Self-consistent initial conditions for
primordial black hole formation

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Introduction

・ Large-amplitude inhomogeneities in the early universe collapse to form Primordial Black Holes.

・ Some inflationary models predict large-amplitude curvature perturbation on small scales, leading to production of PBHs.

・ CMB or LSS probes primordial perturbation on large scales, while information about that on small scales is scarce.
Abundance of PBHs has been constrained by gravitational lensing, gravitational waves, etc...

By investigating the condition for PBH formation in detail, PBH abundance can be correctly predicted assuming some inflationary model.

Combined with observational data, the prediction can be used to probe primordial perturbation on small scales.
curv. pert. $\rightarrow$ density pert. $\delta$

If $\delta > \delta_c$, PBH is formed

curv. pert. is time-independent

outline of PBH formation

Hubble radius

scale of pert.

If $\delta > \delta_c$, PBH is formed

inflation

radiation domination

time

PBH formation

analytic calc.

numerical calc.
When $\epsilon \equiv \frac{\text{Hubble radius}}{\text{scale of pert.}} \rightarrow 0$, pert. is time-independent. So, initial curv. pert. profile is represented by $K_i(r)$.

When $\epsilon \ll 1$, the solution of the Einstein eqs. can be obtained by asymptotic expansion over $\epsilon$.

- First order asymptotic expansion: Polnarev, Musco 2007

- We have obtained higher order expansion.
  Polnarev, Nakama, Yokoyama, JCAP09(2012)027
Benefits of higher order expansion

- Hubble radius
- Scale of pert.
- Time
- Inflation
- Radiation domination
- First order
- Higher order
- Analytic calc.
- Numerical calc.
- PBH formation
- Fast
- Accurate
\[ ds^2 = -\frac{a^2}{R} dt^2 + b^2 dr^2 + R^2 (d\theta^2 + \sin^2 \theta d\phi^2), \]
\[ u \equiv \frac{\dot{R}}{a}, \quad \text{(definition of } u) \]

Einstein eqs. (Misner & Sharp, 1964)

\[
\begin{align*}
\ddot{b} &= a u', \\
\ddot{b} &= \frac{a u'}{R'}, \\
\frac{a'}{a} &= -\frac{\gamma \rho'}{1 + \gamma \rho}, \\
M' &= 4\pi \rho R^2 R', \\
\dot{M} &= -4\pi \rho R^2 \dot{R}, \\
\frac{R'^2}{b^2} &= 1 + u^2 - \frac{2GM}{R}.
\end{align*}
\]
consider perturbed region surrounded by the flat FLRW univ. 

\[ ds^2 = -dt^2 + d(Sr)^2 + (Sr)^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \]

• definition of curvature profile \( K(t, r) \):

\[ ds^2 = -a^2 dt^2 + \frac{dR^2}{1-K(t,r)r^2} + R^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \]

• initial curvature profile: \( K(0, r) \equiv K_i(r) (< \frac{1}{r^2}) \)

• boundary condition: \( K(t, \infty) = 0 \)

The metric above coincides with the flat FLRW metric at spatial infinity.
example of initial curvature profile

central large curvature region

$$K_i(r) = \left[ 1 + \frac{B}{2} \left( \frac{r}{\sigma} \right)^2 \right] \exp \left[ -\frac{1}{2} \left( \frac{r}{\sigma} \right)^2 \right],$$

$$(B, \sigma) = (1, 0.7)$$

flat FLRW universe at spatial infinity
• decompose all the quantities

\[ X(t, r) = X_0(t, r)\tilde{X}(t, r) \]

FLRW solution \quad deviation from FLRW solution

• expansion parameter:

\[ \epsilon \equiv \left( \frac{H_0^{-1}}{S(t)r_i} \right)^2 \propto t \quad \text{(R. D.)} \]

comoving radius of perturbed region

• Rewriting the Einstein eqs. in terms of \( \tilde{X} \) and plugging

\[ \tilde{X}(t, r) = \sum_{n=0}^{\infty} \epsilon^n(t)\tilde{X}_n(r) \]

recurrence relations to calculate \( \tilde{X}_n \) are obtained.
\( \tilde{\rho}(t, r) \) up to second order in \( \varepsilon \)

\[
1 + \frac{2}{9} \ri^2 \varepsilon (3 K_i[r] + r K_i'[r])
\]

\[
- \frac{1}{540} \frac{\ri^4 \varepsilon^2}{r} (-132 r K_i[r]^2 - 8 r^3 K_i'[r]^2 + 2 K_i'[r] (-16 + r^4 K_i''[r]))
\]

\[
- 4 r (7 K_i''[r] + r K_i^{(3)}[r]) + 2 r^2 K_i[r] (-20 K_i'[r] + 2 r (8 K_i''[r] + r K_i^{(3)}[r]))
\]

first order

second order
Letting $\Delta$ be the required accuracy, the maximum of $\epsilon^{N+1} \tilde{X}_{(N+1)}(r)$ is less than $\Delta$, where $\epsilon^{N+1} M_{(N+1)} < \Delta$. The accuracy of asymptotic expansion is given by $\text{ERR} \left( \sum_{n=0}^{N} \epsilon^n \tilde{X}_{(n)} \right) \equiv \tilde{X} - \sum_{n=0}^{N} \epsilon^n \tilde{X}_{(n)} \sim O(\epsilon^{N+1} \tilde{X}_{(N+1)})$. 
Higher order expansions are accurate even when $\epsilon \sim 1$.
Letting $\Delta$ be the required accuracy,

\[ \epsilon^{N+1} M_{(N+1)} < \Delta, \]

