Shinpei Kobayashi, JGRG 22(2012)111217

"Fuzzy objects in noncommutative geometry and their applications"

RESCEU SYMPOSIUM ON

GENERAL RELATIVITY AND GRAVITATION

JGRG 22

November 12-16 2012

Koshiba Hall, The University of Tokyo, Hongo, Tokyo, Japan

Fuzzy Objects in Noncommutative Geometry and Their Applications

Shinpei Kobayashi
(Gunma National College of Technology)

2012/11/12, JGRG22 @ RESCEU, University of Tokyo

Motivation

- Resolution of singularity
- String inspired
  - non-BPS D2/D0-system
  - 2-dimensional space with constant magnetic flux
- (an effective theory of) quantum gravity

Realization of Noncommutativity

- Noncommutativity between space coordinates

$$[x, y] = i\theta, \quad \theta$$: constant parameter

- a realization: Wick-Voros product

$$(f \star g)(z, \bar{z}) = \exp\left(\frac{\partial}{\partial \bar{z}'}\frac{\partial}{\partial z''}\right) f(z', \bar{z}')g(z'', \bar{z}'')\Big|_{z'=z''=z}$$

⇩

$$[z, \bar{z}] = z \star \bar{z} - \bar{z} \star z = 1 \qquad \left(z = \frac{x + iy}{\sqrt{2\theta}}, \ \bar{z} = \frac{x - iy}{\sqrt{2\theta}}\right)$$

Field Theory on noncommutative space

commutative:

$$S = \int dt d^2x \left(\partial_z \phi \partial_{\bar{z}} \phi + \frac{m}{2} \phi^2 + \cdots \right)$$

$$\Downarrow$$

noncommutative:

$$S = \int dt d^2x \left(\partial_z \phi \star \partial_{\bar{z}} \phi + \frac{m}{2} \phi \star \phi + \cdots \right)$$

Fuzzy Objects

Nontrivial Solutions
in Noncommutative Geometry

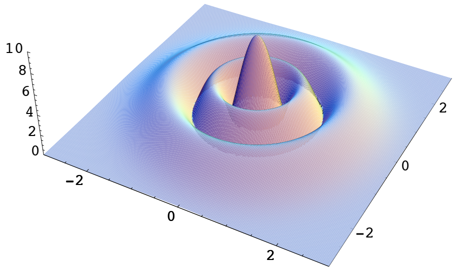
Noncommutative Solitons

- scalar field theory on NC plane: GMS solitons
  [Gopakumar- Minwalla- Strominger, Kraus-Larsen, …]

$$E = \int_D d^2 z V_\star(\Phi)$$

$$V_\star(\Phi) = \frac{b_2}{2}\Phi \star \Phi + \frac{b_3}{3}\Phi \star \Phi \star \Phi + \cdots$$

- circular symmetric, connection to D-branes



Weyl-Wigner Correspondence

function $f(\bar{z}, z) = \sum_{n=0}^{\infty} f_{mn}^{\text{Tay}} \bar{z}^m z^n$

Weyl projection

operator $\hat{f}(\hat{a}^\dagger, \hat{a}) = \sum_{n=0}^{\infty} f_{mn}^{\text{Tay}} \hat{a}^{\dagger m} \hat{a}^n$

Weyl-Wigner Correspondence

algebra

function: $f(x,y)$

star product: $f \star g$

$$[x,y] = i\theta$$

↔

algebra

operator: $\hat{f}(\hat{x},\hat{y})$

product
with an ordering: $\hat{f} \cdot \hat{g}$

$$[\hat{x},\hat{y}] = i\theta \quad [\hat{a},\hat{a}^\dagger] = 1$$

operators which act
on the Fock space
of a harmonic oscillator

Weyl-Wigner Correspondence

function $$f(\bar{z}, z) = \sum_{n=0}^{\infty} f_{mn}^{\mathrm{Tay}} \bar{z}^m z^n$$

Weyl projection

operator $$\hat{f}(\hat{a}^\dagger, \hat{a}) = \sum_{n=0}^{\infty} f_{mn}^{\mathrm{Tay}} \hat{a}^{\dagger m} \hat{a}^n$$

