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“Self-consistent initial conditions for primordial black hole”

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Self-consistent initial conditions for primordial black hole formation

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Collaboration with
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Introduction

- Large-amplitude inhomogeneities in the early universe collapse to form Primordial Black Holes.

- Some inflationary models predict large-amplitude curvature perturbation on small scales, leading to production of PBHs.

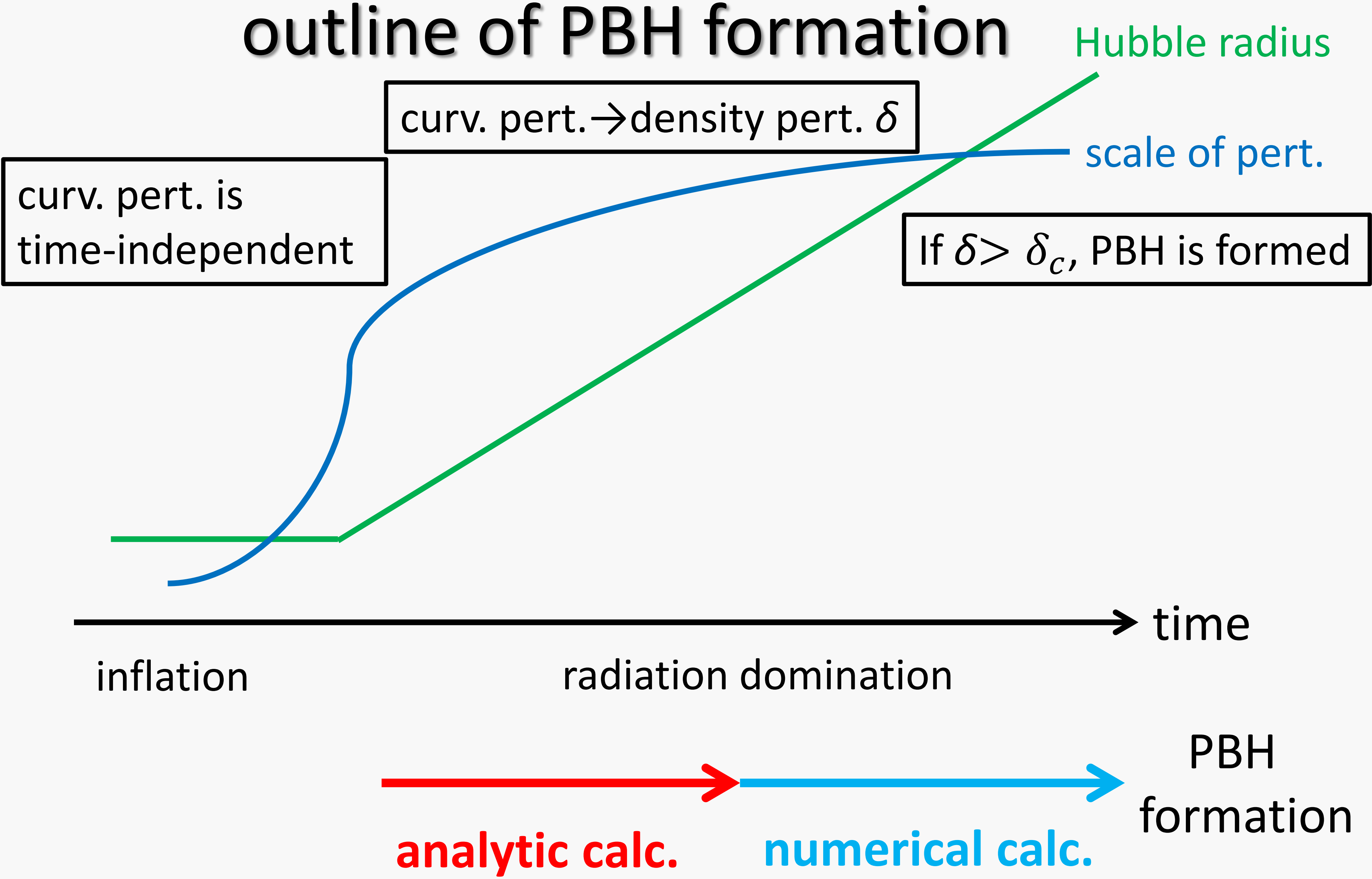
- CMB or LSS probes primordial perturbation on large scales, while information about that on small scales is scarce.

- Abundance of PBHs has been constrained by gravitational lensing, gravitational waves, etc…

- By investigating the condition for PBH formation in detail, PBH abundance can be correctly predicted assuming some inflationary model.

- Combined with observational data, the prediction can be used to probe primordial perturbation on small scales.

outline of PBH formation



Hubble radius

curv. pert.→density pert. δ

scale of pert.

curv. pert. is
time-independent

If $\delta > \delta_c$, PBH is formed

time

inflation

radiation domination

PBH
formation

analytic calc.

numerical calc.

- When $\epsilon \equiv \frac{\text{Hubble radius}}{\text{scale of pert.}} \to 0$, pert. is time-independent.

  So, initial curv. pert. profile is represented by $\boxed{K_i(r).}$

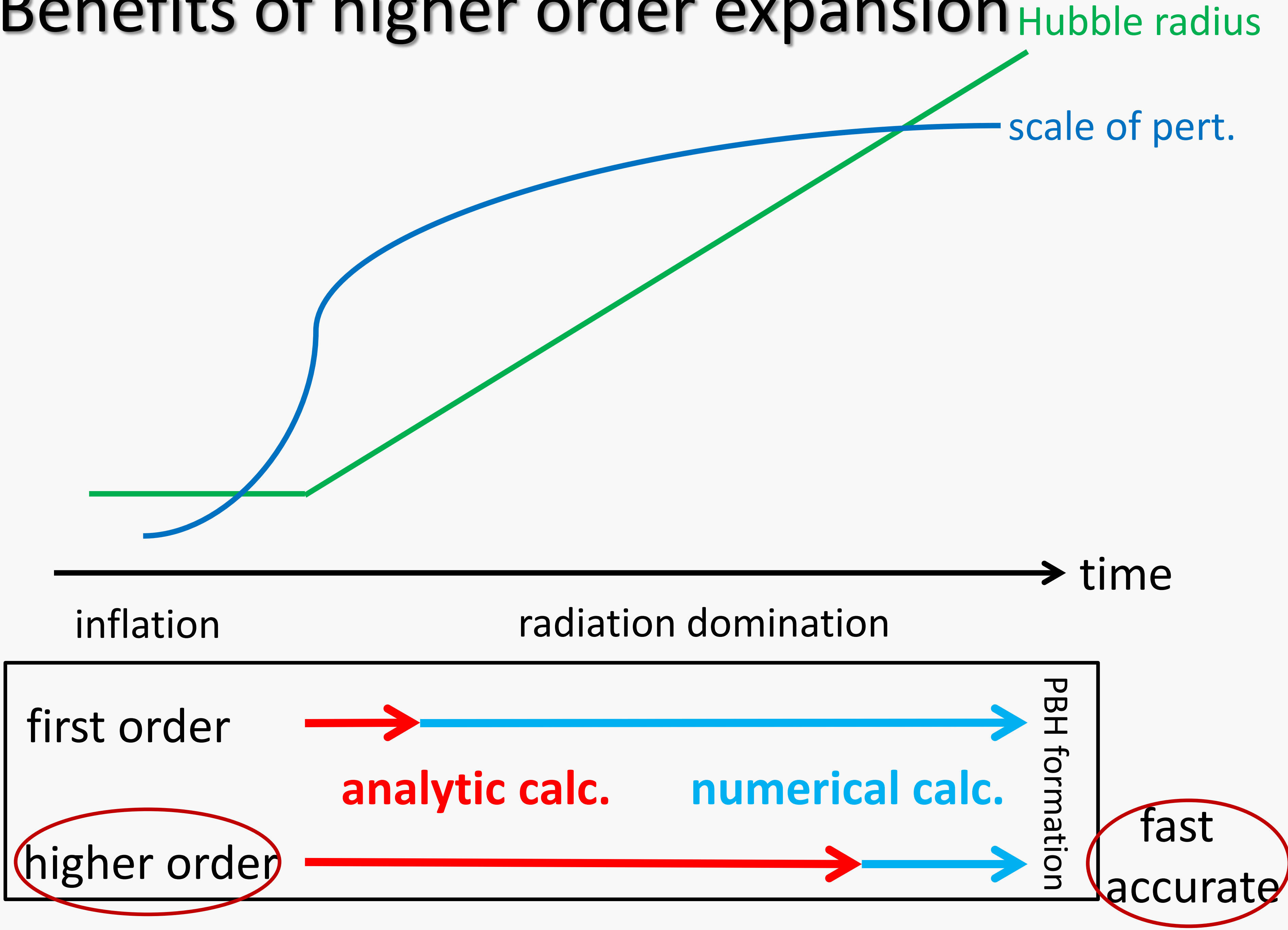
  When $\epsilon \ll 1$, the solution of the Einstein eqs.
  can be obtained by asymptotic expansion over ϵ.

