Stable Bound Null Orbit around a Black Ring

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Introduction

Higher-dim. Black Hole is a key for a verification of extra dimensions?

\[ D = 4 \]

Kerr Black Hole

\[ D = 5 \]

Black Hole

Myers & Perry (1986)

\[ S^2 \times S^1 \]

Black Ring

Emparan & Reall (2002)
Geodesics around a Black Hole

Geodesic particles are important probes of gravitational field around a black hole.

\[ D = 4 \]
- ISCO appears for timelike particles
- Unstable circular orbits exist for null particles
- Geodesic equations in Kerr geometry are separable

\[ D = 5 \]
For spherical black holes (the Myers-Perry metric)
- No stable circular orbit exists for timelike particles
- Unstable circular orbits exist for null and timelike particles
- Geodesic equations are separable

For black rings (the Emparan-Real metric)
- Stable stationary orbits exist for timelike particles
- Geodesic equations would not be separable

S.Grunau, V.Kagramanova, J.Kunz, C.Lammerzahl(2012)

V.P.Frolov and D.Stojkovic(2003)
## Stable Bound Orbits

Orbits stable against small perturbations, and bounded in a finite domain outside the black hole horizon.

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4–dim. Schwarzschild BH case

Effective potential:

\[ V_4(r; L) = \frac{L^2}{r^2} - \frac{M}{r} - \frac{ML^2}{r^3} \]

Diagram showing \( V_4(r; L) \) with curves indicating behavior as \( r \) changes.
Particle Motion around a Black Hole

D–dim. Schwarzschild BH case

Effective potential: \[ V_D(r; L) = \frac{L^2}{r^2} - \frac{M}{r^{D-3}} - \frac{ML^2}{r^{D-1}} \]

\[ V_4(r) \propto \frac{L^2}{r^2} \]
\[ V_5(r) \propto \frac{L^2}{r^2} \]
\[ \propto -\frac{M}{r} \]
\[ \propto -\frac{M}{r^2} \]

No stable bound orbits for \( D \geq 5 \)
**Stationary Points Set for Effective Potential**

**Effective potential for D-dim. Schwarzschild BH**

\[ V_D(r; L) = \frac{L^2}{2r^2} - \frac{M}{r^{D-3}} - \frac{ML^2}{r^{D-1}} \]

**Stationary condition**

\[ \partial_r V_D(r; L) = -\frac{L^2}{r^3} + (D-3)\frac{M}{r^{D-2}} + (D-1)\frac{ML^2}{r^D} = 0 \]

**Stationary Points Set**

\[ L(r) = \sqrt{\frac{(D - 3)Mr^2}{r^{D-3} - (D - 1)M}} \]

![Graphs](image)

**Local max.**  
**Local min.**  
**ISCO**  
**D = 4**  
**D = 5**  

*no stable bound orbits*
How about

Black Ring?

**5D Singly Rotating Black Ring**

**Metric**

\[
ds^2 = -\frac{F(y)}{F(x)} \left( dt - CR \frac{1 + y}{F(y)} d\psi \right)^2 + \frac{R^2}{(x - y)^2} F(x) \left( -\frac{G(y)}{F(y)} d\psi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)} d\phi^2 \right),
\]

\[
F(\xi) = 1 + \lambda \xi, \quad G(\xi) = (1 - \xi^2)(1 + \nu \xi), \quad C = \sqrt{\lambda(\lambda - \nu) \frac{1 + \lambda}{1 - \lambda}}, \quad \lambda = \frac{2\nu}{1 + \nu^2}
\]

**Free parameters**

- \( R \) : ring radius
- \( \nu \) : thickness \( 0 < \nu < 1 \)

**Killing vectors**

\( \partial_t, \quad \partial_\phi, \quad \partial_\psi \)

**Horizon topology**

\( S^2 \times S^1 \)
Coordinates for the Black Ring

(t, \Phi, \Psi) are suppressed.
(\rho, \Phi), (\zeta, \Psi): polar coordinates on the independent two planes, respectively.
Hamiltonian formalism for a Null particle

Constants of motion

\[ E = -k_t, \quad L_\phi = k_\phi, \quad L_\psi = k_\psi \]

Hamiltonian

\[ H = \frac{1}{2} g^{\alpha\beta} k_\alpha k_\beta \]

\[ = \frac{1}{2} \left( g^{\xi\xi} k_\xi^2 + g^{\rho\rho} k_\rho^2 + E^2 U \right) = 0, \]

Effective Potential

\[ U(\xi, \rho; l_\psi, l_\phi) = g^{tt} + g^{\phi\phi} l_\phi^2 + g^{\psi\psi} l_\psi^2 - 2 g^{t\psi} l_\psi \]

\[ (l_\phi := \frac{L_\phi}{E}, \quad l_\psi := \frac{L_\psi}{E}) \]
Stationary Orbits

Stationary Solutions

\[
\dot{\rho} = \dot{\zeta} = 0 \quad \longrightarrow \quad k_\rho = k_\zeta = 0
\]

\[
\longrightarrow \quad \partial_\zeta U(\zeta, \rho; l_\psi, l_\phi) = \partial_\rho U(\zeta, \rho; l_\psi, l_\phi) = 0
\]

\[
\longrightarrow \quad l_\psi = l_\psi(\zeta, \rho), \quad l_\phi = l_\phi(\zeta, \rho)
\]

Stationary Points Set

\[
\Sigma = \{(\zeta, \rho, l_\psi(\zeta, \rho), l_\phi(\zeta, \rho))\}
\]

defines 2-dimensional surface embedded in the 4-dimensional space

\[
\mathcal{N} = \{(\zeta, \rho, l_\psi, l_\phi)\}
\]

Null condition for the stationary orbits is \(U|_\Sigma = 0\)
Stability conditions

Local minimum of $U(\zeta, \rho; l_\psi, l_\phi)$ \iff Stable Stationary orbit

Two eigenvalues of Hessian matrix

$$\mathcal{H}(U) = \begin{pmatrix}
\partial_\zeta^2 U & \partial_\zeta \partial_\rho U \\
\partial_\rho \partial_\zeta U & \partial_\rho^2 U
\end{pmatrix}$$

are positive at the stationary point.

$$\det \mathcal{H}(U) \big|_\Sigma > 0 \quad \text{and} \quad \text{tr} \mathcal{H}(U) \big|_\Sigma > 0$$
Existence of Stable Stationary Orbits

Projection of \( \Sigma \) into the \( \zeta - \rho \) plane for \( \nu = 0.1 \)
The stable stationary orbit is tangent to a null Killing vector
\[ \partial_t + \alpha(l_\psi, l_\phi)\partial_\psi + \beta(l_\psi, l_\phi)\partial_\phi. \]
The projection of a orbit on a \( t = \text{const} \) surface is a toroidal spiral on \( S^1 \times S^1 \).
Critical Thickness

$\nu = 0.1 \quad \nu = 0.13224 \simeq \nu_c \quad \nu = 0.2$

$p$ : Innermost Stable Toroidal Spiral Orbit (ISTSO)
$q$ : Outermost Stable Toroidal Spiral Orbit (OSTSO)
Cusps

Projection of \( \Sigma \) into the \( l_\psi - l_\phi \) plane for \( \nu = 0.1 \)
Stationary Points Set

2-dim. surface embedded in 4-dimensional space

\[ \det \mathcal{H}|_{\Sigma} = 0 \]
Stationary Points
Stationary Points
Bounded Null Orbits

Non stationary bounded orbits appear
Summary

Stable bound null orbits exist around a Black Ring.

for $\nu \leq \nu_c = 0.13224 \ldots$

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