Weyl-Wigner Correspondence

function $f(\bar{z}, z) = \langle\, z \,|\, \hat{f} \,|\, z \,\rangle$

inverse Weyl projection

$\hat{a}\,|\, z \,\rangle = z \,|\, z \,\rangle$: coherent state

operator $\hat{f}(\hat{a}^\dagger, \hat{a}) = \sum_{n=0}^{\infty} f_{mn}^{\mathrm{Tay}} \hat{a}^{\dagger m} \hat{a}^n$

Weyl-Wigner Correspondence

In operator formalism:

$$\hat{N} \,|\, n \,\rangle = n \,|\, n \,\rangle$$

$$\hat{a} = \frac{\hat{x} + i\hat{y}}{\sqrt{2\theta}} \quad \hat{a}^\dagger = \frac{\hat{x} - i\hat{y}}{\sqrt{2\theta}} \quad \hat{N} = \hat{a}^\dagger \hat{a} = \frac{\hat{x}^2 + \hat{y}^2}{2\theta}$$

$$\mathcal{H} = \mathrm{span}\{|\,0\,\rangle\,, |\,1\,\rangle\,, |\,2\,\rangle\,, \cdots\}$$

Projection operators as NC solitons

[GMS (2000)]

- EOM: $0 = \frac{\partial V_\star}{\partial \Phi} = b_2\Phi + b_3\Phi\star\Phi + b_4\Phi\star\Phi\star\Phi + \cdots$
- non-trivial sln: $\Phi = \lambda_* p_n(z,\bar{z})$ with $p_n^2(z,\bar{z}) = p_n(z,\bar{z})$

⇒ projection operator $\hat{p}_n = |\, n\,\rangle\langle\, n\,|$

⇒ $p_n(r) = \langle\, z\,|\, n\,\rangle\langle\, n\,|\, z\,\rangle = e^{-\frac{r^2}{2\theta}}\frac{r^{2n}}{n!(2\theta)^n}$:circular symmetric

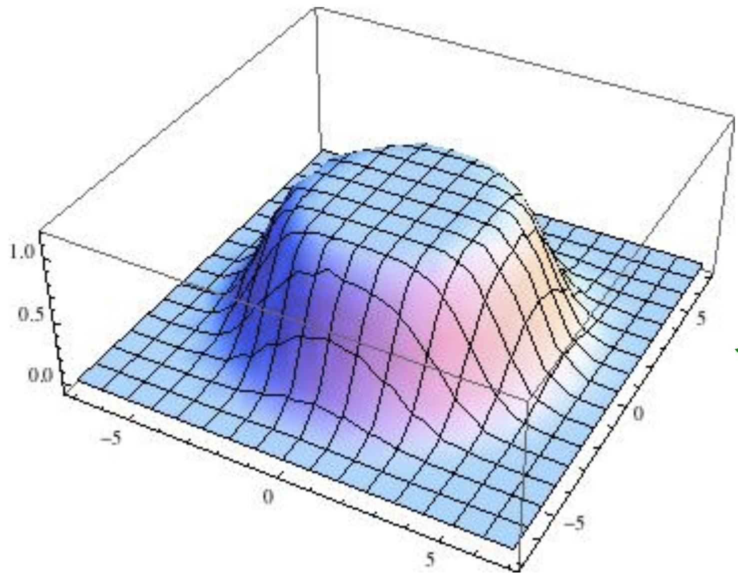
Fuzzy Disc [Lizzi, Vitale, Zampini (2003)]

- Def.: finite dim. truncation of a noncommutative plane

$$\mathcal{H}_N = \mathrm{span}\{|\,0\,\rangle\,,|\,1\,\rangle\,,|\,2\,\rangle\,,\cdots\,,|\,N-1\,\rangle\}$$

- Two parameters:
  - noncommutativity: θ
  - fuzzyness : N
- applications:
  - matrix model
  - quantum Hall effect

A Fuzzy Disc (N=10, θ =1)