- first order asymptotic expansion: Polnarev, Musco 2007

- We have obtained higher order expansion.

  Polnarev, Nakama, Yokoyama, JCAP09(2012)027

Benefits of higher order expansion



first order

analytic calc. **numerical calc.**

higher order

PBH formation

fast
accurate

$$ds^2 = -a^2dt^2 + b^2dr^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2),$$

- $u \equiv \dfrac{\dot{R}}{a},$ (definition of u)

Einstein eqs. (Misner&Sharp, 1964)

$$\left\{\begin{array}{l}
\dfrac{\dot{b}}{b} = \dfrac{au'}{R'}, \\[2ex]
\dfrac{a'}{a} = -\dfrac{\gamma}{1+\gamma}\dfrac{\rho'}{\rho}, \\[2ex]
M' = 4\pi\rho R^2 R'. \\[2ex]
\dot{M} = -4\pi p R^2 \dot{R}. \\[2ex]
\dfrac{R'^2}{b^2} = 1 + u^2 - \dfrac{2GM}{R}.
\end{array}\right.$$

- consider perturbed region surrounded by the flat FLRW univ.

$$ds^2 = -dt^2 + d(Sr)^2 + (Sr)^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)$$

- definition of curvature profile $K(t,r)$:

$$ds^2 = -a^2 dt^2 + \frac{dR^2}{1 - \boxed{K(t,r)} r^2} + R^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)$$

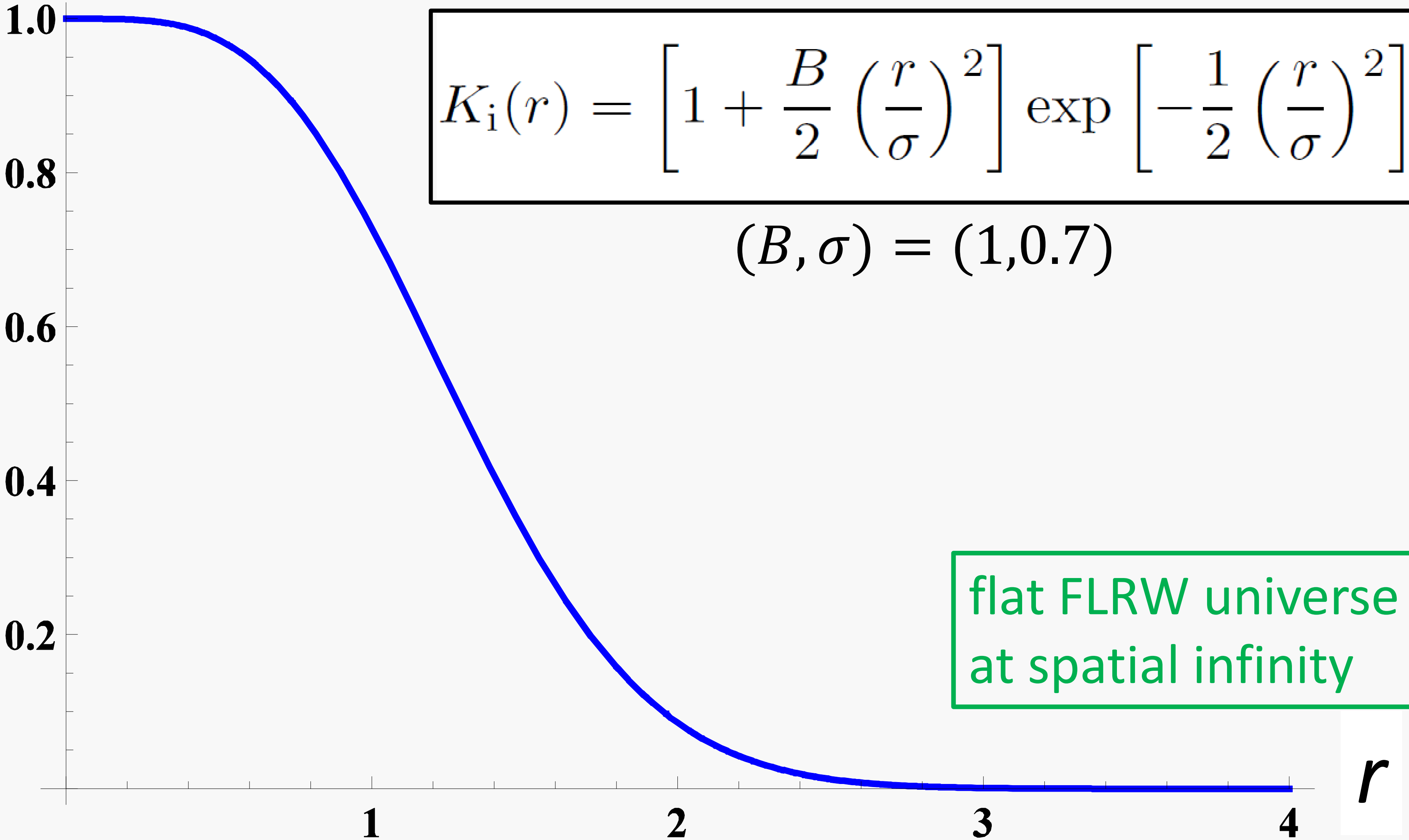
- initial curvature profile: $\boxed{K(0,r) \equiv K_i(r)} \left(< \frac{1}{r^2}\right)$

- boundary condition: $K(t,\infty) = 0$

The metric above coincides with the flat FLRW metric at spatial infinity.

example of initial curvature profile

central large curvature region



$$K_{\rm i}(r) = \left[1 + \frac{B}{2}\left(\frac{r}{\sigma}\right)^2\right] \exp\left[-\frac{1}{2}\left(\frac{r}{\sigma}\right)^2\right],$$

$(B, \sigma) = (1{,}0.7)$

flat FLRW universe
at spatial infinity

- decompose all the quantities

$$\boxed{X(t,r) = X_0(t,r)\tilde{X}(t,r)}$$

FLRW solution

deviation from FLRW solution

- expansion parameter : $\boxed{\epsilon \equiv \left(\frac{H_0{}^{-1}}{S(t)r_i}\right)^2} \propto t$ (R. D.)

