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Nonlinear superhorizon perturbation and nonlinear gauge transformation

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Abstract

We develop a theory of nonlinear cosmological perturbations on superhorizon scales for a multi-component scalar field with a general kinetic term and a general form of the potential in the context of inflationary cosmology. We employ the ADM formalism and the spatial gradient expansion approach valid up to second-order in the expansion. We provide a formalism to obtain the solution in the multi-field case and also derive fully nonlinear gauge transformation rules valid through second-order. These fully nonlinear gauge transformation rules can be used to derive the solution in a desired gauge from the one in a gauge where computations are much simpler.

1 Introduction

Recent observations of the cosmic microwave background anisotropy show a very good agreement of the observational data with the predictions of conventional, single-field slow-roll models of inflation, that is, adiabatic Gaussian random primordial fluctuations with an almost scale-invariant spectrum. Nevertheless, possible non-Gaussianities from inflation has been a focus of much attention in recent years. To study possible origins of non-Gaussianity, one must go beyond the linear perturbation theory. In particular, in the classical production on superhorizon scales case, the δN formalism [1] turned out to be a powerful tool for computing non-Gaussianities thanks to its full non-linear nature, where one can employ the spatial gradient expansion approach. The length scale of the perturbation is longer than the Hubble radius. In the context of inflation, based on the leading order in gradient expansion, the δN formalism [1] was developed. This leading order in the expansion provides a general conclusion for the evolution on superhorizon scales that the adiabatic growing mode is conserved on the comoving hypersurface. In multi-field inflation, a non-slow-roll stage may appear when there is a change in the dominating component of the scalar field. The previous analyses are essentially based on the δN formalism and it is in general necessary to extend it to second-order in the expansion, that is, to the *beyond* δN formalism. As for a single scalar field, it has been developed in [2]. We focus on the case of a multi-component scalar field following recent our paper [3].

2 Basics

We develop a theory of nonlinear cosmological perturbations on superhorizon scales. For this purpose we employ the ADM formalism and the gradient expansion approach. In the ADM decomposition, the metric is expressed as $ds^2 = -\alpha^2 dt^2 + \hat{\gamma}_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$, where α is the lapse function, β^i is the shift vector and Latin indices run over 1,2 and 3. We introduce the extrinsic curvature K_{ij} defined by $K_{ij} = (\partial_t \hat{\gamma}_{ij} - \hat{D}_i\beta_j - \hat{D}_j\beta_i)/2\alpha$, where $\hat{D}$ is the covariant derivative with respect to the spatial metric $\hat{\gamma}_{ij}$. In addition to the standard ADM decomposition, the spatial metric and the extrinsic curvature are further decomposed so as to separate trace and trace-free parts $\hat{\gamma}_{ij} = a^2(t)e^{2\psi}\gamma_{ij}$ and $K_{ij} = a^2(t)e^{2\psi}(K\gamma_{ij}/3 + A_{ij})$, where $a(t)$ is the scale factor of a fiducial homogeneous FLRW spacetime, the determinant of γ_{ij} is normalised to be unity and A_{ij} is trace free. As for a matter field, let us focus on a minimally-coupled multi-component scalar field, described by $S_m = \int d^4x\sqrt{-g}P(X^{IJ}, \phi^K)$, $X^{IJ} \equiv -g^{\mu\nu}\partial_\mu\phi^I\partial_\nu\phi^J$, with ϕ^K denoting the K-th component of the scalar field. Note that we do not assume a specific form of both the kinetic term and potential, which are arbitrary functions of X^{IJ} and ϕ^K. The

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equation of motion for the scalar field is given by $\frac{2}{\sqrt{-g}}\partial_\mu\big(\sqrt{-g}P_{(IJ)}g^{\mu\nu}\partial_\nu\phi^J\big)+P_I=0$, where the subscript I in P_I represents a derivative with respect to ϕ^I and $P_{(IJ)}$ is defined as $P_{(IJ)}=(\frac{\partial P}{\partial X^{IJ}}+\frac{\partial P}{\partial X^{JI}})/2$. Now we write down the Einstein equations. In the ADM decomposition, the Einstein equations are separated into four constraints, the Hamiltonian constraint and three momentum constraints, and six dynamical equations for the spatial metric.