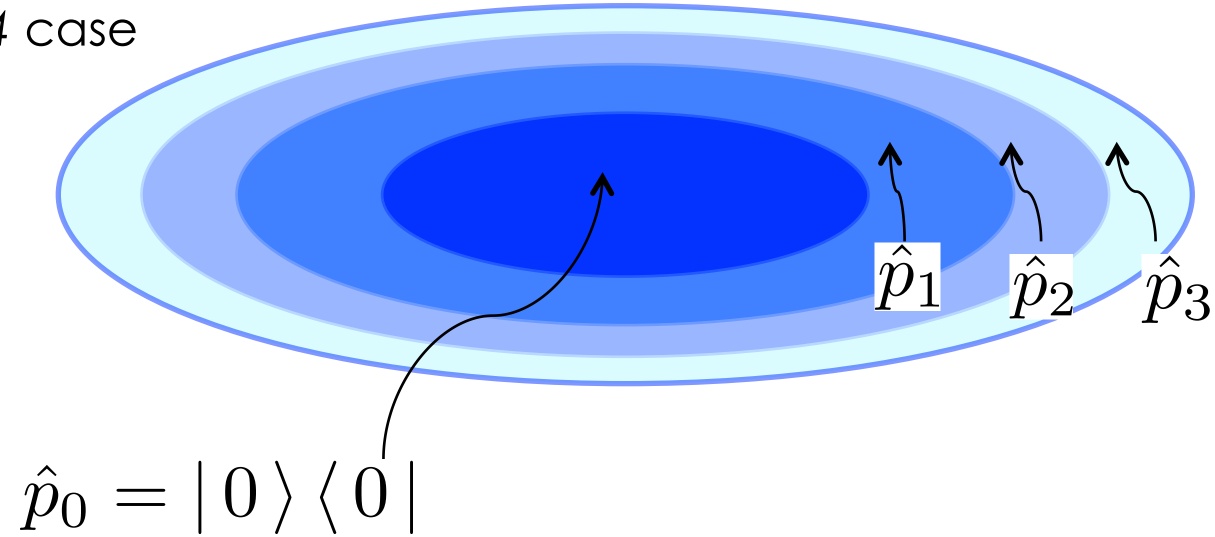
$$\hat{\mathcal{A}}_{10} = \hat{P}_{10}\hat{\mathcal{A}}\hat{P}_{10}$$

$$\hat{P}_{10} = \hat{p}_0 + \hat{p}_1 + \cdots + \hat{p}_9$$

$$p_0 + p_1 + \cdots + p_9 = \sum_{n=0}^{9} e^{-\frac{r^2}{2\theta}} \frac{r^{2n}}{n!(2\theta)^n}$$

GMS solitons and fuzzy disc

N=4 case



Another orthonormal basis : angle states?

- number basis: concentric cutting of space
  - $\hat{N}$ ~ radius $(\hat{N} \sim \sqrt{\hat{x}^2 + \hat{y}^2})$
- another basis: radial cutting of space
  - $\hat{\varphi}$ ~ angle

Angles in the fuzzy disc

SK-Asakawa, arXiv:1206.6602

Angle Operator and States

- The angle operator: $\hat{\varphi} = \sum_{m=0}^{N-1} \varphi_m \,|\, \varphi_m \,\rangle \langle\, \varphi_m \,|$

with help of Pegg-Barnett phase operator

- Eigen states of the angle operator: $\hat{\varphi} \,|\, \varphi_m \,\rangle = \varphi_m \,|\, \varphi_m \,\rangle$
- Relation to the number state $|\, \varphi_m \,\rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{in\varphi_m} \,|\, n \,\rangle$
- Orthonormality: $\langle\, \varphi_m \,|\, \varphi_n \,\rangle = \delta_{mn}$
- Angular projection operators:

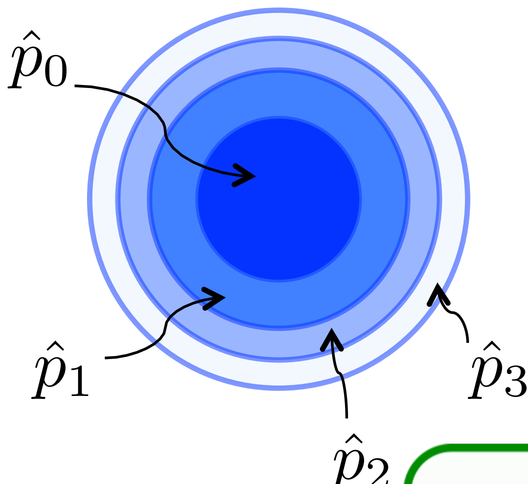
$$\hat{\pi}_m = |\, \varphi_m \,\rangle \langle\, \varphi_m \,|$$