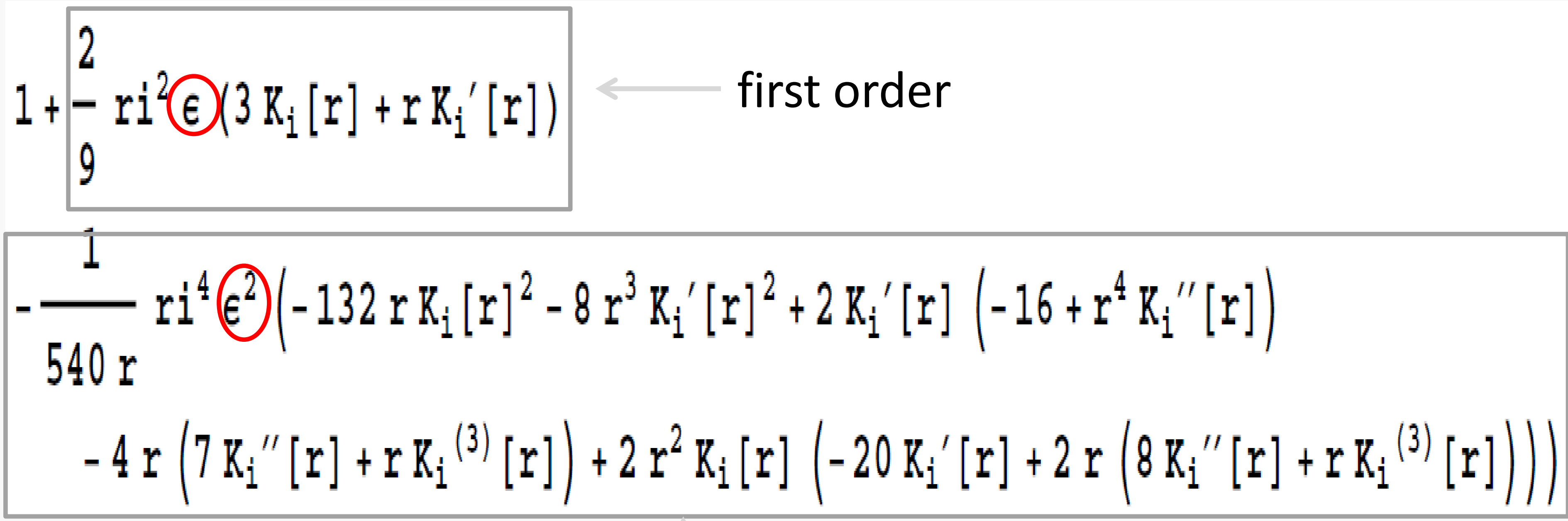
comoving radius of perturbed region

- Rewriting the Einstein eqs. in terms of $\tilde{X}$ and plugging

$$\boxed{\tilde{X}(t,r) = \sum_{n=0}^{\infty} \epsilon^n(t)\tilde{X}_{(n)}(r)}$$

recurrence relations to calculate $\tilde{X}_{(n)}$ are obtained.

$\tilde{\rho}(t,r)$ up to second order in ϵ



$$1+\boxed{\frac{2}{9}\,ri^{2}\,\epsilon\,(3\,K_i[r]+r\,K_i'[r])}\;\boxed{-\frac{1}{540\,r}\,ri^{4}\,\epsilon^{2}\Big(-132\,r\,K_i[r]^{2}-8\,r^{3}\,K_i'[r]^{2}+2\,K_i'[r]\,\big(-16+r^{4}\,K_i''[r]\big)-4\,r\,\big(7\,K_i''[r]+r\,K_i^{(3)}[r]\big)+2\,r^{2}\,K_i[r]\,\big(-20\,K_i'[r]+2\,r\,\big(8\,K_i''[r]+r\,K_i^{(3)}[r]\big)\big)\Big)}$$

first order

second order

accuracy of asymptotic expansion

$$\mathrm{ERR}\left(\sum_{n=0}^{N} \epsilon^n \tilde{X}_{(n)}\right) \equiv \tilde{X} - \sum_{n=0}^{N} \epsilon^n \tilde{X}_{(n)} \sim O(\epsilon^{N+1} \tilde{X}_{(N+1)}).$$

- Letting Δ be the required accuracy,

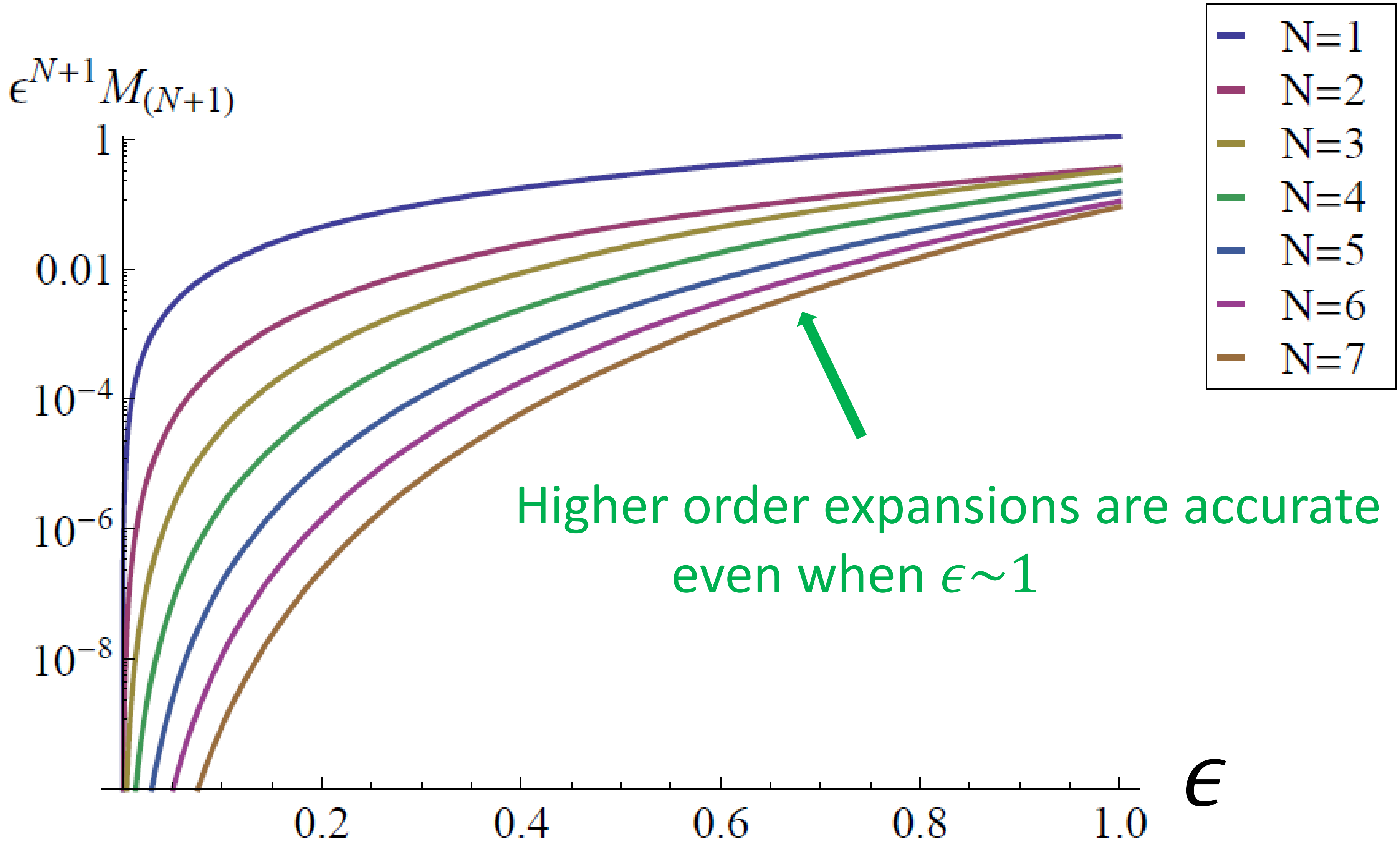
$$\boxed{\epsilon^{N+1} M_{(N+1)}} < \Delta,$$

