Adiabatic evolution of resonant orbits on Kerr space time

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Extreme Mass Ratio Inspiral

A compact object inspirals into a massive black hole: Promising sources of gravity waves (Tests GR etc...)

Required: understand the evolution of general orbits on a Kerr black hole

Kerr $\approx O(10^6)M_\odot$

Compact object
$\approx O(10)M_\odot$ [from S.Drasco]

Property of Kerr geodesics

1. Three constants of motion

\[ E = -u^\alpha \xi^{(t)}_{\alpha} \]
\[ L_z = -u^\alpha \xi^{(\phi)}_{\alpha} \]
\[ Q = K_{\alpha\beta} u^\alpha u^\beta \quad \text{Carter constant} \]

2. Bi-periodic: $(\Upsilon_r, \Upsilon_{\theta})$

\[ \left( \frac{d r}{d \lambda} \right)^2 = R(r) \]
\[ \left( \frac{d \theta}{d \lambda} \right)^2 = \Theta(\theta) \]
**Inspirals in “resonance”**

There is a special geodesic: **resonant orbit**

\[
\frac{\gamma_r}{\beta_r} = \frac{\gamma_\theta}{\beta_\theta} := \gamma
\]

The two orbital frequencies are **no longer independent**

The dependence appears in the difference between the origin of \( r \) and \( \theta \) motion:

\[
\Delta \lambda := \lambda_{\theta 0} - \lambda_{r 0}
\]

\( \Delta \lambda = 0 \) : Non resonance

How to understand the radiation reaction effects?
Adiabatic approximation

(Orbital period) \( \ll \) (Timescale of radiation reaction)

\[ \approx O(M) \quad \approx O(M^2/\mu) \]

Leading orbital evolution with radiation reaction:

Long time averaged change: \( I^i := \{E, L, Q\} \)

Expand in the Fourier series:

\[
\langle F(\lambda) \rangle := \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{+T} d\lambda' F(\lambda')
\]

\[
\frac{dI^i}{d\lambda} = \left\langle \frac{dI^i}{d\lambda} \right\rangle + \sum_{(k,n) \neq (0,0)} J_{k,n}^{(i)} (I^i [\lambda]) e^{i(k \gamma_r + n \gamma_\theta) \lambda}
\]

Accumulate for long time \hspace{1cm} Oscillatory: not accumulate
For resonant orbits:

There is only one common frequency

\[
\frac{\gamma_r}{\beta_r} = \frac{\gamma_\theta}{\beta_\theta} := \gamma
\]

\[
\frac{dI^i}{d\lambda} = \left\langle \frac{dI^i}{d\lambda} \right\rangle_{\hat{N}=0} + \sum_{\hat{N} \neq 0} J_N^{(i)} (I^i [\lambda]) e^{iN\gamma \lambda}
\]

Accumulate for long time

Oscillatory

\[
\hat{N} := \{(k, n)| \beta_r k + \beta_\theta n = N\} \quad N = 0 \text{ even if } (k, n) \neq 0
\]

The term accumulating for long time is different

Resonant orbits evolve quite differently.

Calls for better understandings of resonance
Evolution of consts. of motion

- **Conservation laws** with global Killing vectors

\[
\frac{dE}{dt} = - \frac{dE^{(GW)}}{dt}
\]

\[
E^{(GW)} := \int d\Sigma^\mu t^{(GW)}_{\mu \nu} \xi^\nu(t)
\]

\[
\left( \frac{dE}{dt}, \frac{dL}{dt} \right)
\]

Easy to compute via GWs flux.

- There is no “Carter constant GW flux” thus…

\[
\frac{dQ}{d\lambda} = 2 K^\nu_\mu u^\mu f^{(ret)}_\nu
\]

Needs Gravitational self-forces. (Singular at particle location)

Impossible to practical calculation for Kerr orbit, But…
Averaged value is still calculable in simple manner if the orbit is NOT in resonance

\[
\left\langle \frac{dQ}{dt} \right\rangle = 2 \left\langle \frac{(r^2 + a^2)P(r)}{\Delta(r)} \right\rangle \left\langle \frac{dE}{dt} \right\rangle - 2 \left\langle \frac{aP(r)}{\Delta(r)} \right\rangle \left\langle \frac{dL_z}{dt} \right\rangle + 2 \sum_{\#} \frac{k\Omega_{r}}{\omega_{\#}} |Z_{\#}|^2
\]

\[P(r) = E(r^2 + a^2) - aL\]

Asymptotic amplitudes of GWs. at the infinity and at the horizon

Our goal: Deduce the similar formula to compute the long time averaged evolution of the Carter constant even when the orbit is in resonance.
Simplify the discussion, use the scalar toy model as the first attempt.
Mode decomposition

- Scalar field equation (separable):

\[
\Box \Phi(x) := \mathcal{T}(x) \quad \mathcal{T}(x) := -q \int_{-\infty}^{+\infty} d\tau \frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g(z(\tau))}}
\]

\[
\pi_{\omega \ell m}(t, r, \theta, \phi) := \frac{2}{\sqrt{r^2 + a^2}} e^{-i\omega t} e^{im\phi} \theta_{\omega \ell m}(\theta) u_{\omega \ell m}(r)
\]

