Inflation in Bimetric Gravity

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What is bimetric gravity?

Introduce massive graviton
→ ”massive gravity theory”

When there is general covariance, graviton cannot have mass.

Introduce “reference metric”.

It breaks general covariance, then graviton can have mass.

* reference metric is non dynamical in massive gravity

Make the reference metric dynamical

bi(metric)gravity
Bigravity action

\[ S = \frac{M_g^2}{2} \int d^4 x \sqrt{-g} R[g_{\mu\nu}] + \frac{M_f^2}{2} \int d^4 x \sqrt{-f} R[f_{\mu\nu}] + m^2 M_e^2 \int d^4 x \sqrt{-g} F_2 \]

EH action of \( g_{\mu\nu} \) \quad EH action of \( f_{\mu\nu} \) \quad Interaction term

\[ g_{\mu\nu} : \text{physical metric} \quad f_{\mu\nu} : \text{reference metric} \]

\[ F_2[L^\mu_\nu] = \frac{1}{2} ([L]^2 - [L^2]) \]

trace

\[ L^\mu_\nu = \delta^\mu_\nu - (\sqrt{g^{-1} f})^\mu_\nu \]

\[ m^2 : \text{coupling constant} \]

\[ M_e^2 = \left( \frac{1}{M_g^2} + \frac{1}{M_f^2} \right)^{-1} \]

: reduced Plank scale
The motivation of our study

We would like to investigate dynamics of spacetime with matter in bimetric gravity.

As the simple case, we think of bimetric theory with cosmological constants.

\[
S = \frac{M_g^2}{2} \int d^4x \sqrt{-g} (R[g_{\mu\nu}] - 2\Lambda_g) + \frac{M_f^2}{2} \int d^4x \sqrt{-f} (R[f_{\mu\nu}] - 2\Lambda_f) \\
+ m^2 M_e^2 \int d^4x \sqrt{-g} F_2[L^\mu_\nu],
\]

* We can think of cosmological constants as scalar fields in slow roll approximation.

We can also discuss inflation.
de Sitter solution

homogeneous metric ansatz

\[ ds^2 = -N^2(t)dt^2 + e^{2\alpha(t)}[dx^2 + dy^2 + dz^2] \]
\[ ds'^2 = -M^2(t)dt^2 + e^{2\beta(t)}[dx^2 + dy^2 + dz^2] \]

From variational principle of action

\[ \left( \frac{\alpha'}{N} \right)' - \xi a_g (M - N\epsilon) \left( \frac{3}{2} - \epsilon \right) = 0, \quad \text{EoM of } \alpha \]
\[ \left( \frac{\beta'}{M} \right)' + \xi (1 - a_g)\epsilon^{-3} (M - N\epsilon) \left( \frac{3}{2} - \epsilon \right) = 0, \quad \text{EoM of } \beta \]
\[ \left( \frac{\alpha'}{N} \right)^2 = \lambda_g + \xi a_g (2 - \epsilon)(\epsilon - 1), \quad \text{constraint (from variation with respect to N)} \]
\[ \left( \frac{\beta'}{M} \right)^2 = \lambda_f + \xi (1 - a_g)\epsilon^{-3} (1 - \epsilon), \quad \text{constraint (from variation with respect to M)} \]
\[ \xi \left( \frac{3}{2} - \epsilon \right) \left( \frac{\beta' e^\beta}{M} - \frac{\alpha' e^\alpha}{N} \right) = 0 \quad \text{consistency relation (secondary constraint)} \]

de Sitter solution is represented as positive roots of \( g(\epsilon) : \epsilon_0 = \text{const.} \)
\[ g(\epsilon) = (\lambda_f + \xi a_g)\epsilon^3 - 3\xi a_g \epsilon^2 + \left[ -\lambda_g + 2\xi a_g - \xi (1 - a_g) \right] \epsilon + \xi (1 - a_g) = 0. \]
de Sitter solution

homogeneous metric ansatz

\[ ds^2 = -N^2(t)dt^2 + e^{2\alpha(t)}[dx^2 + dy^2 + dz^2] \]
\[ ds'^2 = -M^2(t)dt^2 + e^{2\beta(t)}[dx^2 + dy^2 + dz^2] \]