↑

The maximum of $\epsilon^{N+1}\tilde{X}_{(N+1)}(r)$

Time dependence of errors

$(B, \sigma) = (1, 0.7)$



Higher order expansions are accurate even when $\epsilon \sim 1$

Letting Δ be the required accuracy,

$$\boxed{\epsilon^{N+1} M_{(N+1)}} < \Delta,$$

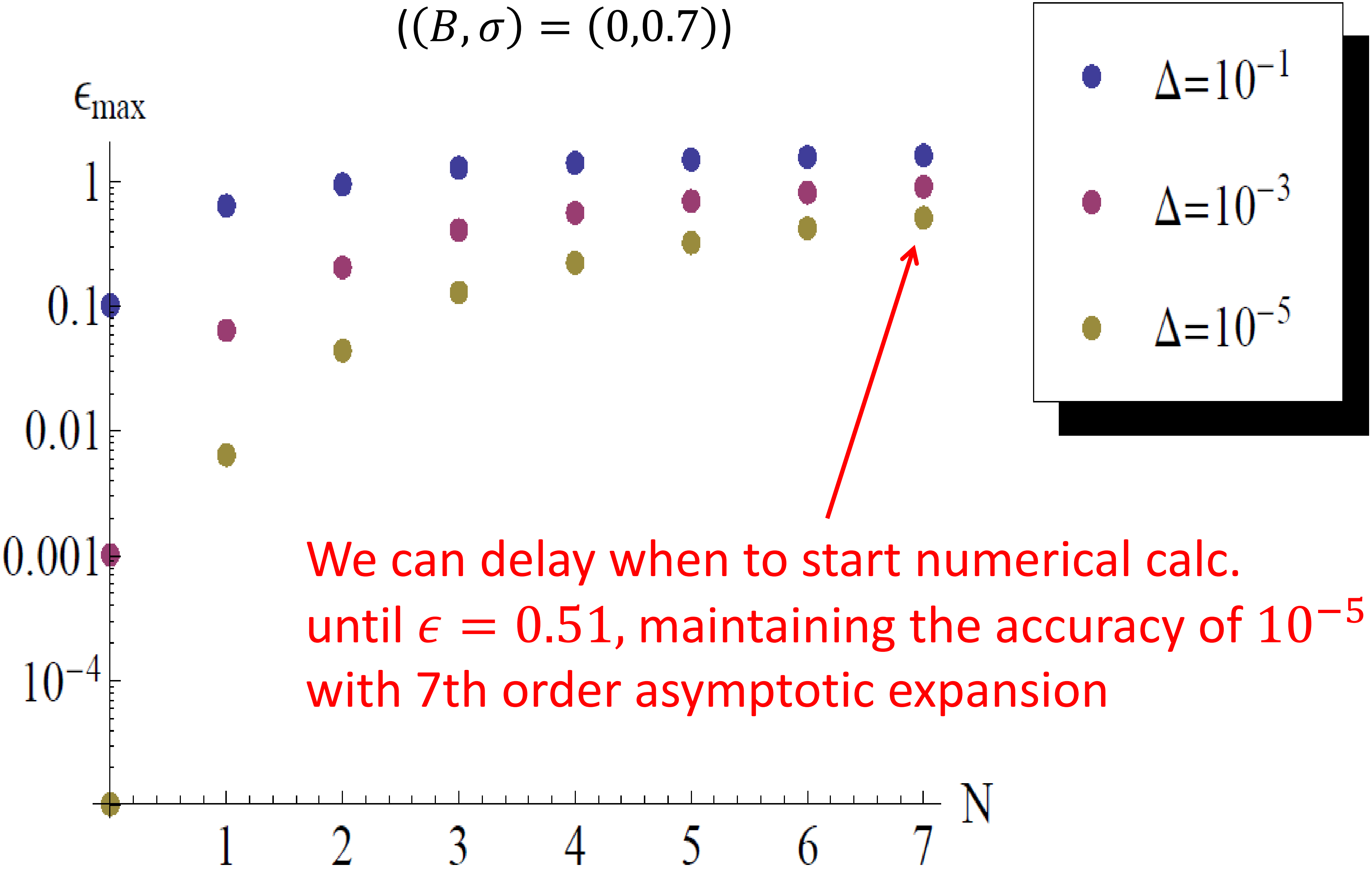
$\uparrow$

The maximum of $\epsilon^{N+1}\tilde{X}_{(N+1)}(r)$

The error is less than Δ if

$$\epsilon < \epsilon_{\max} \equiv \sqrt[N+1]{\frac{\Delta}{M_{(N+1)}}}.$$

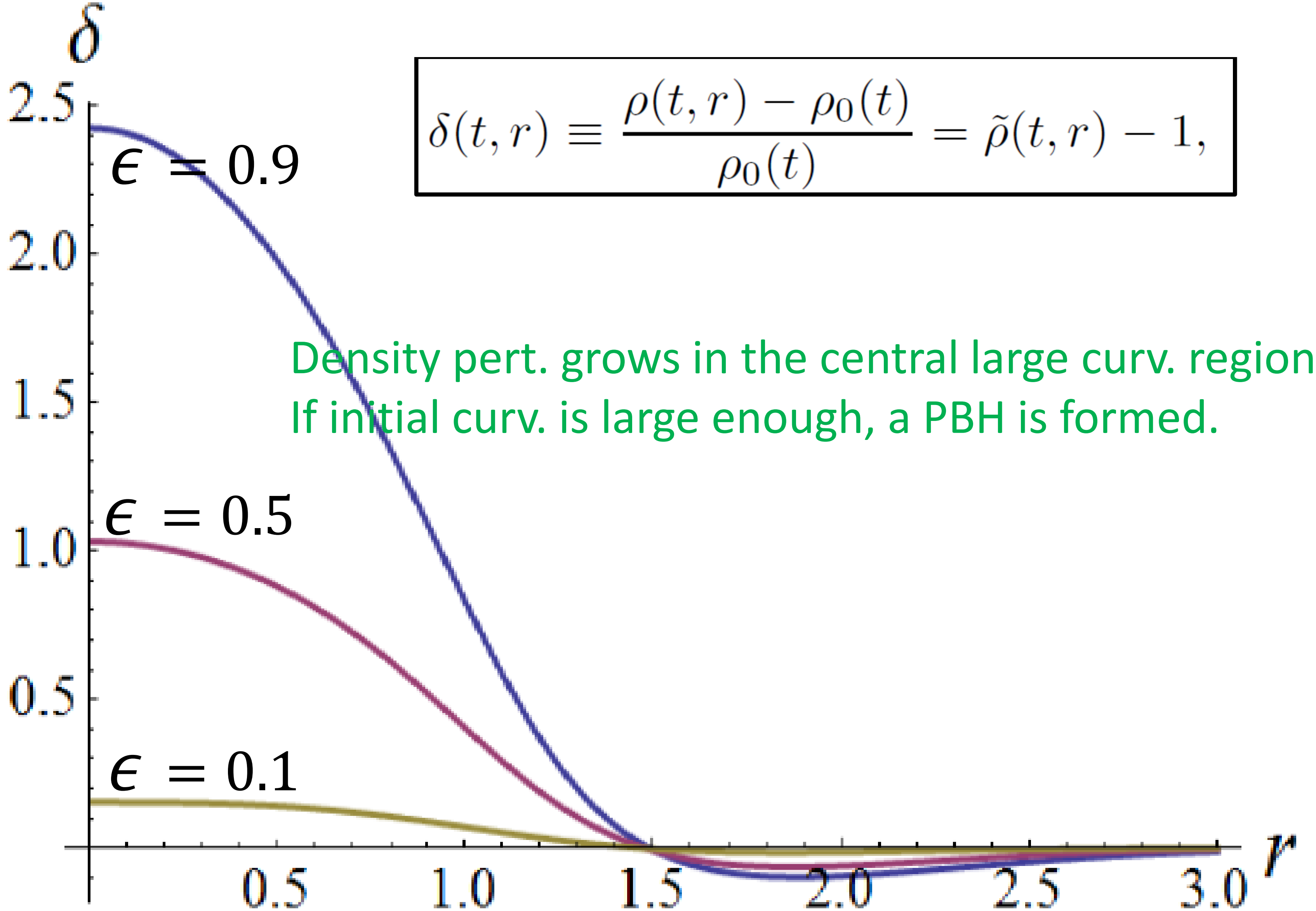
When to start numerical computation



We can delay when to start numerical calc. until $\epsilon = 0.51$, maintaining the accuracy of 10^{-5} with 7th order asymptotic expansion

time evolution of density pert. profile

((*B*,*σ*)=(1,0.7),7th order expansion is used)



$$\delta(t,r) \equiv \frac{\rho(t,r)-\rho_0(t)}{\rho_0(t)} = \tilde{\rho}(t,r)-1,$$

Density pert. grows in the central large curv. region.
If initial curv. is large enough, a PBH is formed.

summary

- Spherically symmetric large amplitude perturbation embedded in the flat FLRW universe is investigated.
- The solution of the Einstein eqs. is obtained by asymptotic expansion over ϵ.
- The solution is valid while the perturbed region is outside the horizon.
- Initial curvature profile, $K_i(r)$, generates the solution.
- Since the solution is accurate even when $\epsilon \sim 1$, we can delay the time when numerical computation is started.

work in progress

backups

- define $\widetilde{K}$, which describes time evolution of K

$$1 - K(t,r)r^2 = (1 - K_{\rm i}(r)r^2)\tilde{K}(t,r)$$

- $\widetilde{K}(t,\infty)$=1 since $K(t,\infty) = 0,\ K_i(\infty) = 0.$

- $\widetilde{K}(0,r)$=1 since $K(0,r) \equiv K_i(r).$

- $\widetilde{K} > 0$, so we can define $\widehat{K} \equiv \log\widetilde{K}$

The Einstein equations are rewritten in terms of $\tilde{X}$, $\hat{X}$.

$$\tilde{\mu} = \tilde{H}^2 + \epsilon \left[\frac{r_{\rm i}^2}{r^2} + e^{\hat{K}} \left(K_{\rm i} r_{\rm i}^2 - \frac{r_{\rm i}^2}{r^2} \right) \right] e^{-2\hat{R}}$$

$$\frac{\partial \tilde{\mu}}{\partial \xi} = 2(\tilde{\mu} - \tilde{\Phi} \tilde{f}),$$

$$\tilde{\rho} = \tilde{\mu} + \frac{1}{3} D_r \tilde{\mu},$$