A bound orbit only excites discretized frequencies.

\[
\omega_{mN} := m\Omega_\phi + N\Omega
\]

\[
\frac{\Omega_r}{\beta_r} = \frac{\Omega_\theta}{\beta_\theta} =: \Omega
\]

Deduce the retarded solution via discrete mode sum

\[
\Phi_{\text{ret}}(x) = \int d^4x' \sqrt{-g(x')} G^{(\text{ret})}(x, x') \mathcal{T}(x')
\]

\[
\Phi_{\text{ret}}(x) = \frac{1}{16\pi i} \sum_{\hat{N}=-\infty}^{+\infty} \sum_{\ell=0}^{+\infty} \sum_{m=-\infty}^{+\infty} \frac{\omega_{mN}}{|\omega_{mN}|} Z_{\ell m N}^\text{in} (\Delta \lambda) \pi_{\ell m N}^\text{up} (x)
\]

\( r > r_{\text{max}} \)

Amplitude depends on an orbit Mode function
Evolution of the Carter constant

- We return to the definition of the Carter constant

\[ \frac{dQ}{d\lambda} = 2\Sigma K^\nu_\mu u^\mu f^{(\text{ret})}_\nu \quad f^\mu := \frac{q}{\mu} (g^{\mu\nu} + u^\mu u^\nu) \Phi ;_\nu \]

\[ d\lambda := d\tau / \Sigma \]

With geodesic equations

\[ \frac{dQ}{d\lambda} = \left[ \left( -\frac{P(r)}{\Delta} \{ (r^2 + a^2) \partial_t + a \partial_\phi \} - \frac{dr}{d\lambda} \partial_r \right) \Sigma(x) \Phi^{(\text{ret})}(x) \right] \]

\[ P(r) = E(r^2 + a^2) - aL \quad \Delta = r^2 - 2Mr + a^2 \quad \Sigma := r^2 + a^2 + \cos^2 \theta \]

Note: the retarded field is singular at the particle location
Radiative and symmetric field

With the advanced field, decompose the retarded field into two pieces: radiative and symmetric field

\[ \Phi_{\text{ret}}(x) := \Phi_{\text{rad}}(x) + \Phi_{\text{sym}}(x) \]

\[ \Phi_{\text{rad}} := \frac{1}{2} (\Phi_{\text{ret}} - \Phi_{\text{adv}}) \quad \Phi_{\text{sym}} := \frac{1}{2} (\Phi_{\text{ret}} + \Phi_{\text{adv}}) \]

Singular structure at particle location is common both in retarded and advanced field.

The radiative field is regular everywhere.
Radiation reaction formula

The part from radiation field is essentially the same expression as the non resonant case, if \( \Delta \lambda = 0 \)

\[
\Phi_{\text{rad}} := \frac{1}{2} (\Phi_{\text{ret}} - \Phi_{\text{adv}})
\]

Calculable with asymptotic amplitude of scalar waves

1. Radiation field part (Regular piece)

\[
\left< \frac{dQ}{dt} \right>_{(\text{rad})} = 2 \left< \frac{(r^2 + a^2)P(r)}{\Delta} \right> \left< \frac{dE}{dt} \right> - 2 \left< \frac{aP(r)}{\Delta} \right> \left< \frac{dL_z}{dt} \right> + \frac{\gamma_r}{64 \pi^2 \mu} \sum_Y \frac{\omega_{mN}}{\omega_{mN}} \left\{ Z_{\ell m N}^{\text{out}}(\Delta \lambda) Y_{\ell m N}^{*}(\Delta \lambda) + \frac{\omega_{mN} p_m N}{\omega_{mN} p_m N} \left| \tau_{\ell m N} \right|^2 Z_{\ell m N}^{\text{down}}(\Delta \lambda) Y_{\ell m N}^{\text{down}*}(\Delta \lambda) \right\}
\]

\[
Y_{\ell m N}(\Delta \lambda) := \sum_{\tilde{N}} k Z_{\ell m N}^{b}(\Delta \lambda)
\]

\( \tilde{N} := \{(k, n) \mid \beta_r k + \beta_\theta n = N\} \)

Some terms are modified fitting to the resonant case
In resonant case, the contribution from symmetric field exists.

\[
\Phi_{\text{sym}} := \frac{1}{2} (\Phi_{\text{ret}} + \Phi_{\text{adv}})
\]

Diverges at particle location.