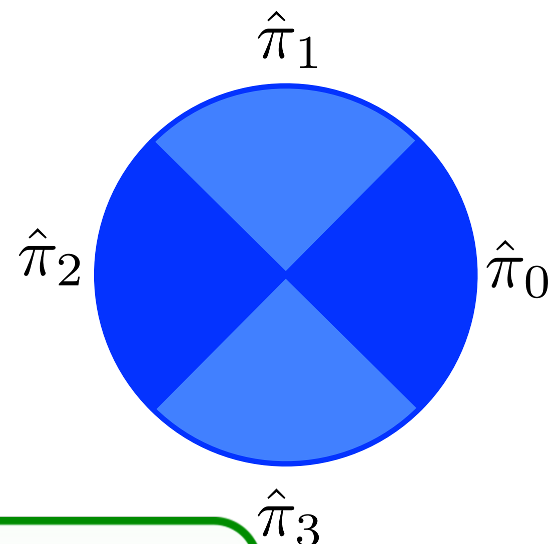
→ angular "delta function" peaked at $\varphi_m = \frac{2\pi}{N} m$

Two descriptions of fuzzy disc

- baum-kuchen vs shortcake

$$N = 4,\ \mathcal{H}_4 = \{|\,0\,\rangle\,, |\,1\,\rangle\,, |\,2\,\rangle\,, |\,3\,\rangle\}$$

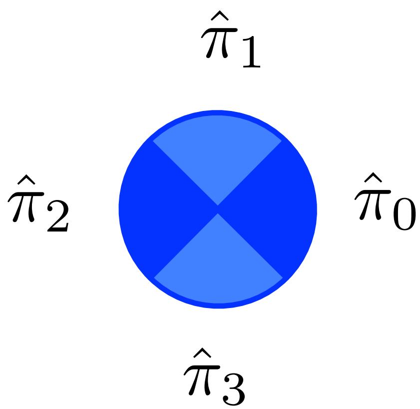




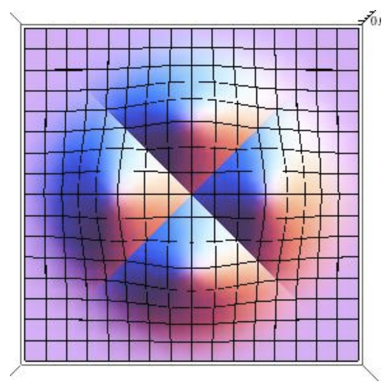
$$\sum_{n=0}^{N-1} \hat{p}_n = \sum_{m=0}^{N-1} \hat{\pi}_m$$

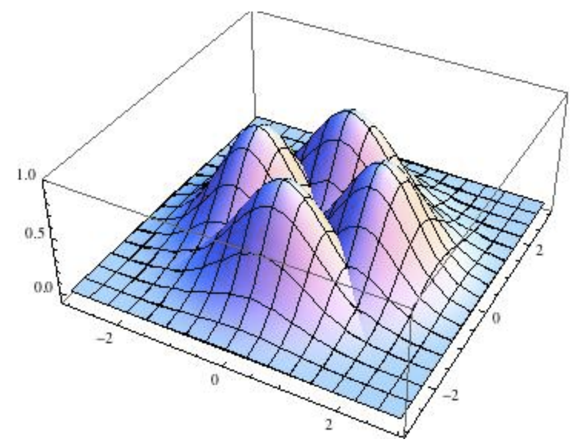
angular projection operators

$$\pi_k^{(N)}(r,\varphi) = \frac{1}{N}\sum_{m,n=0}^{N-1} e^{-\frac{r^2}{2\theta}} \frac{r^{m+n}}{\sqrt{m!n!(2\theta)^{m+n}}} e^{-i(m-n)(\varphi-\varphi_k)}$$