$$\hat{a} = -\frac{\gamma}{1+\gamma} \hat{\rho},$$

$$\frac{\partial \hat{R}}{\partial \xi} = \frac{2}{3(1+\gamma)} (\tilde{\Phi} - 1),$$

$$\frac{\partial \hat{K}}{\partial \xi} = -\frac{4\gamma}{3(1+\gamma)^2} \tilde{\Phi} D_r \hat{\rho}.$$

- construct solution by asymptotic expansion

$$\tilde{X}(t,r) = \sum_{n=0}^{\infty} \in^{n}\ (t)\tilde{X}_{(n)}(r)$$

$$\hat{X}(t,r) = \sum_{n=0}^{\infty} \in^{n}\ (t)\hat{X}_{(n)}(r)$$

- by definition, $\tilde{X}_{(0)} = 1$, $\hat{X}_{(0)} = 0$.

- By plugging the expansions into the Einstein eqs, recurrence relations to calculate the expansion coefficients $\widetilde{X}_{(n)}$ are obtained.

- example of recurrence relations

$$\boxed{\tilde{\rho}_{(n)} = \tilde{\mu}_{(n)} + \frac{r}{3}\tilde{\mu}'_{(n)} + W_{3(n)},}$$

$$W_{3(n)} \equiv S_{(n)}[(\tilde{\mu} - \tilde{\rho})(r\tilde{R})'] + \frac{r}{3}S_{(n)}[\tilde{\mu}'\tilde{R}],$$

$$S_{(n)}[\tilde{X}_1\tilde{X}_2] \equiv \sum_{i=1}^{n-1} \tilde{X}_{1(i)}\tilde{X}_{2(n-i)}.$$

- By plugging these expansions into the Einstein equations, we can obtain recursive relations to calculate $\widetilde{X}_{(n)}$.
- We define

$$S_{(n)}[\tilde{X}_1\tilde{X}_2] \equiv \sum_{i=1}^{n-1} \tilde{X}_{1(i)}\tilde{X}_{2(n-i)}.$$

$$S^*_{(n)}[\tilde{X}F] = \sum_{m=1}^{n-1} m\tilde{X}_m \underset{\uparrow}{F}_{(n-m)}.$$

some function of $\tilde{X}$

- $S_{(n)}$, $S^*{}_{(n)}$ depend only on the expansion coefficients of up to n-1 order.

$$\boxed{\tilde{\mu}_{(n)} = \frac{1}{1+A_n}\left(F_{(n)} + W_{1(n)} - W_{2(n)}\right),}$$

$$A_n \equiv \frac{2}{1+\gamma}\left[\left(\gamma + \frac{1}{3}\right) n - \gamma\right],$$

$$F_{(n)} \equiv \delta_n^1 r_i^2 K_i - 2 r_\mathrm{i}^2 K_\mathrm{i} \hat{R}_{(n-1)} + r_\mathrm{i}^2 \left(K_\mathrm{i} - \frac{1}{r^2}\right) \hat{K}_{(n-1)},$$

$$\begin{aligned} W_{1(n)} \equiv\ & S_{(n)}[\tilde{H}\tilde{H}] + r_\mathrm{i}^2 \left(K_\mathrm{i} - \frac{1}{r^2}\right) S_{(n-1)}[e^{\hat{K}} e^{-2\hat{R}}] \\ & + \frac{1}{n-1}\left\{ r_\mathrm{i}^2 \left(K_\mathrm{i} - \frac{1}{r^2}\right) S^*_{(n-1)}[\hat{K} e^{\hat{K}}] - 2 r_\mathrm{i}^2 K_\mathrm{i} S^*_{(n-1)}[\hat{R} e^{-2\hat{R}}] \right\}. \end{aligned}$$