2. Symmetric field part (Singular piece)

\[
\left< \frac{dQ}{dt} \right>_{t}^{(\text{sym})} = -\lim_{\epsilon \to 0} \frac{1}{128i\pi^2 \mu} \int d(\Delta \lambda) \sum_{\hat{N} = -\infty}^{+\infty} \sum_{\ell = 0}^{+\infty} \sum_{m = -\ell}^{+\ell} \frac{\omega_{mN}}{|\omega_{mN}|} \omega_{mN} := m\Omega_\phi + N\Omega
\]

\[
\times \{ Z_{\ell m N}^{\text{out}} (\Delta \lambda; \epsilon/2) Z_{\ell m N}^{\text{up}*} (\Delta \lambda; -\epsilon/2) + Z_{\ell m N}^{\text{down}} (\Delta \lambda; \epsilon/2) Z_{\ell m N}^{\text{in}*} (\Delta \lambda; -\epsilon/2) \}
\]

Only makes sense in the resonance case. \((\Delta \lambda \neq 0)\)
Avoid divergence, introduced point splitting regularization

\[ z^\mu(\lambda) \rightarrow z^\mu(\lambda) \pm \frac{\epsilon}{2} \xi^\mu(\zeta) \]

\[ \xi^\mu(\zeta) := \cos \zeta \xi^{(t)\mu} + (\Omega_\phi \cos \zeta - \Omega_t \sin \zeta) \xi^{(\phi)\mu} \]

We can factorize the regularization terms.

\[ Z_{\ell m N}^b(\Delta \lambda; \pm \epsilon/2) := e^{\pm i(\cos \zeta N + \sin \zeta m) \Omega \epsilon/2} Z_{\ell m N}^b(\Delta \lambda) \]

Rewrite symmetric part as the double Fourier series.

\[ \Psi^{(\text{sym})}(\Delta \lambda; \epsilon) = \sum_{Nm} e^{i(\epsilon_1 N + \epsilon_2 m) \Omega} \Psi^{(\text{sym})}_{Nm}(\Delta \lambda) \]

\[ \epsilon_1 = \epsilon \cos \zeta \]

\[ \epsilon_2 = \epsilon \sin \zeta \]
Mode sum Regularizations

Read out physical information, we also need to subtract the singular portion from the symmetric field.

\[
\Psi^{(P)}(\Delta \lambda; \epsilon) := \Psi^{(\text{Sym})}(\Delta \lambda; \epsilon) - \frac{\psi(\zeta)}{\sqrt{\epsilon_1^2 + \epsilon_2^2}} + O(\epsilon)
\]

The singular portion can be “smeared” by inverse above Fourier transformation

\[
\Psi_{Nm}(\Delta \lambda) = \frac{\Omega^2}{4\pi^2} \int_{-\frac{\pi}{\Omega}}^{\frac{\pi}{\Omega}} d\epsilon_1 \int_{-\frac{\pi}{\Omega}}^{\frac{\pi}{\Omega}} d\epsilon_2 e^{-i(\epsilon_1 N + \epsilon_2 m)\Omega} \Psi(\Delta \lambda; \epsilon)
\]

\[
\Psi^{(P)}_{Nm}(\Delta \lambda) := \Psi^{(\text{Sym})}_{Nm}(\Delta \lambda) - \Psi^{(S)}_{Nm}(\Delta \lambda)
\]

We can subtract mode by mode, which are regular.
Summary

- We derive the formula for the long time averaged evolution of the Carter constant, applicable to a resonant orbit.

- In the resonant case, the symmetric field also contributes the evolution of the Carter constant.

- Despite the divergence in the symmetric field, we can control it via mode sum regularization.
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お疲れ様でした。
Evolution of consts. of motion

[ Mino (2003), Sago+ (2005), Drasco+ (2005)]

- Conservation laws with global Killing vectors

\[ \frac{dE}{dt} = - \frac{dE^{(GW)}}{dt} \]

\[ E^{(GW)} := \int d\Sigma^\mu t^{(GW)}_{\mu\nu} \xi^\nu_{(t)} \]

\[ (E, L) \text{ must always balance to GWs flux.} \]

- For the Carter constant, there is no Killing vector…

\[ \frac{dQ}{d\lambda} = 2 K^\nu_\mu u^\mu f^{(ret)}_{\nu} \text{ Needs Gravitational self-forces.} \]

(Singular at particle location)

Impossible to practical calculation for Kerr orbit, But…
Radiation reaction formula

Retarded force and radiative force give the same long time average of the change of the Carter constant if and only if the orbit is off resonance.

since the orbit is mapped to itself: $ (t, r, \theta, \phi) \rightarrow (-t, r, \theta, -\phi) $

$$ f_{\mu}^{(\text{rad})} := \frac{1}{2} \left( f_{\mu}^{(\text{ret})} - f_{\mu}^{(\text{adv})} \right) $$

Regular at particle location

$$ \langle \frac{dQ}{dt} \rangle = 2 \left( \frac{r^2 + a^2}{\Delta(r)} \right) \langle \frac{dE}{dt} \rangle - 2 \left( \frac{a}{\Delta(r)} \right) \langle \frac{dL_z}{dt} \rangle + 2 \sum_{\#} \frac{k_i \Omega_r}{\omega_{\#}} |Z_{\#}|^2 $$

Asymptotic amplitudes of GWs.

Numerically calculable formula [Drasco+(2005), Fujita+(2009)]