From variational principle of action

\[ (\frac{\alpha'}{N})' - \xi a_g (M - N\epsilon) \left( \frac{3}{2} - \epsilon \right) = 0, \quad : \text{EoM of } \alpha \]
\[ (\frac{\beta'}{M})' + \xi (1 - a_g) \epsilon^{-3} (M - N\epsilon) \left( \frac{3}{2} - \epsilon \right) = 0, \quad : \text{EoM of } \beta \]
\[ (\frac{\alpha'}{N})^2 = \lambda_g + \xi a_g (2 - \epsilon) (\epsilon - 1), \quad : \text{constraint (from variation with respect to N)} \]
\[ (\frac{\beta'}{M})^2 = \lambda_f + \xi (1 - a_g) \epsilon^{-3} (1 - \epsilon). \quad : \text{constraint (from variation with respect to M)} \]
\[ \xi \left( \frac{3}{2} - \epsilon \right) \left( \frac{\beta' e^\beta}{M} - \frac{\alpha' e^\alpha}{N} \right) = 0 \quad : \text{constraint} \]

Expansion rate (Hubble)

\[ H_0^2 = \lambda_g + \xi a_g (2 - \epsilon_0) (\epsilon_0 - 1) \]

de Sitter solution is represented as positive roots of \( g(\epsilon) \): \( \epsilon_0 = \text{const.} \)

\[ g(\epsilon) = (\lambda_f + \xi a_g) \epsilon^3 - 3\xi a_g \epsilon^2 + [-\lambda_g + 2\xi a_g - \xi (1 - a_g)] \epsilon + \xi (1 - a_g) = 0 \]
The conditions for the existence of de Sitter solution

**Condition ①**: There exist positive roots of $g(\epsilon)$

**Condition ②**: The roots satisfy $H_0^2 > 0$

The region where de Sitter solution exists

\[ \frac{\lambda_f}{\xi} \]

\[ \frac{\lambda_f}{\xi} = \lambda_+ \]

\[ \lambda_f = \lambda_- \]

where $\lambda_+ = \lambda_+(\xi, a_g, \lambda_g)$, $\lambda_c^{\pm} = \xi(1 - a_g)\left(\frac{1}{2} \pm \sqrt{\frac{\lambda_g}{\xi a_g} + \frac{1}{4}}\right)/(\frac{3}{2} \pm \sqrt{\frac{\lambda_g}{\xi a_g} + \frac{1}{4}})^3$
Anisotropic perturbation

Anisotropic ansatz

\[ ds^2 = -N^2(t) dt^2 + e^{2\alpha(t)} \left[ e^{-4\sigma(t)} dx^2 + e^{2\sigma(t)} (dy^2 + dz^2) \right], \]
\[ ds'^2 = -M^2(t) dt^2 + e^{2\beta(t)} \left[ e^{-4\lambda(t)} dx^2 + e^{2\lambda(t)} (dy^2 + dz^2) \right], \]

From variational principle of action,

\[ \sigma'' + 3H_0 \sigma' - \xi a_g \epsilon_0 (3 - 2\epsilon_0) q = 0 \quad : \text{EoM of } \sigma \]
\[ \lambda'' + 3H_0 \lambda' + \xi (1 - a_g) \frac{1}{\epsilon_0} (3 - 2\epsilon_0) q = 0 \quad : \text{EoM of } \lambda \]

From the difference of EoMs,

\[ q'' + 3H_0 q' + \xi \left[ a_g \epsilon_0 + (1 - a_g) \frac{1}{\epsilon_0} \right] (3 - 2\epsilon_0) q = 0 \]

\[ \omega_0^2 = \xi \left[ a_g \epsilon_0 + (1 - a_g) \frac{1}{\epsilon_0} \right] (3 - 2\epsilon_0) \quad : \text{Effective mass of massive graviton} \]
The stability towards the anisotropic perturbation

\[ \omega_0^2 = \xi \left[ a_g \epsilon_0 + (1 - a_g) \frac{1}{\epsilon_0} \right] (3 - 2\epsilon_0) \]