$$W_{2(n)} \equiv 2\left(S_{(n)}[\tilde{a}\tilde{H}] + S_{(n)}[\tilde{\Phi}\tilde{f}] + \frac{\gamma}{n(1+\gamma)} S^*_{(n)}[\hat{\rho}(\tilde{\rho} - \tilde{a})] \right).$$

$$\tilde{\rho}_{(n)} = \tilde{\mu}_{(n)} + \frac{r}{3}\tilde{\mu}'_{(n)} + W_{3(n)},$$

$$\tilde{H}_{(n)} = -\frac{1}{2(1+A_n)}[A_n(F_{(n)} + W_{1(n)}) + W_{2(n)}],$$

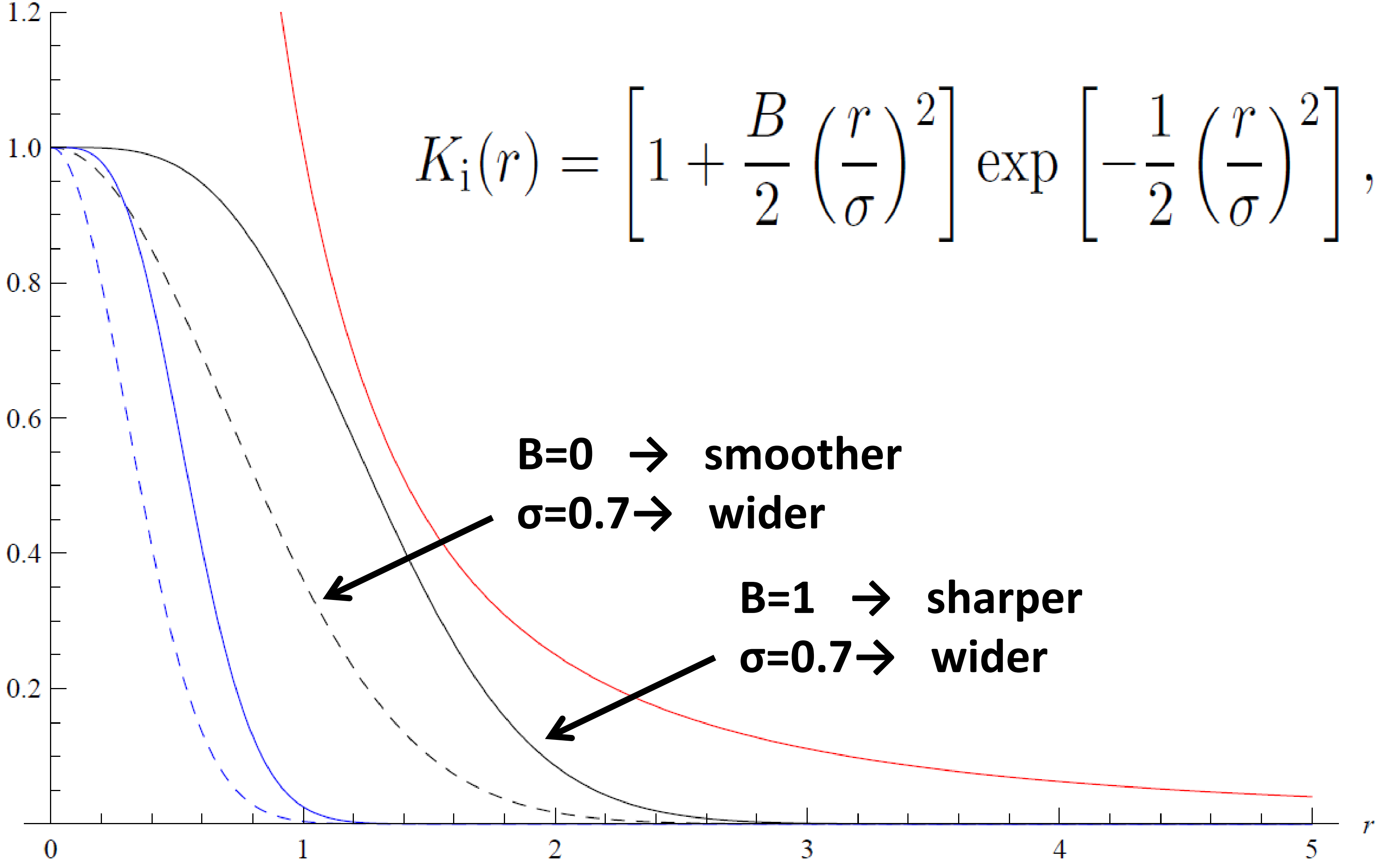
$$\tilde{a}_{(n)} = -\frac{\gamma}{1+\gamma}\left(\tilde{\mu}_{(n)} + \frac{r}{3}\tilde{\mu}'_{(n)}\right) + W_{4(n)},$$

$$\hat{R}_{(n)} = \frac{1}{(1+3\gamma)n}(\tilde{a}_{(n)} + \tilde{H}_{(n)} + W_{5(n)}),$$

$$\hat{K}_{(n)} = -\frac{2\gamma}{(1+\gamma)(1+3\gamma)n}(r\hat{\rho}'_{(n)} + W_{6(n)}),$$

$$\tilde{R}_{(n)} = \hat{R}_{(n)} + \frac{1}{n}S^*_{(n)}[\hat{R}\tilde{R}],$$

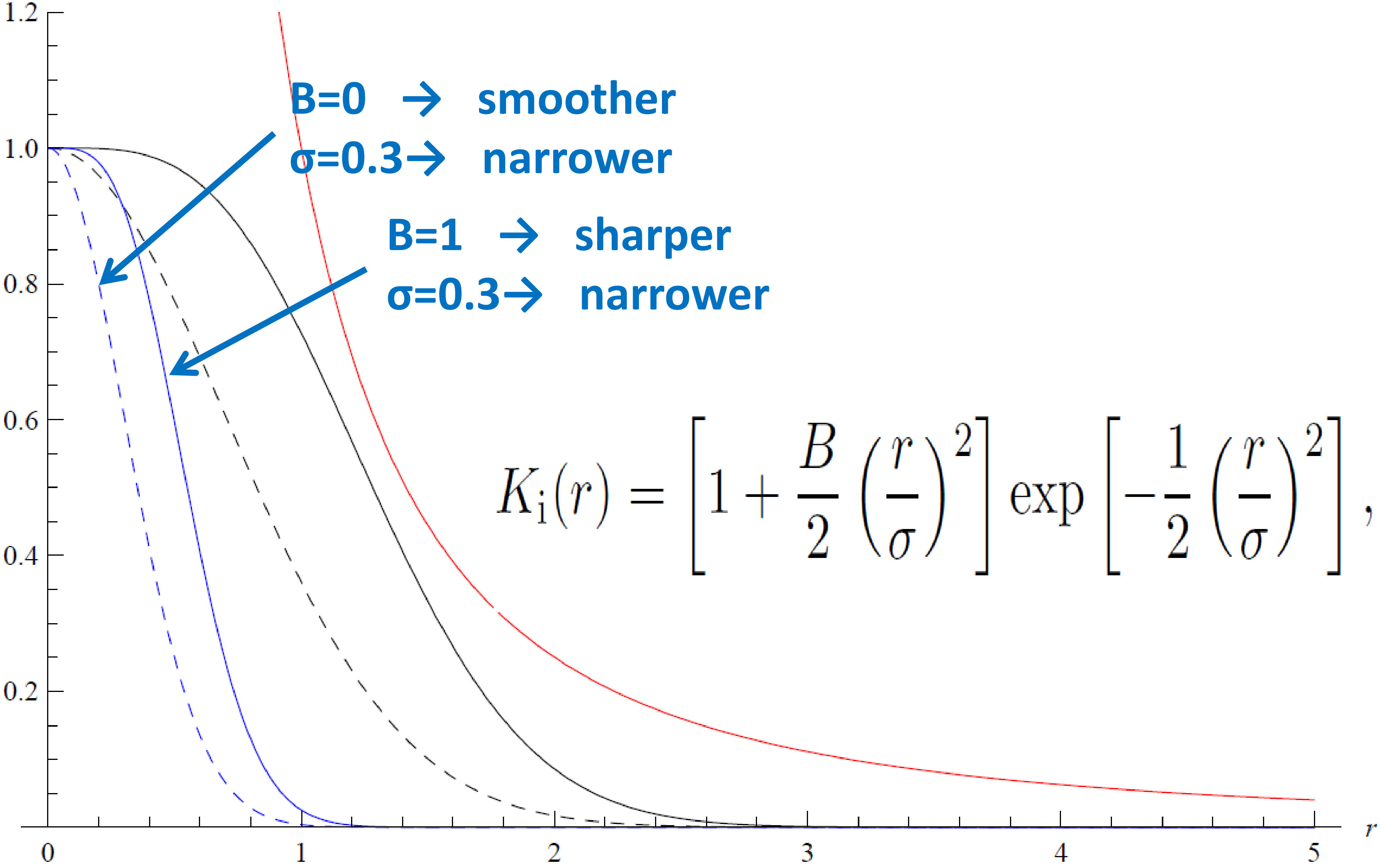
$$\tilde{K}_{(n)} = \hat{K}_{(n)} + \frac{1}{n}S^*_{(n)}[\hat{K}\tilde{K}].$$



