U}^2 + \alpha U'^2 + \frac{e^{4U}}{4t^2}(\dot{A}^2 + \alpha A'^2) + \frac{\alpha e^{2(\eta-U)}B^2}{t^4}, \tag{6}$$

$$\frac{\eta'}{t} = 2\dot{U}U' + \frac{e^{4U}}{2t^2}\dot{A}A' - \frac{\alpha'}{2t\alpha}, \tag{7}$$

$$\dot{\alpha} = -\frac{4\alpha^2 e^{2(\eta-U)}B^2}{t^3}. \tag{8}$$

Dot and prime denote derivative with respect to t and θ, respectively.
Evolution equations

$$\ddot{U} - \alpha U'' = -\frac{\dot{U}}{t} + \frac{\dot{\alpha}\dot{U}}{2\alpha} + \frac{\alpha' U'}{2} + \frac{e^{4U}}{2t^2}(\dot{A}^2 - \alpha A'^2) + \frac{\alpha e^{2(\eta-U)}B^2}{t^4}, \tag{9}$$

$$\ddot{A} - \alpha A'' = \frac{\dot{A}}{t} + \frac{\dot{\alpha}\dot{A}}{2\alpha} + \frac{\alpha' A'}{2} - 4(\dot{A}\dot{U} - \alpha A'U'), \tag{10}$$

We also use the following auxiliary equation for P

$$\ddot{P} - \alpha P'' = \left(-\frac{1}{t} + \frac{\dot{\alpha}}{2\alpha}\right)\dot{U} + \frac{\alpha' P'}{2} + e^{2P}(\dot{A}^2 - \alpha A'^2), \tag{11}$$

where $P = 2U - \ln t$. We call this system **magnetic Gowdy system** [13].

2.2 Wave map

The system of the evolution equations is equivalent with the following system of nonlinear wave equations (wave map $u : (\mathcal{M}^{2+1}, G) \mapsto (\mathcal{N}^2, h)$):

$$S_{\rm MG} = \int_{S^1} dt d\theta \sqrt{-G} \left(G^{\alpha\beta} h_{AB} \partial_\alpha u^A \partial_\beta u^B + \frac{\alpha e^{2(\eta-U)} B^2}{t^4} \right), \tag{12}$$

where

$$G = -dt^2 + \frac{1}{\alpha} d\theta^2 + t^2 d\psi^2, \qquad 0 \le \theta, \psi \le 2\pi,$$

and

$$h = dU^2 + \frac{e^{4U}}{4t^2} dA^2.$$

Every functions depend on time t and θ. Note that the evolution parts in the Einstein equations is obtained by varying this action. The energy-momentum tensor $T_{\alpha\beta}$ for this system is given of the form:

$$T_{\alpha\beta} = h_{AB} \left(\partial_\alpha u^A \partial_\beta u^B - \frac{1}{2} G_{\alpha\beta} \partial_\lambda u^A \partial^\lambda u^B \right) - G_{\alpha\beta} \frac{\alpha e^{2(\eta-U)} B^2}{2t^4}. \tag{13}$$

Now, we can define the energy is defined as follows:

$$\begin{aligned} E(t) &= \int_{S^1} T_{tt} \frac{d\theta}{\sqrt{\alpha}} \\ &= \frac{1}{2} \int_{S^1} \left[h_{AB} \left(\partial_t u^A \partial_t u^B + \partial_\theta u^A \partial_\theta u^B \right) + \frac{\alpha e^{2(\eta-U)} B^2}{t^4} \right] \frac{d\theta}{\sqrt{\alpha}} \\ &= \int_{S^1} \left[\mathcal{E} + \frac{\alpha e^{2(\eta-U)} B^2}{2t^4} \right] \frac{d\theta}{\sqrt{\alpha}}. \end{aligned}$$

2.3 Global existence

Theorem 3 (N.[9]) *Let (M, g) be the maximal Cauchy development of C^∞ initial data for the magnetic Gowdy system. Then, M can be covered by compact Cauchy surfaces of constant areal time t with each value in the range $(0, \infty)$.* ◇

To prove this theorem, monotonicity for the energy is shown and the light cone estimate is used (see [3, 7, 8, 12]).