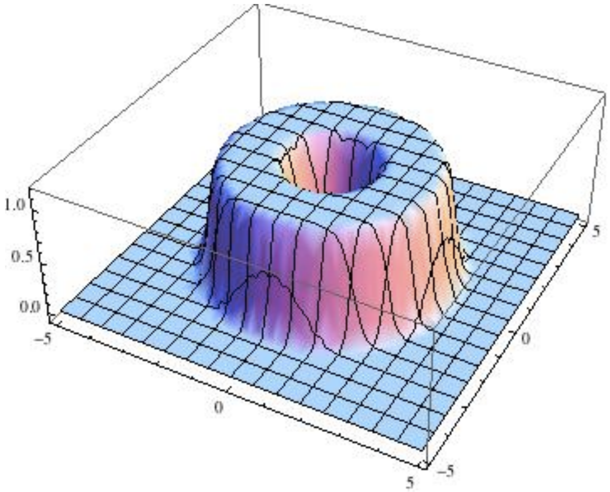
N=4 case





not concentric, but fan-shaped, like pieces of cake

Other fuzzy objects: e.g.) fuzzy Annulus

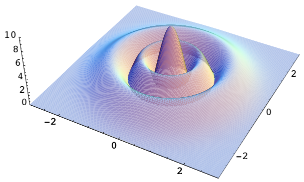


$$\hat{P}_N^M := \hat{p}_M + \hat{p}_{M+1} + \cdots + \hat{p}_{M+N-1}$$

any set of N orthonormal operators is allowed for truncation

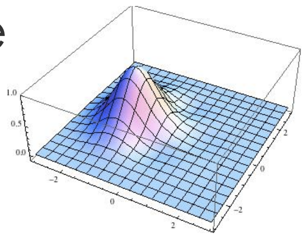
Noncommutative Solitons as D0-branes

- scalar field on the NC plane
  = tachyon filed on a non-BPS D2-brane



- The solution $\Phi = \lambda_* \hat{p}_n$
  = a D0-brane (rank $\hat{p}_n = 1 \rightarrow$ same tension)

- Same thing can be said to our case:
  the solution $\Phi = \lambda_* \hat{\pi}_m$ also can be seen as a D0-brane
  with very different shape (fan-shaped)



- Commutative limit (with Nθ fixed),
  angular NC soliton becomes thinner and thinner
  (A D0-brane is twisted into a string!)

Angular NC Solitons in Gravity

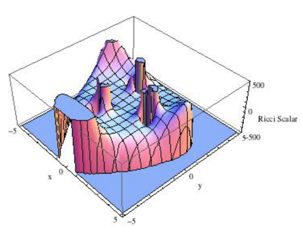
$$S = -\frac{\Lambda}{\kappa^2}\int dt d^2z \; E^\star \qquad E^\star = \det_\star E = \frac{1}{3!}\epsilon^{\mu\nu\rho}\epsilon_{abc}E^a_\mu \star E^b_\nu \star E^c_\rho$$

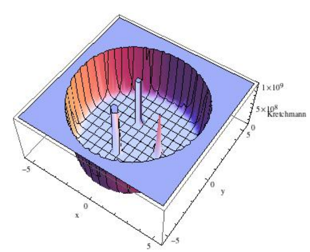
$$\epsilon^{\mu\nu\rho}\epsilon_{abc}\{E^b_\nu, E^c_\rho\}_\star = 0$$

$$E^a_\mu = \begin{pmatrix} E^0_0 & 0 & 0 \\ 0 & E^1_1 & 0 \\ 0 & 0 & E^2_2 \end{pmatrix} = \begin{pmatrix} \alpha_0\pi_0^{(N)} & 0 & 0 \\ 0 & \alpha_1\pi_1^{(N)} & 0 \\ 0 & 0 & \alpha_2\pi_2^{(N)} \end{pmatrix}$$

$$ds^2 = -\alpha_0^2\pi_0^{(3)}dt^2 + \alpha_1^2\pi_1^{(3)}dx^2 + \alpha_2^2\pi_2^{(3)}dy^2$$

$$\pi_k^{(3)}(r,\varphi) = \frac{1}{3}e^{-r^2/\theta}\left[1 + \frac{2r}{\theta^{1/2}}\cos(\varphi - \varphi_k^{(3)}) + \frac{r^2}{\theta}\left\{1 + \sqrt{2}\cos[2(\varphi - \varphi_k^{(3)})]\right\} + \frac{\sqrt{2}r^3}{\theta^{3/2}}\cos(\varphi - \varphi_k^{(3)}) + \frac{r^4}{\theta^2}\right]$$