$$K_\mathrm{i}(r) = \left[1 + \frac{B}{2}\left(\frac{r}{\sigma}\right)^2\right] \exp\left[-\frac{1}{2}\left(\frac{r}{\sigma}\right)^2\right],$$

B=0 → smoother
σ=0.7→ wider

B=1 → sharper
σ=0.7→ wider



$B=0$ → smoother

$\sigma=0.3$ → narrower

$B=1$ → sharper

$\sigma=0.3$ → narrower

$$K_\mathrm{i}(r) = \left[1+\frac{B}{2}\left(\frac{r}{\sigma}\right)^2\right]\exp\left[-\frac{1}{2}\left(\frac{r}{\sigma}\right)^2\right],$$

$$\delta(t, r_{\boldsymbol{od}})=0$$

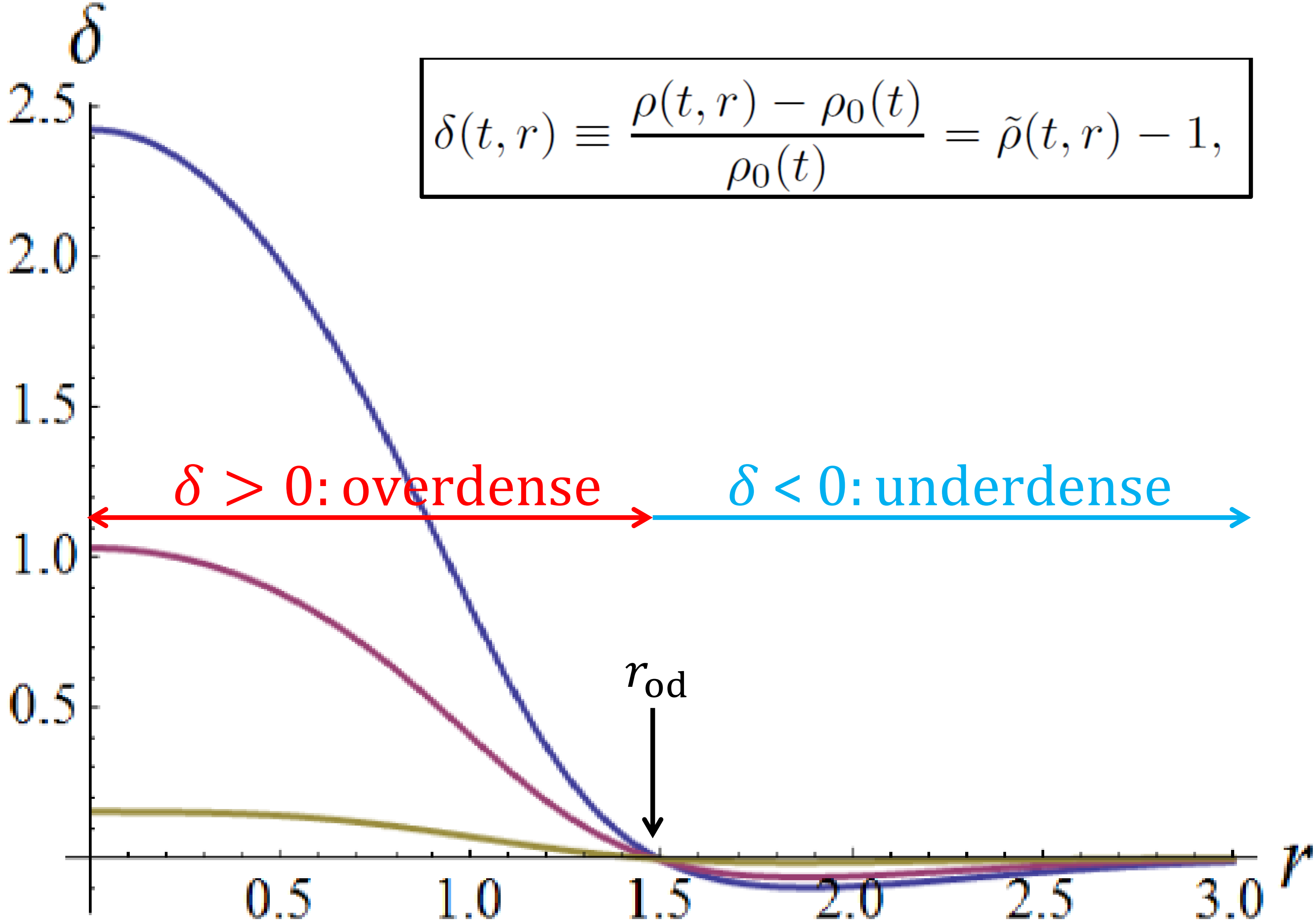
r_i is defined as $r_{\boldsymbol{od}}$
at an sufficiently early time.

$$\delta_{(1)}(t, r_i)=\frac{(1+\gamma){r_i}^2(3K_i(r)+rK_i'(r))}{5+3\gamma}\epsilon=0$$