In the gradient expansion approach we suppose that the characteristic length scale L of a perturbation is longer than the Hubble length scale $1/H$ of the background, i.e. $HL \gg 1$. Therefore, $\epsilon \equiv 1/(HL)$ is regarded as a small parameter and we can systematically expand equations in the order of ϵ, identifying a spatial derivative is of order ϵ, $\partial_i Q = \mathcal{O}(\epsilon)Q$. To clarify the order of gradient expansion, we introduce the superscript (n). For example, ${}^{(2)}\alpha$ means the lapse function at second order in gradient expansion. As a background spacetime, we consider a FLRW universe. This leads to the following condition on the spatial metric: $\partial_t\gamma_{ij}=\mathcal{O}(\epsilon^2)$. Since we adopt this assumption, the spatial metric at leading order is given by an arbitrary spatial function of the spatial coordinates. Throughout this paper, in order to simplify the equations, we set the shift vector to zero up to second order in gradient expansion, $\beta^i=\mathcal{O}(\epsilon^3)$. Let us call this choice of the spatial coordinates as the *time-slice-orthogonal threading.*

2.1 Leading order in gradient expansion

In this subsection, we study the leading order gradient expansion and make clear the correspondence between the leading order equations and background equations. This correspondence can be used to construct the solution at leading order in gradient expansion in terms of the background solution. We will introduce the proper time τ by $\tau(t,x^i) \equiv \int_{x^i=const.} dt\ \alpha(t,x^i)$. In terms of τ, the expression of K is simplified under the time-slice-orthogonal threading condition $K=\frac{1}{\alpha}\frac{\partial_t(a^3e^{3\psi})}{a^3e^{3\psi}}=3\frac{\partial_\tau(ae^\psi)}{ae^\psi}$. Under the identification, $ae^\psi \Leftrightarrow a$ and $\tau \Leftrightarrow t$, one also has the correspondence, $K \Leftrightarrow 3H$. This means that the basic equations at leading order take exactly the same form as those in the background modulo above identifications. Namely, given a background solution, $\phi^I(t)\big|_{\rm background}=\phi^I_{\rm BG}\big[t,\,\phi^I_0(t_0)\big]$, one can construct the solution at leading order in gradient expansion as $\phi^I(t,x^i)\big|_{\rm gradient}=\phi^I_{\rm BG}\big[\tau,\,\phi^I_0(\tau_0)\big]$. All the information of inhomogeneities is contained in the initial condition as well as in the proper time τ. Thus it is sufficient to solve the background equations to obtain the solution at leading order in gradient expansion. In passing, we note that the e-folding number is often used as the time coordinate to describe the background evolution. For convenience, we define it as the number of e-folds counted backward in time from a fixed final time. That is, $N=\int_t^{t_0}dt'H(t')$. Accordingly, the scale factor is expressed as $a(N)=a_0e^{-(N-N_0)}$. By replacing t with τ and H with $K/3$ we can generalise the e-fold number to the one defined locally in space as $\mathcal{N}\equiv\int_t^{t_0}\alpha(t',x^i)dt'\,\frac{1}{3}K(t,x^i)\big|_{x^i=const.}$. Again one can check the validity of the above correspondence in terms of $\mathcal{N}$ as the time coordinate.