The maximum of $\epsilon^{N+1} \tilde{X}_{(N+1)}(r)$

The error is less than $\Delta$ if

\[ \epsilon < \epsilon_{\text{max}} \equiv \sqrt[ N+1 ]{ \frac{ \Delta }{ M_{(N+1)} } }. \]
When to start numerical computation

\((B, \sigma) = (0, 0.7)\)

We can delay when to start numerical calc. until \(\epsilon = 0.51\), maintaining the accuracy of \(10^{-5}\) with 7th order asymptotic expansion.
Density pert. grows in the central large curv. region. If initial curv. is large enough, a PBH is formed.

\[ \epsilon = 0.9 \]

\[ \epsilon = 0.5 \]

\[ \epsilon = 0.1 \]
summary

• Spherically symmetric large amplitude perturbation embedded in the flat FLRW universe is investigated.
• The solution of the Einstein eqs. is obtained by asymptotic expansion over $\epsilon$.
• The solution is valid while the perturbed region is outside the horizon.
• Initial curvature profile, $K_i(r)$, generates the solution.
• Since the solution is accurate even when $\epsilon \sim 1$, we can delay the time when numerical computation is started.

work in progress
backups
define $\tilde{K}$, which describes time evolution of $K$

\[
1 - K(t, r)r^2 = (1 - K_i(r)r^2)\tilde{K}(t, r)
\]

- $\tilde{K}(t, \infty) = 1$ since $K(t, \infty) = 0, K_i(\infty) = 0$.

- $\tilde{K}(0, r) = 1$ since $K(0, r) \equiv K_i(r)$.

- $\tilde{K} > 0$, so we can define $\tilde{K} \equiv \log\tilde{K}$
The Einstein equations are rewritten in terms of $\tilde{X}$, $\hat{X}$.

\[
\tilde{\mu} = \tilde{H}^2 + \epsilon \left[ \frac{r_1^2}{r^2} + e^K \left( K_i r_i^2 - \frac{r_1^2}{r^2} \right) \right] e^{-2\hat{R}}
\]

\[
\frac{\partial \tilde{\mu}}{\partial \xi} = 2(\tilde{\mu} - \tilde{\Phi} \tilde{f}),
\]

\[
\tilde{\rho} = \tilde{\mu} + \frac{1}{3} D_r \tilde{\mu},
\]

\[
\hat{a} = -\frac{\gamma}{1 + \gamma} \hat{\rho},
\]

\[
\frac{\partial \hat{R}}{\partial \xi} = \frac{2}{3(1 + \gamma)} (\tilde{\Phi} - 1),
\]

\[
\frac{\partial \hat{K}}{\partial \xi} = -\frac{4\gamma}{3(1 + \gamma)^2} \tilde{\Phi} D_r \hat{\rho}.
\]
• construct solution by asymptotic expansion

\[
\tilde{X}(t, r) = \sum_{n=0}^{\infty} \in^n (t) \tilde{X}(n)(r)
\]

\[
\hat{X}(t, r) = \sum_{n=0}^{\infty} \in^n (t) \hat{X}(n)(r)
\]

• by definition, \( \tilde{X}(0) = 1 \), \( \hat{X}(0) = 0 \).
• By plugging the expansions into the Einstein eqs, recurrence relations to calculate the expansion coefficients \( \tilde{X}(n) \) are obtained.

• example of recurrence relations

\[
\tilde{\rho}(n) = \tilde{\mu}(n) + \frac{r}{3} \tilde{\mu}'(n) + W_{3}(n),
\]

\[
W_{3}(n) \equiv S_{(n)}[(\tilde{\mu} - \tilde{\rho})(r \tilde{R})'] + \frac{r}{3} S_{(n)}[\tilde{\mu}' \tilde{R}],
\]

\[
S_{(n)}[\tilde{X}_{1}\tilde{X}_{2}] \equiv \sum_{i=1}^{n-1} \tilde{X}_{1(i)} \tilde{X}_{2(n-i)}.
\]
By plugging these expansions into the Einstein equations, we can obtain recursive relations to calculate $\tilde{X}(n)$.