The stability is determined by the sign of oscillational term

whether \( \epsilon_0 \) is larger/smaller than \( \frac{3}{2} \)

Inner root

Outer root

\[ \lambda_f / \xi \]

stable

unstable

\[ \lambda_f = \lambda_{3/2} \]

where

\[ \lambda_{3/2} = \frac{4}{27} \left[ 3(\lambda_g + \frac{\xi a_g}{4}) + \xi (1 - a_g) \right] \]
Evaluation of effective mass $\omega_0^2$

$$\beta = \frac{\omega_0^2}{H_0^2} = \frac{\xi[a_g \varepsilon_0 + (1 - a_g) \frac{1}{\varepsilon_0}](3 - 2\varepsilon_0)}{\lambda_g + \xi a_g (2 - \varepsilon_0)(\varepsilon_0 - 1)}$$

: the ratio of effective mass to Hubble scale

- **Inner root** $\rightarrow \beta > 2$
- **Outer root** $\rightarrow \beta < 2$

Effective mass exactly equals to $2H_0^2$ on $\lambda_f = \lambda_+$!
There are two series of solutions: inner root and outer root.

(1) homogeneous isotropic metric ansatz
   the condition that de Sitter solution exists

   There are two series of solutions: inner root and outer root.

(2) anisotropic perturbation around de Sitter sol.
   the stability for the perturbation
   - inner root $\rightarrow$ stable
   - outer root $\rightarrow$ stable for $\lambda_f > \lambda_3^2$ and unstable for $\lambda_f < \lambda_3^2$

   effective mass of massive graviton corresponding to the anisotropy.
   - inner root $\rightarrow \omega_0^2 > 2H_0^2$
   - outer root $\rightarrow \omega_0^2 < 2H_0^2$

For inner root, effective mass is bounded above Hubble scale. If we consider inflation then the anisotropy decays in inflation time scale.
When we consider perturbations on de Sitter background in massive gravity, the square of graviton mass should be larger than $2H_0^2$ (Higuchi bound) 


In our analysis,

Effective mass exactly equals to $2H_0^2$ on $\lambda_f = \lambda_+$!

i.e. the critical condition for the existence of de Sitter solutions

$\downarrow$ coincide with

Higuchi bound

Is there really the relation between them?

$\bigcirc$ explicitly calculating the mass bound of massive graviton in bimetric gravity
\[
S = \frac{M_g^2}{2} \int d^4x \sqrt{-g} R[g_{\mu\nu}] + \frac{M_f^2}{2} \int d^4x \sqrt{-f} R[f_{\mu\nu}]
+ m^2 M_e^2 \int d^4x \sqrt{-g} \sum_{n=0}^4 \beta_n e_n \left( \sqrt{g^{-1} f} \right)^\mu \nu
\]

\[
e_n [X^\mu_\nu] = \frac{(-1)^n}{n!} \sum_{\sigma \in S_n} \text{sign} (\sigma) X^{\mu_1\sigma(1)}_{\mu_2\sigma(2)} \cdots X^{\mu_n\sigma(n)}_{\nu}
\]

\[
\beta_0 = \alpha_0 - 4\alpha_1 + 6\alpha_2 - 4\alpha_3 + \alpha_4
\]
\[
\beta_1 = \alpha_1 - 3\alpha_2 + 3\alpha_3 - \alpha_4
\]
\[
\beta_2 = \alpha_2 - 2\alpha_3 + \alpha_4
\]
\[
\beta_3 = \alpha_3 - \alpha_4
\]
\[
\beta_4 = \alpha_4
\]

\[
S = \frac{M_g^2}{2} \int d^4x \sqrt{-g} R[g_{\mu\nu}] + \frac{M_f^2}{2} \int d^4x \sqrt{-f} R[f_{\mu\nu}]
+ m^2 M_e^2 \int d^4x \sqrt{-g} \sum_{n=0}^4 \alpha_n e_n \left( 1 - \sqrt{g^{-1} f} \right)^\mu \nu
\]

equivalent
The condition for the existence of de Sitter solution

* We fix $\xi, a_g, \lambda_g$ and vary $\lambda_f$ in the following. Then root of $g(\epsilon)$ is function of $\lambda_f$.