2.2 Towards the next-to-leading order in gradient expansion

One needs to specify the gauge condition to study perturbations in perturbation theory or in gradient expansion. Since spatial coordinates have been already fixed by the time-slice-orthogonal threading, one has to determine the time-slicing condition. Here, let us list various slicings and their definitions, Uniform expansion; $K(t,x^i)=3H(t)$, Uniform e-folding number; $\mathcal{N}(t,x^i)=N(t)$. Hereinafter, we call the uniform expansion and uniform e-folding number slicings as the uniform K and uniform $\mathcal{N}$ slicings, respectively. We mention that there is a remaining gauge degree of freedom in the synchronous or uniform $\mathcal{N}$ slicing, while the time slices are completely fixed in the uniform K slicing. As for the uniform $\mathcal{N}$ slicing, the gauge condition demands $\partial_t\psi$ to vanish. As we have seen in subsection 2.1, the leading order solutions are given by functions of τ in terms of the background solutions. At next-to-leading order in gradient expansion, terms with spatial derivatives of the leading order solution appear in the evolution equations. To evaluate those terms, one needs to calculate the spatial derivative of the lapse function, for example in $\partial_i\phi$, $\partial_i\phi_{BG}(\tau)=\partial_\tau\phi_{BG}(\tau)\,\partial_i\tau=\partial_\tau\phi_{BG}(\tau)\int dt\partial_i\alpha$. However the leading order ${}^{(0)}\alpha$ is in general given explicitly only after solving the following equation for α:

$$\alpha=f\Big[t,\,\phi(\tau)\Big]=f\Big[t,\,\phi\Big(\int\alpha dt\Big)\Big]. \tag{1}$$

It is clearly shown that it is almost impossible to solve this equation, at least in an analytical way. This problem did not appear in the single-field case. It is because one can show that various different slicings become identical at leading order in gradient expansion. It means that for all the slicings, we may set $\alpha = 1$ if desired. On the other hand, one has to face this problem in the case of multi-field inflation. We overcome this problem by uniform $\mathcal{N}$ slicing, which gives us a *homogeneous time* coordinate. On these slicings, one can evaluate the spatial derivatives of the leading order solution which appear as source terms and construct a solution to next-to-leading order in gradient expansion by integrating those terms.

3 Nonlinear gauge transformation

We derive the gauge transformation rules for the metric, its derivative (K and A_{ij}) and the scalar field. We consider a nonlinear gauge transformation from a coordinate system with vanishing shift vector $\beta^i = 0$, to another coordinate system in which the new shift vector also vanishes, $\tilde{\beta}^i = 0$. We note that once the time slicing is changed, the shift vector appears in the new slicing in general. So the spatial coordinates also need to be changed to eliminate thus appeared shift vector. We use the background e-folding number N as the time coordinate and define the temporal and spatial shift, n and L^i, respectively, $\tilde{N} + \tilde{n}(\tilde{N}, \tilde{x}^i) = N$, $\tilde{x}^i + \tilde{L}^i(\tilde{N}, \tilde{x}^i) = x^i$. Under the change of the coordinates, the line element should remain invariant, $ds^2 = -\frac{\alpha^2}{H^2(N)}dN^2 + a^2(N)e^{2\psi}\gamma_{ij}dx^i dx^j = -\frac{\tilde{\alpha}^2}{H^2(\tilde{N})}d\tilde{N}^2 + a^2(\tilde{N})e^{2\tilde{\psi}}\tilde{\gamma}_{ij}d\tilde{x}^i d\tilde{x}^j$. Equating the coefficients of $d\tilde{N}^2$, $d\tilde{N}d\tilde{x}^i$, and $d\tilde{x}^i d\tilde{x}^j$ on both sides of the above, we obtain the nonlinear gauge transformation rules. As a result, let us summarize the derived rules. The leading order transformation rules are

$$\tilde{\alpha}(\tilde{N}, \tilde{x}^i) = \frac{H(\tilde{N})}{H({}^{(0)}\tilde{N})}(1 + \partial_{\tilde{N}}{}^{(0)}\tilde{n})\ \alpha + \mathcal{O}(\epsilon^2)\,,\ \tilde{\psi}(\tilde{N}, \tilde{x}^i) = \psi - {}^{(0)}\tilde{n} + \mathcal{O}(\epsilon^2)\,, \tag{2}$$

$$\tilde{\gamma}_{ij}(\tilde{N}, \tilde{x}^i) = \gamma_{ij} + \mathcal{O}(\epsilon^2)\,,\ \tilde{K}(\tilde{N}, \tilde{x}^i) = K + \mathcal{O}(\epsilon^2)\,,\ \tilde{\phi}(\tilde{N}, \tilde{x}^i) = \phi + \mathcal{O}(\epsilon^2)\,. \tag{3}$$

