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Revisiting perturbations of a scalar field in an anisotropic universe

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Abstract

We revisit the issue on signatures of pre-inflationary background anisotropy by considering the quantization of a massless and minimally coupled scalar field in an axially symmetric Kasner background, mimicking cosmological perturbations. We show that the power spectrum of the scalar field fluctuation has a negligible difference from the standard inflation in the non-planar directions, but it has a sharp peak around the symmetry plane. For the non-planar high-momentum modes, we use the WKB approximation for the first period and the asymptotic approximation based on the de Sitter solution for the next period. At the boundary, two mode functions have the same accuracy with error of $O(H_i/k)$. We calculate the approximation up to the order of $(H_i/k)^6$ and show that the power spectrum of the scalar field fails to get corrections until we execute the approximation up to 6^{th} order. This note is based on our recent paper [1].

1 Introduction

Inflation has become one of the paradigms of modern cosmology. First of all, inflation elegantly solves many problems which are present in the standard Big-Bang model such as the horizon and flatness problems. Second, it accounts for the origin of the large scale structure of the universe in terms of the quantum fluctuations originating from the adiabatic vacuum structure in early universe. Remarkably, the nature of the primordial fluctuations is understood in terms of symmetries of the de Sitter spacetime. In general, we need n-point correlation functions to characterize the statistical nature of primordial fluctuations. However, these symmetries lead the power spectrum of a scale invariant form. These predictions from symmetries are robust and universal in inflationary scenarios. In fact, the above predictions have been confirmed, e.g., by the measurements of cosmic microwave background (CMB). As the observational precision increases, we have to go beyond the power spectrum to look at fine structure of the primordial fluctuations. Since in the realistic inflationary universe the symmetries of the de Sitter spacetime do not hold exactly, violation of them provides a measurable effect. One possibility is introducing spatially homogeneous models violating the spatial isotropy [2–4], where the Copernican principle is kept since there is no privileged positions in the universe. The universe has a privileged direction. From the observational point of view, a lot of anomalies indicating the statistical anisotropy are reported although its statistical significance is uncertain.

We start from the discussion on the evolution of anisotropic universe in the Einstein gravity minimally coupled to a massive scalar field, where the scalar field plays the role of the inflaton. To obtain a sufficiently long period of inflation, one usually imposes the slow rolling condition, $\phi_0 \gg M_P$, where ϕ_0 is the initial value of the inflaton. Under the assumption, as discussed in Ref. [5], the background metric can be approximated by the Kasner spacetime with a positive cosmological constant, $\Lambda(= 3{H_i}^2)$. Among all, we are mainly interested in the the regular Kasner spacetimes with two dimensional axial symmetry with metric,

$$ds^2 = -d\tau^2 + \sinh^{\frac{2}{3}}(3H_i\tau)\left[\tanh^{-\frac{2}{3}}\left(\frac{3H_i\tau}{2}\right)(dx_1^2 + dx_2^2) + \tanh^{\frac{4}{3}}\left(\frac{3H_i\tau}{2}\right)dx_3^2\right]. \quad (1)$$

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The spacetime has a privileged axis x_3 orthogonal to the symmetry plane. The spacetime has a Rindler-like event horizon at $\tau = 0$ since $g_{11}(= g_{22})$ approaches to a finite value and g_{33} goes to zero linearly. This spacetime is a good testing ground in analyzing the properties of anisotropic universes because of a couple of reasons. First, it bears various important features of the whole anisotropic universes including large anisotropy at $\tau = 0$. Second, only in this symmetric case of all anisotropic expansions, we can impose proper anisotropic vacuum state in terms of the zeroth order WKB approximation [6].

If one uses the Sasaki-Mukhanov variable, except for the complication due to the mixing of tensor-scalar modes, the evolutions of the metric perturbations are not much different from that of a massless scalar field [7]. In Ref. [5], it was also shown that the mode mixing modifies the power spectrum only by a proportionality factor with a small correction term of order m/H_i, where m is the mass of the inflaton. Therefore, as a formulation level, it is good to deal with the scalar field rather than the metric perturbation itself. Hence in this work, we are interested in the evolution of a massless, minimally coupled scalar field propagating on the background anisotropic universe (1) with action

$$S_\phi = -\frac{1}{2}\int d^4x\sqrt{-g}\Big(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\Big). \tag{2}$$

This scalar field is not the inflaton but just a *mimic of metric perturbations.*

2 Quantization of the Scalar field on anisotropic vacuum

The canonical quantization of the minimally coupled massless scalar field with the action (2) in the anisotropic spacetime (1) is done in the standard manner:

$$\phi = \int d^3k\Big(u_{\mathbf{k}}a_{\mathbf{k}} + u^*_{\mathbf{k}}a^\dagger_{\mathbf{k}}\Big), \tag{3}$$

where the creation and annihilation operators satisfy the commutation relations $\big[a_{\mathbf{k_1}}, a^\dagger_{\mathbf{k_2}}\big] = \delta(\mathbf{k_1}-\mathbf{k_2})$ (others are zero) and $u_{\mathbf{k}} = e^{i\mathbf{kx}}\phi_{\mathbf{k}}/(2\pi)^{3/2}$. The details of the quantization process depend on the choice of the mode function $u_{\mathbf{k}}$. We normalize the mode function as $\phi_{\mathbf{k}}\partial_\tau\phi^*_{\mathbf{k}} - \big(\partial_\tau\phi_{\mathbf{k}}\big)\phi^*_{\mathbf{k}} = \frac{i}{e^{3\alpha}}$.