3 Existence of Kasner-like solutions near spacetime singularity

Recently, a new Fuchsian technique has been develoved by Ames-Beyer-Isenberg-LeFloch [1] (see also [4]). By using this technique, we can construct Kasner-like solutions to the Einstein equations. For our problem, the leading-order term for Kasner-like solution:

$$U = \frac{1}{2} (1 - k(\theta)) \ln t + U_{**}(\theta) + \cdots = U_0 + \cdots, \tag{14}$$

$$A = A_*(\theta) + t^{2k(\theta)} A_{**}(\theta) + \cdots = A_0 + \cdots, \tag{15}$$

$$\eta = \frac{1}{4} (1 - k(\theta))^2 \ln t + \eta_*(\theta) + \cdots = \eta_0 + \cdots, \tag{16}$$

$$\alpha = \alpha_*(\theta) + \cdots = \alpha_0 + \cdots, \tag{17}$$

Then we have the following theorem, which means that there exists a Kasner-like solution to the Einstein-Maxwell equations in *polarized* magnetic Gowdy spacetimes.

Theorem 4 (N. [9]) *Suppose that an asymptotic data constant A_* and a set of asymptotic data functions $k, U_{**}, \alpha_*(>0), A_{**} \in C^\infty(T^1)$ which satisfy the integrability condition*

$$\int_0^{2\pi} \left((1-k(\theta))U'_{**}(\theta) - \frac{1}{2}(\ln\alpha_*)'(\theta)\right) d\theta = 0$$

with either

1. *$k(\theta) > 3$ for arbitrary $A_{**}(\theta)$, or*
2. *$k(\theta) > 3$ or $k(\theta) < -3$ for $A_{**}(\theta) \equiv 0$,*

at each point $\theta \in T^1$ Then, there is a $\delta > 0$, and a magnetic Gowdy solution U, A, η, α of the Einstein-maxwell equations of the form

$$(U, A, \eta, \alpha) = (U_0, A_0, \eta_0, \alpha_0) + W,$$

where the leading-order terms $(U_0, A_0, \eta_0, \alpha_0)$ is given by equations (16)-(15) and

$$\eta_*(\theta) := \eta_0 + \int_0^\theta \left((1-k(\theta))U'_{**}(\Theta) - \frac{1}{2}(\ln\alpha_*)'(\Theta)\right) d\Theta.$$

The remainder W is contained in $X_{\delta,\mu,\infty}$ for some exponent vector $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$ with $\mu_1, \mu_2 - 2k, \mu_3, \mu_4 > 0$. This solution is unique among all solutions with the same leading-order term and with the remainder $W \in X_{\delta,\mu,\infty}$.

3.1 Function spaces

In this section, we define some function spaces. Fix a smooth vector valued function $\mu : T^1 \to \mathrm{R}^\mathrm{d}$ with the corresponding diagonal matrix

$$\mathcal{R}[\mu](t,\theta) := diag(t^{-\mu_1(\theta)}, \cdots, t^{-\mu_d(\theta)}),$$

and then define the norm

$$\| w \|_{\delta,\mu,q} := \sup_{0<t\leq\delta} \| \mathcal{R}[\mu](t,\cdot)w(t,\cdot) \|_{H^q(T^1)},$$

for vecor-valued function $w(t,\theta) \in C^l((0,\delta], H^m(U))$. Here, $\| \cdot \|_{H^q(T^1)}$ denotes the standard q-order Sobolev norm on T^1. From this, we define the Banach space $X_{\delta,\mu,q}(T^1) = X_{\delta,\mu,q}$. We also define the space

$$X_{\delta,\mu,\infty} := \bigcap_{p=0}^{l} X_{\delta,\mu,q}.$$

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