The next-to-leading order transformation rules are

$$\begin{aligned}{}^{(2)}\tilde{\alpha}(\tilde{N}, \tilde{x}^i) =& \frac{\alpha H(\tilde{N})}{H({}^{(0)}\tilde{N})}(1 + \partial_{\tilde{N}}{}^{(0)}\tilde{n})\left(\frac{{}^{(2)}\alpha}{\alpha} + {}^{(2)}\tilde{n}\frac{\partial_N \alpha}{\alpha} - {}^{(2)}\tilde{n}\frac{\partial_N H({}^{(0)}\tilde{N})}{H({}^{(0)}\tilde{N})} + \frac{\partial_{\tilde{N}}{}^{(2)}\tilde{n}}{1 + \partial_{\tilde{N}}{}^{(0)}\tilde{n}}\right)\\ &- \frac{1}{2}\frac{\alpha^3 H(\tilde{N})(1 + \partial_{\tilde{N}}{}^{(0)}\tilde{n})}{a^2({}^{(0)}\tilde{N})e^{2\psi}H^3({}^{(0)}\tilde{N})}\gamma^{ij}\partial_{\tilde{i}}{}^{(0)}\tilde{n}\partial_{\tilde{j}}{}^{(0)}\tilde{n} + \tilde{L}^i(\partial_{\tilde{i}}\alpha)\frac{H(\tilde{N})}{H({}^{(0)}\tilde{N})}(1 + \partial_{\tilde{N}}{}^{(0)}\tilde{n}) + \mathcal{O}(\epsilon^4)\,,\end{aligned} \tag{4}$$

$${}^{(2)}\tilde{\psi}(\tilde{N}, \tilde{x}^i) = {}^{(2)}\psi - {}^{(2)}\tilde{n} + {}^{(2)}\tilde{n}\partial_N\psi + \tilde{L}^i\partial_{\tilde{i}}\psi + \frac{1}{3}\partial_{\tilde{i}}\tilde{L}^i - \frac{1}{6}\frac{\alpha^2\gamma^{ij}\partial_{\tilde{i}}{}^{(0)}\tilde{n}\partial_{\tilde{j}}{}^{(0)}\tilde{n}}{a^2({}^{(0)}\tilde{N})e^{2\psi}H^2({}^{(0)}\tilde{N})} + \mathcal{O}(\epsilon^4)\,, \tag{5}$$

$$\begin{aligned}{}^{(2)}\tilde{\gamma}_{ij}(\tilde{N}, \tilde{x}^i) =& {}^{(2)}\gamma_{ij} + {}^{(2)}\tilde{n}\partial_{\tilde{N}}\gamma_{ij} + \tilde{L}^k\partial_{\tilde{k}}\gamma_{ij} + \gamma_{jk}\partial_{\tilde{i}}\tilde{L}^k + \gamma_{ik}\partial_{\tilde{j}}\tilde{L}^k - \frac{2}{3}\partial_{\tilde{k}}\tilde{L}^k\gamma_{ij}\\ &- \frac{\alpha^2}{a^2({}^{(0)}\tilde{N})e^{2\psi}H^2({}^{(0)}\tilde{N})}\left(\partial_{\tilde{i}}{}^{(0)}\tilde{n}\partial_{\tilde{j}}{}^{(0)}\tilde{n} - \frac{1}{3}\gamma^{kl}\partial_{\tilde{k}}{}^{(0)}\tilde{n}\partial_{\tilde{l}}{}^{(0)}\tilde{n}\gamma_{ij}\right) + \mathcal{O}(\epsilon^4)\,,\end{aligned} \tag{6}$$