$\epsilon_0 \equiv \epsilon_0(\lambda_f)$

Condition(1) : There exist positive roots of $g(\epsilon)$

$\lambda_f \leq \lambda_+(\xi, a_g, \lambda_g)$

Condition(2) : The roots satisfy $H_0^2 > 0$

Expansion rate is determined from constraint.

$H_0^2 = \lambda_g + \xi a_g (2 - \epsilon_0)(\epsilon_0 - 1)$

$\epsilon_{c-} < \epsilon_0(\lambda_f) < \epsilon_{c+}$

where $\epsilon_{c\pm} = \frac{3}{2} \pm \sqrt{\frac{\lambda_g}{\xi a_g} + \frac{1}{4}}.$
The behavior of the roots of $g(\epsilon)$

- $\lambda_f = \lambda_+$

There exists a positive multiple root $\epsilon_*$

This root satisfies

\[ \epsilon_{c-} < \epsilon_* < \epsilon_{c+} \]

- $\lambda_f < \lambda_+$

There ordinarily exist two positive roots.

$\epsilon_{in}$ and $\epsilon_{out}$ (inner root and outer root)

As we decrease $\lambda_f$,

\[ \begin{align*}
\epsilon_{in} & \text{ decreases.} \\
\epsilon_{out} & \text{ increases.}
\end{align*} \]

Sometime,

\[ \begin{align*}
\epsilon_{in} & \text{ becomes smaller than } \epsilon_{c-} \\
\epsilon_{out} & \text{ becomes larger than } \epsilon_{c+}
\end{align*} \]

The critical $\lambda_f$ is $\lambda_{c-}$, $\lambda_{c+}$, respectively.
we can rewrite the following value as

\[ \omega_0^2(\epsilon_0) - 2H_0^2(\epsilon_0) = (\epsilon_* - \epsilon_0) \times \text{(positive definite)} \]

\[
\begin{align*}
\omega_0^2 &> 2H_0^2 \quad \Rightarrow \quad \epsilon_0 < \epsilon_* \\
\omega_0^2 &= 2H_0^2 \quad \Rightarrow \quad \epsilon_0 = \epsilon_* \\
\omega_0^2 &< 2H_0^2 \quad \Rightarrow \quad \epsilon_0 > \epsilon_*
\end{align*}
\]

From

\[ \begin{align*}
\epsilon_{\text{in}} &< \epsilon_* \\
\epsilon_{\text{out}} &> \epsilon_*
\end{align*} \]

inner root satisfies \( \omega_0^2 > 2H_0^2 \)

outer root satisfies \( \omega_0^2 < 2H_0^2 \)
Bigravity action

\[ S = \frac{M_g^2}{2} \int d^4 x \sqrt{-g} R[g_{\mu\nu}] + \frac{M_f^2}{2} \int d^4 x \sqrt{-f} R[f_{\mu\nu}] + m^2 M_e^2 \int d^4 x \sqrt{-g} F_2 \]

\( g_{\mu\nu} \) : physical metric
\( f_{\mu\nu} \) : reference metric

\[ F_2[L^\mu_\nu] = \frac{1}{2} ([L]^2 - [L^2]) \]

\[ L^\mu_\nu = \delta^\mu_\nu - (\sqrt{g^{-1} f})^\mu_\nu \]

* when we make the general coordinate transformation at the same time for both metrics, the action is unchanged.
(there are only 4 DOF as general covariance.)

\[ m^2 : \text{coupling constant} \]

\[ M_e^2 = (\frac{1}{M_g^2} + \frac{1}{M_f^2})^{-1} \]

: reduced Plank scale