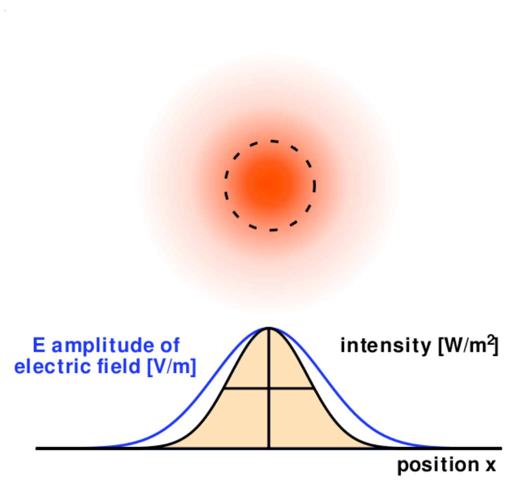


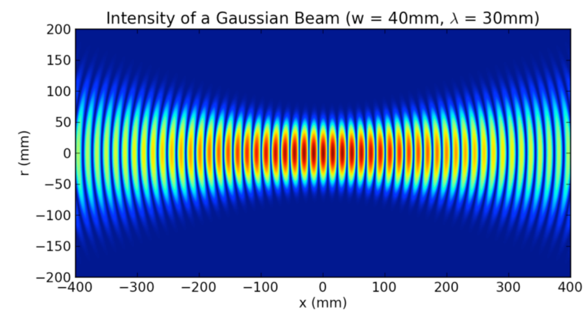


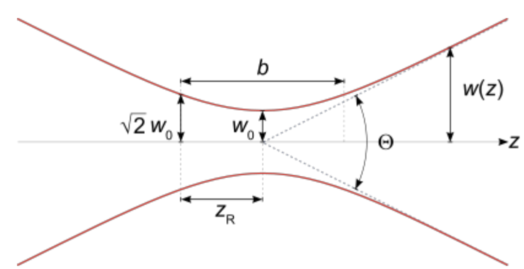
Experiment with laser

- Gaussian beam

$$I(r,z) = \frac{|E(r,z)|^2}{2\eta} = I_0 \left(\frac{w_0}{w(z)} \right)^2 \exp\left(\frac{-2r^2}{w^2(z)} \right) \ ,$$