and

$$\begin{aligned}{}^{(2)}\tilde{K}(\tilde{N}, \tilde{x}^i) =& {}^{(2)}K + {}^{(2)}\tilde{n}\partial_N{}^{(0)}K + \tilde{L}^i\partial_{\tilde{i}}{}^{(0)}K + \frac{3H({}^{(0)}\tilde{N})}{(1 + \partial_{\tilde{N}}{}^{(0)}n)\alpha}\left(\partial_{\tilde{N}}{}^{(0)}\tilde{n}\frac{{}^{(2)}\alpha}{\alpha} - \frac{\partial_{\tilde{N}}{}^{(2)}\tilde{n}}{1 + \partial_{\tilde{N}}{}^{(0)}\tilde{n}}\right)\\ &+ \frac{3\alpha\gamma^{ij}\partial_{\tilde{i}}{}^{(0)}\tilde{n}\partial_{\tilde{j}}{}^{(0)}\tilde{n}}{2(1 + \partial_{\tilde{N}}{}^{(0)}n)a^2({}^{(0)}\tilde{N})e^{2\psi}H({}^{(0)}\tilde{N})} + \frac{\partial_{\tilde{i}}{}^{(0)}\tilde{n}\partial_{\tilde{j}}{}^{(0)}\tilde{n}}{2\alpha a^2({}^{(0)}\tilde{N})H({}^{(0)}\tilde{N})}\partial_N\left(\frac{\alpha^2}{e^{2\psi}}\gamma^{ij}\right)\\ &+ \frac{3H({}^{(0)}\tilde{N})}{(1 + \partial_{\tilde{N}}{}^{(0)}n)\alpha}\left[\partial_{\tilde{N}}{}^{(2)}\tilde{n}(1 - \partial_N\psi) - (\partial_{\tilde{N}}\tilde{L}^i)\partial_{\tilde{i}}\psi - \frac{1}{3}\partial_{\tilde{i}}\partial_{\tilde{N}}\tilde{L}^i\right]\\ &+ \frac{H({}^{(0)}\tilde{N})\alpha\gamma^{ij}}{2(1 + \partial_{\tilde{N}}{}^{(0)}n)e^{2\psi}}\partial_{\tilde{N}}\left(\frac{\partial_{\tilde{i}}{}^{(0)}\tilde{n}\partial_{\tilde{j}}{}^{(0)}\tilde{n}}{a^2({}^{(0)}\tilde{N})H^2({}^{(0)}\tilde{N})}\right) + \mathcal{O}(\epsilon^4)\,,\end{aligned} \tag{7}$$

$$\begin{aligned}\tilde{A}_{ij}(\tilde{N}, \tilde{x}^i) =& A_{ij} - \frac{H({}^{(0)}\tilde{N})}{2(1 + \partial_{\tilde{N}}{}^{(0)}\tilde{n})\alpha}\left(\partial_{\tilde{N}}\tilde{L}^k\partial_{\tilde{k}}\gamma_{ij} + \gamma_{jk}\partial_{\tilde{i}}\partial_{\tilde{N}}\tilde{L}^k + \gamma_{ik}\partial_{\tilde{j}}\partial_{\tilde{N}}\tilde{L}^k\right)^{TF}\\ &- \frac{H({}^{(0)}\tilde{N})}{2(1 + \partial_{\tilde{N}}{}^{(0)}\tilde{n})\alpha}\partial_{\tilde{N}}\left[\frac{\alpha^2}{a^2({}^{(0)}\tilde{N})e^{2\psi}H^2({}^{(0)}\tilde{N})}\left(\partial_{\tilde{i}}{}^{(0)}\tilde{n}\partial_{\tilde{j}}{}^{(0)}\tilde{n}\right)^{TF}\right] + \mathcal{O}(\epsilon^4)\,,\end{aligned} \tag{8}$$