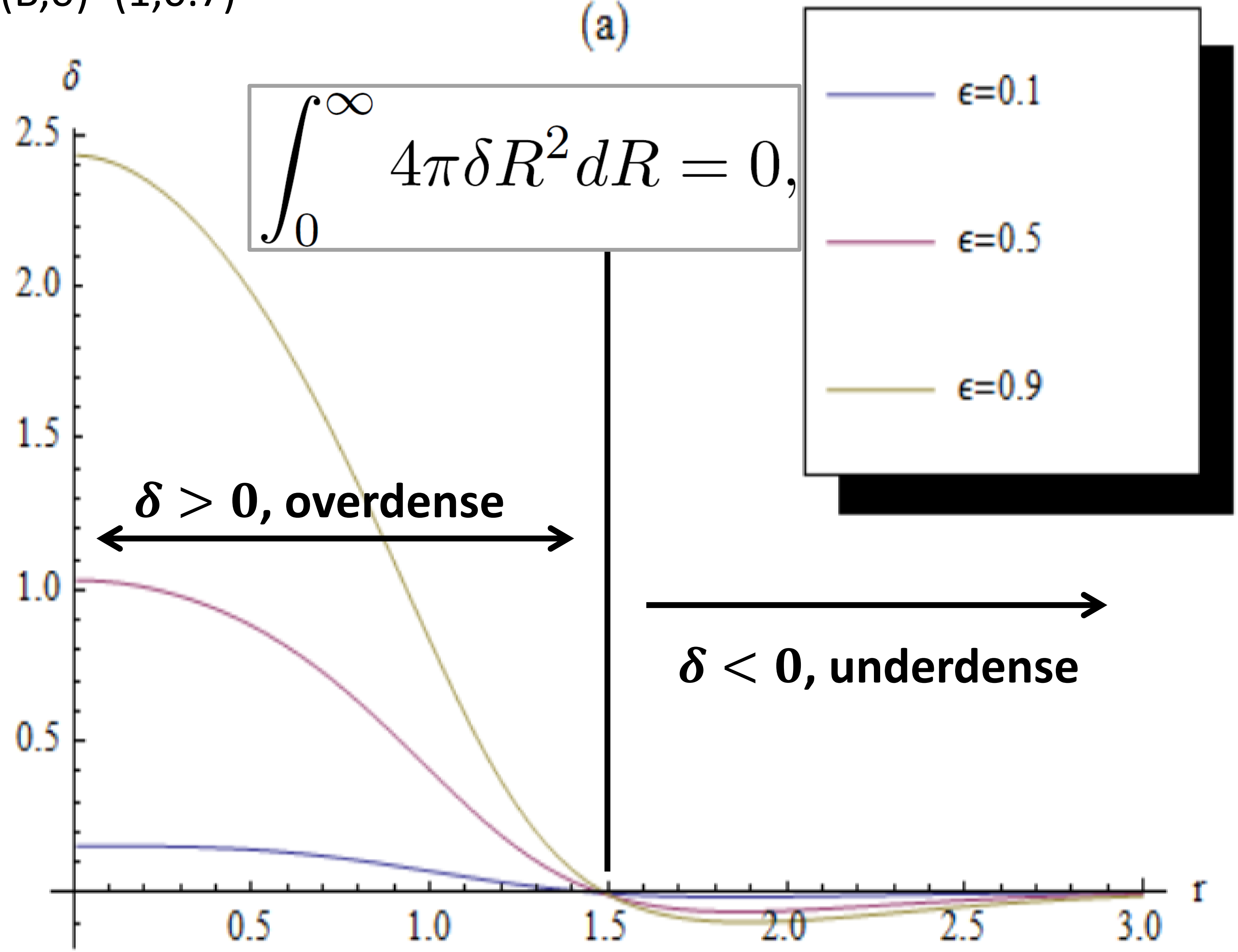
time evolution of density pert. profile

((*B*,*σ*)=(1,0.7),7th order expansion is used)

$$\delta(t,r) \equiv \frac{\rho(t,r)-\rho_0(t)}{\rho_0(t)} = \tilde{\rho}(t,r)-1,$$



(B,σ)=(1,0.7)



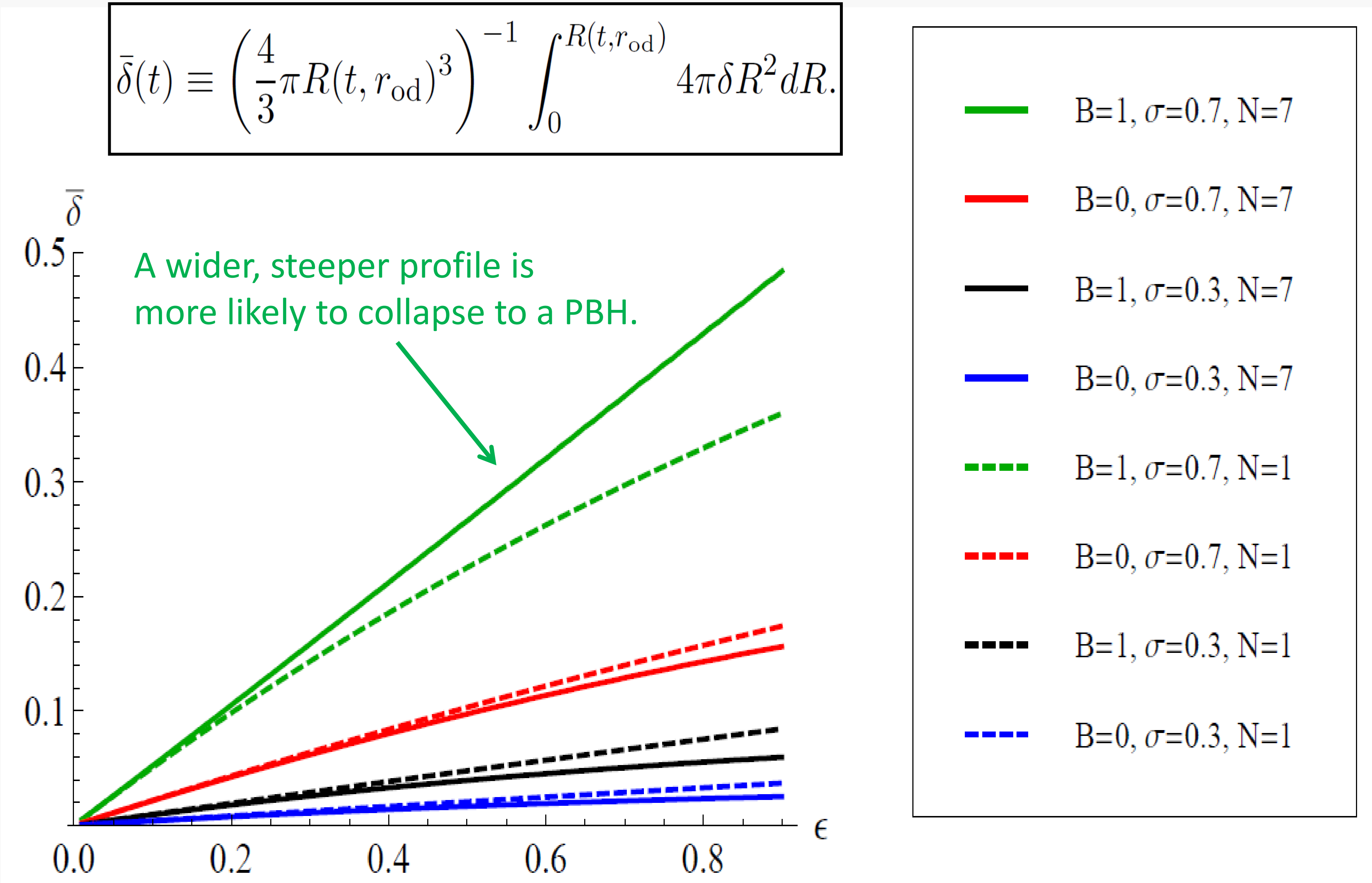
$$\int_0^\infty 4\pi\delta R^2 dR = 0,$$

$\delta > 0$, overdense

$\delta < 0$, underdense

Time evolution of averaged overdensity

$$\bar{\delta}(t) \equiv \left(\frac{4}{3}\pi R(t, r_{\rm od})^3\right)^{-1} \int_0^{R(t,r_{\rm od})} 4\pi\delta R^2 dR.$$



A wider, steeper profile is more likely to collapse to a PBH.