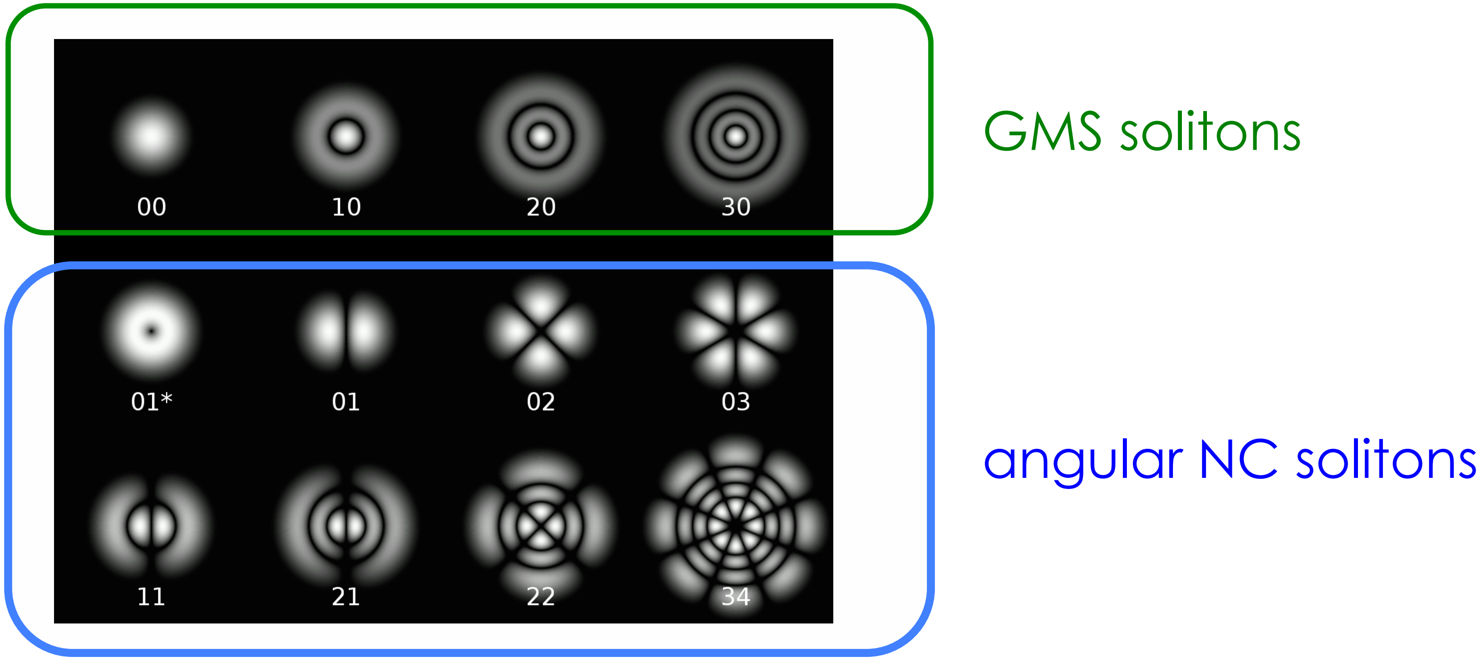




Experiment with laser

- Laguerre-Gaussian beam

$$u(r,\phi,z) = \frac{C_{lp}^{LG}}{w(z)}\left(\frac{r\sqrt{2}}{w(z)}\right)^{|l|}\exp\left(-\frac{r^2}{w^2(z)}\right)L_p^{|l|}\left(\frac{2r^2}{w^2(z)}\right)\exp\left(ik\frac{r^2}{2R(z)}\right)\exp(il\phi)\exp\left[-i(2p+|l|+1)\zeta(z)\right],$$



Noncommutative Solitons with Time-Dependence

Weyl-Wigner Correspondence

algebra

function: $f(x,y)$

star product: $f \star g$

$$[x,y] = i\theta$$

algebra

operator: $\hat{f}(\hat{x},\hat{y})$

product
with an ordering: $\hat{f} \cdot \hat{g}$

$$[\hat{x},\hat{y}] = i\theta \quad [\hat{a},\hat{a}^\dagger] = 1$$

operators which act

no restriction as long as $[\hat{x},\hat{y}] = i\theta$

of a harmonic oscillator

Noncommutative Solitons with Time-Dependence

- Time-dependent Harmonic Oscillator

e.g.,

$$\hat{H}(x,p,t) = \frac{\hat{p}^2}{2m(t)} + \frac{1}{2}m(t)\omega^2(t)\hat{x}^2$$

$\rightarrow$ we can solve this system analytically by the LR method

Lewis-Reisenfeld Method

- Time-dependent Schroedinger equation

$$i\hbar\frac{\partial}{\partial t}\psi = \hat{H}\psi$$

- Invariant operator $\hat{I}$

$$\frac{d\hat{I}}{dt} = \frac{\partial \hat{I}}{\partial t} + \frac{1}{i\hbar}[\hat{I}, \hat{H}] = 0$$

- Eigenvalue problem

$$\hat{I}\phi_n(x,p,t) = \lambda_n \phi_n(x,p,t)$$

$$\begin{cases} \psi(x,p,t) = e^{i\epsilon(t)}\phi(x,p,t) \\ \hbar\dot{\epsilon} = \langle\, \phi_n(t)\,|\left(i\hbar\frac{\partial}{\partial t} - \hat{H}\right)|\,\phi_n(t)\,\rangle \end{cases}$$

Time-Dependent NC Solitons

- general, quadratic, time-dependent Hamiltonian

[Choi-Gweon (2004)]