$${}^{(2)}\tilde{\phi}(\tilde{N}, \tilde{x}^i) = {}^{(2)}\phi + {}^{(2)}\tilde{n}\partial_N{}^{(0)}\phi + \tilde{L}^i\partial_{\tilde{i}}{}^{(0)}\phi + \mathcal{O}(\epsilon^2)\,, \tag{9}$$

where $\tilde{L}^i$ is given by

$$\tilde{L}^i = l^i(\tilde{x}^i) + \int_{\tilde{N}_0}^{\tilde{N}} d\hat{N} \frac{\alpha^2(1+\partial_{\hat{N}}\tilde{n})}{a^2({}^{(0)}\hat{N})e^{2\psi}H^2({}^{(0)}\hat{N})}\gamma^{ij}\partial_{\tilde{j}}\tilde{n} + \mathcal{O}(\epsilon^3)\,. \tag{10}$$

4 Beyond δN formalism

Let us briefly summarise the five steps in the *Beyond δN formalism*.

1. Write down the basic equations (the Einstein equations and scalar field equation) in the uniform $\mathcal{N}$ slicing with the time-slice-orthogonal threading. For convenience let us call the choice of the coordinates in which one adopts the uniform X slicing with the time-slice-orthogonal threading the X gauge. So the above choice is the $\mathcal{N}$ gauge. In this gauge the metric components at leading order are trivial since both ψ and γ_{ij} are independent of time.
2. First solve the leading order scalar field equation under an appropriate initial condition and then the next-to-leading order scalar field equation which involves spatial gradients of the leading order solution.
3. Solve the next-to-leading order Einstein equations for the metric components and their derivatives.
4. Determine the gauge transformation from the $\mathcal{N}$ gauge to the K gauge and apply the gauge transformation rules to obtained the solution the obtained solution in the K gauge.
5. Evaluate the curvature perturbation $\mathfrak{R} = \psi + \chi/3$ in the K gauge, where χ is to be extracted from γ_{ij}.

5 Summary and discussion

In this paper, we developed a theory of nonlinear cosmological perturbations on superhorizon scales in the context of inflationary cosmology. We considered a multi-component scalar field with a general kinetic term and a general form of the potential. To discuss the superhorizon dynamics, we employed the ADM formalism and the spatial gradient expansion approach.

Different from the single-field case, there is a difficulty in solving the equations in the multi-field case. At leading-order, the equations take the same form as those for the homogeneous and isotropic FLRW background with suitable identifications of variables. In cosmological perturbation theory, the most important quantity to be evaluated is the curvature perturbation on the comoving slices which is conserved on superhorizon scales after the universe has reached the adiabatic limit. This quantity accurate to next-to-leading order may be relatively easily obtained in the single-field case because of the above mentioned coincidence among several temporal slicings. On the other hand, in the multi-field case, such a coincidence between different slicings does not hold.

In this paper, we developed a formalism to go beyond the leading order which avoids the problem. Namely, we first solve the field equations in a slicing in which the lapse function is trivial. The synchronous slicing is one of such slicings, but we adopt the uniform e-folding number slicing in which the time slices are chosen in such a way that the number of e-folds along each orbit orthogonal to the time slices, $\mathcal{N}$, is spatially homogeneous on each time slice. In this slicing we can solve the equations to next-to-leading order without encountering the above mentioned problem. After the solution to next-to-leading order is obtained, we transform it to the one in the uniform expansion slicing which is known to be identical to the comoving slicing on superhorizon scales in the adiabatic limit. Thus the gauge transformation laws play an essential role in our formalism. We derived them which are accurate to next-to-leading order. Note that they are fully nonlinear in nature in the language of the standard perturbation approach.

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