For the later convenience, we introduce a dimensionless time x by $\sinh(\varepsilon x) = \frac{1}{\sinh(3H_i\tau)} = e^{-3\alpha}$, where ε denotes a small expansion parameter which will be specified later. The arrow of time for x is inverted since it varies from ∞ to $0+$ as the comoving time τ increases from $0+$ to ∞.

The equation of motion for the scalar field is written in the form of a time-dependent oscillator

$$\Big(\frac{d^2}{dx^2} + \Omega_{\mathbf{k}}(x)^2\Big)\phi_{\mathbf{k}} = 0\,, \tag{4}$$

where the dimensionless frequency squared is

$$\Omega^2_{\mathbf{k}}(x) = \left(\frac{\varepsilon}{3H_i}\right)^2 \frac{2^{4/3}\big(k_\perp^2 e^{-2\varepsilon x} + k_3^2\big)}{(1-e^{-2\varepsilon x})^{4/3}} = \frac{2(\bar{k}\varepsilon^{2/3})^2}{9}\left(\frac{e^{\varepsilon x}}{\sinh\varepsilon x}\right)^{1/3}\left(\frac{1}{e^{2\varepsilon x}-1} + r^2\right). \tag{5}$$

Here we define a scaled wave-number and a measure of planarity of a given mode by $\bar{k} = \varepsilon^{1/3}\frac{k}{H_i}$, $\quad r = \frac{k_3}{k}$, where $k^2 := k_1^2 + k_2^2 + k_3^2 = k_\perp^2 + k_3^2$. Later in this paper, we omit $\mathbf{k}$ in the frequency squared $\Omega^2_{\mathbf{k}}$ for simplicity. The power spectrum is defined by

$$\big\langle 0|\phi^2|0\big\rangle := \int d\ln k\int\frac{d\theta_{\mathbf{k}}}{2}P, \quad P = \frac{k^3}{2\pi^2}\big|\phi_{\mathbf{k}}\big|^2. \tag{6}$$

In contrast to the case of the standard de Sitter universe the direction dependence would be included in the power spectrum. The vacuum is chosen at the initial anisotropic era: $\tau \to +0$ to satisfy $a_{\mathbf{k}}|0\rangle = 0$. For this purpose, we choose the solution to be purely positive frequency mode with respect to τ at the early stage.

3 Non-planar high-momentum modes

As mentioned in the introduction, we use the WKB solution at early times ($x > x_*$) and asymptotic solution at later times ($x < x_*$). At the matching time $x = x_*$, the accuracies of the two are equal. Later in this paper, we set $x_* = 1$ by choosing ε appropriately. We also assume that $\bar{k}$ is larger than one, which will be satisfied with the modes we are interested in at the present approximation.

The WKB approximation is if $\epsilon(x) := \left|\frac{\frac{d\Omega^2(x)}{dx}}{\Omega^3(x)}\right| \ll 1$. For the early times, the WKB wavefunction is expanded up to an enough adiabatic order, to validate our matching scheme with the solutions in the asymptotic region. The next order of the approximation will improve the accuracy by the order $E_{\rm WKB}(x) \sim \epsilon(x)^2 \simeq \frac{1}{\bar{k}^2\, x^{2/3}}$.

In the later time limit, the space-time approaches that of the de Sitter spacetime. Therefore, it is natural that the zeroth order solution is just the well-known scalar field mode solution in the de Sitter spacetime. The higher order corrections are based on the *asymptotic approximation* based on the limit $\varepsilon x \ll 1$. By series expanding the frequency squared, we approximate the equation of motion (4) to be

$$\left(\frac{d^2}{dx^2} + \sum_{n=0}^{\infty} \varepsilon^n V_n\right)\phi = 0\,, \tag{7}$$

where the order by order corrections of the frequency squared are the same order for $x \sim 1$. The general solution to the differential equation (7) is given by $\phi = A_+ u(x) + A_- v(x)$, where u and v are the positive and negative frequency modes, respectively. We get the approximate solution by series expanding the modes $u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots$ and $v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \cdots$ and then solving the equation of motions order by order. To determine the time when the WKB and the asymptotic approximations will be matched, we need to know the size of the error of the zeroth order solution for a given x. The relative ratio of the correction term to the zeroth order solution gives the error, $E_{\rm asym}(x) \sim \varepsilon \bar{k}\, x^{4/3}$. The approximation will be best if we choose the intermediate time x_* so that the accuracies of the two solution match: $E_{\rm WKB} = E_{\rm asym}$, which gives $x_* = (\varepsilon \bar{k}^3)^{-1}$. For simplicity, we choose to set $x_* = 1$. Therefore, the small expansion parameter becomes $\varepsilon = \bar{k}^{-3} = \left(\frac{H_i}{k}\right)^{3/2}$. Now, the size of error at $x = x_*$ becomes $E_{\rm asym} = \bar{k}^{-2} = H_i/k$, which ensures that the present approximation works well for high momentum modes.