$\rightarrow$ apply to NC solitons

$$\hat{H}(\hat{x}, \hat{y}, t)$$
$$= A(t)\hat{x}^2 + B(t)(\hat{x}\hat{y} + \hat{y}\hat{x}) + C(t)\hat{y}^2 + D(t)\hat{x} + E(t)\hat{y} + F(t)$$

- creation and annihilation operator

$$\hat{a} = \sqrt{\frac{1}{\theta k^{1/2}}} \left\{ \left[\frac{\sqrt{k}}{2\rho} + i\frac{2B\rho - \dot{\rho}}{2A} \right] (\hat{x} - x_p(t)) + i\rho(\hat{y} - y_p(t)) \right\}$$
$$\hat{a}^\dagger = \sqrt{\frac{1}{\theta k^{1/2}}} \left\{ \left[\frac{\sqrt{k}}{2\rho} - i\frac{2B\rho - \dot{\rho}}{2A} \right] (\hat{x} - x_p(t)) - i\rho(\hat{y} - y_p(t)) \right\}$$

they satisfy $[\hat{a}, \hat{a}^\dagger] = 1$

Time-Dependent NC Solitons

- time-dependent circular symmetric solitons

[SK, in progress]

$$| \, \phi_n , t \, \rangle \, \langle \, \phi_n , t \, |$$

$$\Updownarrow$$

$$e^{-|\zeta(t)|^2} \frac{|\zeta(t)|^2}{n!}$$

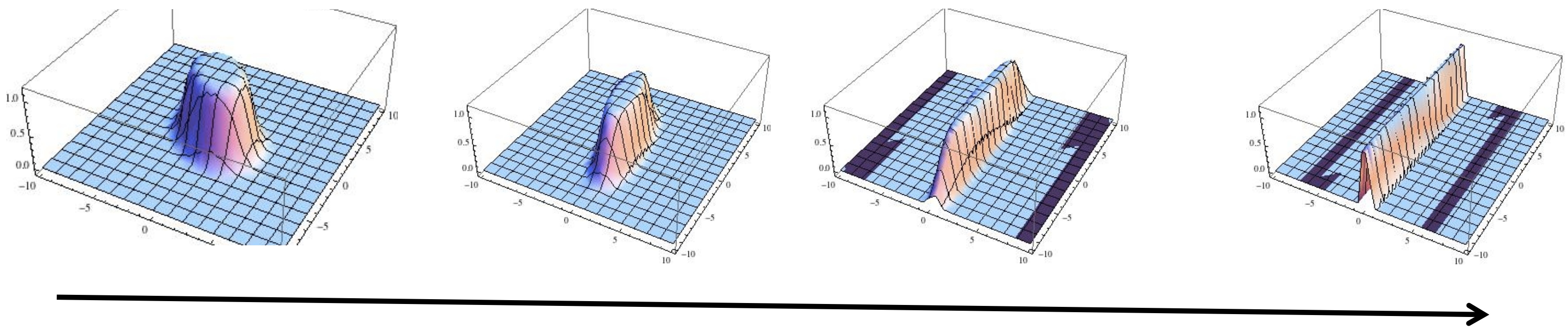
$$|\zeta(t)|^2 = \frac{1}{\theta k^{1/2}} \left\{ \frac{k}{4\rho^2} (x - x_p(t))^2 + \left[\left(\frac{2B\rho - \dot{\rho}}{2A} \right) (x - x_p(t)) + \rho \, (y - y_p(t)) \right]^2 \right\}$$

Time-Dependent NC Solitons

[SK, in progress]

e.g., Caldirola-Kanai oscillator

$$\hat{H} = e^{\gamma t}\hat{x}^2 + e^{-\gamma t}\hat{y}^2$$



$$t = 1/\gamma$$

Summary

- The fuzzy disc:
  - a disc-shaped, finite region in the NC plane
  - a fuzzy approximation by θ
- Introduction of angles to the fuzzy disc
  - angle projection operator and angle states
  - directly relates the boundary to the bulk
- Application
  - angular NC scalar solitons & fan-shaped D-branes
  - angular NC gravitational solitons
  - laser physics
- Time-dependent noncommutative solitons exist