We define

\[
S(n)[\tilde{X}_1 \tilde{X}_2] \equiv \sum_{i=1}^{n-1} \tilde{X}_1(i) \tilde{X}_2(n-i).
\]

\[
S^*(n)[\tilde{X} F] = \sum_{m=1}^{n-1} m \tilde{X}_m F(n-m).
\]

Some function of $\tilde{X}$

$S(n), S^*(n)$ depend only on the expansion coefficients of up to $n-1$ order.
\[ \tilde{\mu}(n) = \frac{1}{1 + A_n} (F(n) + W_1(n) - W_2(n)), \]

\[ A_n \equiv \frac{2}{1 + \gamma} \left[ \left( \gamma + \frac{1}{3} \right) n - \gamma \right], \]

\[ F(n) \equiv \delta_n^1 r_i^2 K_i - 2r_i^2 K_i \hat{R}_{(n-1)} + r_i^2 \left( K_i - \frac{1}{r^2} \right) \hat{K}_{(n-1)}, \]

\[ W_1(n) \equiv S_{(n)}[\hat{H}\hat{H}] + r_i^2 \left( K_i - \frac{1}{r^2} \right) S_{(n-1)}[e^{\hat{K}}e^{-2\hat{R}}] \]

\[ + \frac{1}{n-1} \left\{ r_i^2 \left( K_i - \frac{1}{r^2} \right) S^*_{(n-1)}[\hat{K}e^{\hat{K}}] - 2r_i^2 K_i S^*_{(n-1)}[\hat{R}e^{-2\hat{R}}] \right\}. \]

\[ W_2(n) \equiv 2 \left( S_{(n)}[\tilde{a}\tilde{H}] + S_{(n)}[\tilde{\Phi}\tilde{f}] + \frac{\gamma}{n(1 + \gamma)} S^*_{(n)}[\tilde{\rho}(\tilde{\rho} - \tilde{a})] \right). \]
\[ \tilde{\rho}(n) = \tilde{\mu}(n) + \frac{r}{3} \tilde{\mu}'(n) + W_3(n), \]

\[ \tilde{H}(n) = -\frac{1}{2(1 + A_n)} [A_n(F(n) + W_1(n)) + W_2(n)], \]

\[ \tilde{\alpha}(n) = -\frac{\gamma}{1 + \gamma} \left( \tilde{\mu}(n) + \frac{r}{3} \tilde{\mu}'(n) \right) + W_4(n), \]

\[ \hat{R}(n) = \frac{1}{(1 + 3\gamma)n} (\tilde{\alpha}(n) + \tilde{H}(n) + W_5(n)), \]

\[ \hat{K}(n) = -\frac{2\gamma}{(1 + \gamma)(1 + 3\gamma)n} (r \tilde{\rho}'(n) + W_6(n)), \]

\[ \tilde{R}(n) = \hat{R}(n) + \frac{1}{n} S^*_n [\hat{R}\hat{R}], \]

\[ \tilde{K}(n) = \hat{K}(n) + \frac{1}{n} S^*_n [\hat{K}\hat{K}]. \]
\[ K_i(r) = \left[1 + \frac{B}{2} \left(\frac{r}{\sigma}\right)^2\right] \exp \left[-\frac{1}{2} \left(\frac{r}{\sigma}\right)^2\right], \]

B=0 \rightarrow \text{smoother}

\sigma=0.7 \rightarrow \text{wider}

B=1 \rightarrow \text{sharper}

\sigma=0.7 \rightarrow \text{wider}
\[ K_i(r) = \left[ 1 + \frac{B}{2} \left( \frac{r}{\sigma} \right)^2 \right] \exp \left[ -\frac{1}{2} \left( \frac{r}{\sigma} \right)^2 \right], \]
\[ \delta(t, r_{od}) = 0 \]

\[ r_i \text{ is defined as } r_{od} \]

at an sufficiently early time.

\[ \delta_{(1)}(t, r_i) = \frac{(1+\gamma)r_i^2(3K_i(r)+rK_i'(r))}{5+3\gamma} \quad \varepsilon = 0 \]
time evolution of density pert. profile

\((B, \sigma) = (1, 0.7), 7\text{th order expansion is used}\)

\[\delta(t, r) \equiv \frac{\rho(t, r) - \rho_0(t)}{\rho_0(t)} = \tilde{\rho}(t, r) - 1,\]

\(\delta > 0: \text{overdense}\)

\(\delta < 0: \text{underdense}\)
\[(B, \sigma) = (1, 0.7)\]

\[
\int_0^\infty 4\pi \delta R^2 dR = 0,
\]

\[\delta > 0, \text{ overdense}\]

\[\delta < 0, \text{ underdense}\]
Time evolution of averaged overdensity

\[
\bar{\delta}(t) \equiv \left( \frac{4}{3} \pi R(t, r_{od})^3 \right)^{-1} \int_0^R R(t, r_{od}) \, 4\pi \delta R^2 \, dR.
\]

A wider, steeper profile is more likely to collapse to a PBH.