After the matching, we find that the power spectrum acquires corrections only when we calculate to the adiabatic order $O(\bar{k}^{-12})$. The power spectrum including the corrections becomes

$$P = \frac{{H_i}^2}{4\pi^2}\left\{1 + \frac{9(11 - 90r^2 + 99r^4)}{32}\left(\frac{H_i}{k}\right)^6 + O\left(\left(\frac{H_i}{k}\right)^7\right)\right\}. \tag{8}$$

For mode with $k \sim 10H_i$, the relative size of the correction term is of $O(10^{-6})$. The isotropy violation at initial stage of the universe is not small but its effects on the spectrum for the non-planar modes are suppressed by the long duration of inflation and high momentum effect. The correction $O((H_i/k)^6)$ is highly dependent on k to suppress the anisotropy effect.

4 Planar modes

For the planar modes $r^2 \sim 0$, there appears a region where the WKB approximation may not be valid during a period in $\varepsilon x \gg 1$. We divide the time into three separate regions divided by the times x_1 and x_* (See Fig. 1). In the region $x_1 > x > x_*$ the WKB approximation is valid. For other two regions, we may find approximate solutions. In the case of $r = 0$ exactly, the adiabaticity parameter diverges in the limit of $x \to \infty$ and there is no anisotropic vacuum state. The mode $r = 0$ would behave classically, not quantum mechanically, and will be out of scope.

The characteristic behavior of $\epsilon(x)$ is shown in Fig. 1.

Matching the solutions in the three regions yields the final amplitude and hence the power spectrum

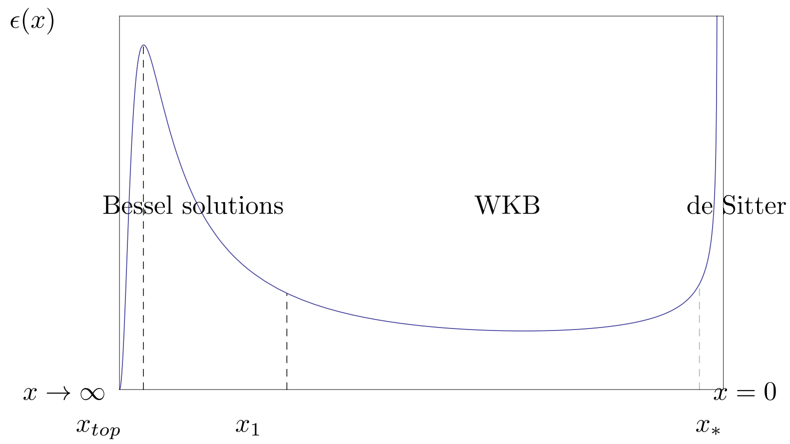

Figure 1: The behavior of the adiabatic parameter in the case of a planar mode. The left hand side ($x \to \infty$) corresponds to the past. The WKB approximation is temporary violated around $x \simeq x_{top}$.

obtained in the isotropic limit:

$$P = \left(\frac{H_i}{2\pi}\right)^2 \left(\coth \pi\bar{r} - \frac{\cos\left(2\Psi\right)}{\sinh \pi\bar{r}}\right), \tag{9}$$

where $\Psi(k) = \bar{k}x_*^{1/3} - \int_{x_1}^{x_*} \Omega(x)dx + \bar{q}e^{-\varepsilon x_1} - \frac{\pi}{4}$. The explicit value of the correction term becomes order of 10^{-3} for $\bar{r} \sim 3$. In the planar limit, the deviation of the power spectrum from the ansatz in Ref. [2] is quite clear.

5 Conclusion

We have reinvestigated the quantization of a massless and minimally coupled scalar field as a way to probe the signature of pre-inflationary background anisotropy in the spectrum of cosmological perturbations.

We first dealt the non-planar modes. We have shown that the power spectrum of the scalar field acquires non-vanishing corrections only when we execute the approximation up to 6^{th} order. Hence, the direction dependence appears only at the order $O((H_i/k)^6)$.

For the planar mode, we have obtained essentially the same result as that in our previous analysis [5, 6], but was confirmed by a more accurate matching. For such a mode, the temporal breaking of the WKB approximation relatively enhances the effects of the primordial anisotropy in the power spectrum.

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