NEW COSMOLOGICAL SOLUTIONS IN MASSIVE GRAVITY

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OUTLINE

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2. Cosmological solutions
3. Generalization of the solutions
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MOTIVATION OF MASSIVE GRAVITY

Our universe is accelerating now!!

- introduce Dark energy
- modify general relativity

Massive gravity is one of modified gravity theory
Linear theory of massive graviton

\[ S = \frac{M_{PL}^2}{2} \int d^4x \left[ -\frac{1}{4} h_{\mu\nu,\rho} h^{\mu\nu,\rho} + \frac{1}{2} h_{\nu\rho,\mu} h^{\mu\rho,\nu} - \frac{1}{2} h_{,\nu} h^{\mu\nu,\mu} + \frac{1}{2} h_{,\mu} h^{,\mu} - \frac{1}{4} m_g^2 (h_{\mu\nu} h^{\mu\nu} - h^2) \right] \]

Linearized Einstein-Hilbert action

**Graviton mass term**

\[ h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} \]

Even if graviton mass is very small, the theory does not agree with linearized general relativity.


This discontinuity disappear in non-linear theory.

Vainshtein (1972)
in generally, non-linear massive gravity have a ghost mode.

Boulware, Deser (1972)

Recently, ghost free generalization was realized

de Rham, Gabadadze, Tolley (2011)
**GHOST FREE MASSIVE GRAVITY**

de Rham, Gabadadze, Tolley (2011)

\[
S_{mass} = \frac{M_{PL}^2}{2} \int d^4x \sqrt{-g} m_g^2 (U_2 + \alpha_3 U_3 + \alpha_4 U_4)
\]

\(U_2, U_3, U_4\) are combination that ghost mode disappear

\[
U_2 = \left[K\right]^2 - \left[K\right]^2 \\
U_3 = \left[K\right]^3 - 3\left[K\right]\left[K^2\right] + 2\left[K^3\right] \\
U_4 = \left[K\right]^4 - 6\left[K^2\right]\left[K^2\right] + 8\left[K^3\right]\left[K\right] + 3\left[K^2\right]^2 - 6\left[K^4\right]
\]

Here, \(K_{\mu\nu}\) is defined by fiducial metric

\[
K_{\nu}^{\mu} = \delta_{\nu}^{\mu} - \sqrt{g^{\mu\rho} \Sigma_{\rho\nu}} \\
\Sigma_{\mu\nu} = \partial_{\mu} \phi^a \partial_{\nu} \phi^b \eta_{ab} : \text{fiducial metric} \\
\phi^a : \text{Stuckelberg scalar fields} \\
\text{ (auxiliary fields introduced to recover general covariance)}
\]

**the form of** \(\Sigma_{\mu\nu}\) **is**

**the component of** Minkowski space time in certain coordinate
EQUATION OF MOTION

The equation of motion is the usual Einstein equation plus mass term:

\[ G_{\mu\nu} + \Lambda g_{\mu\nu} + m_g^2 X_{\mu\nu} = \frac{1}{M_{PL}^2} T_{\mu\nu} \]

where

\[ X_{\mu\nu} = -\frac{1}{2} g_{\mu\nu} U + \mathcal{K}_{\mu\nu} - [\mathcal{K}] g_{\mu\nu} + [\mathcal{K}] \mathcal{K}_{\mu\nu} - \mathcal{K}_{\mu\nu}^2 \]

\[ + \alpha_3 \left( -\frac{3}{2} ([\mathcal{K}]^2 - [\mathcal{K}^2])(g_{\mu\nu} - \mathcal{K}_{\mu\nu}) + 3[\mathcal{K}](\mathcal{K}_{\mu\nu} - \mathcal{K}_{\mu\nu}^2) - 3(\mathcal{K}_{\mu\nu}^2 - \mathcal{K}_{\mu\nu}^3) \right) \]

\[ + \alpha_4 \left( -2([\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3])(g_{\mu\nu} - \mathcal{K}_{\mu\nu}) + 6([\mathcal{K}]^2 - [\mathcal{K}^2])(\mathcal{K}_{\mu\nu} - \mathcal{K}_{\mu\nu}^2) \]

\[ - 12[\mathcal{K}](\mathcal{K}_{\mu\nu}^2 - \mathcal{K}_{\mu\nu}^3) + 12(\mathcal{K}_{\mu\nu}^3 - \mathcal{K}_{\mu\nu}^4) \right) \]

\[ \mathcal{K}_{\nu}^\mu = \delta_{\nu}^\mu - \sqrt{g^{\mu\rho} \Sigma_{\rho\nu}} \]

Mass term is calculated from two metric \( g_{\mu\nu}, \Sigma_{\mu\nu} \) and free parameter \( \alpha_3, \alpha_4 \).
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ABSENCE OF FLAT FLRW SOLUTION

D’Amico , de Rham, Dubovsky, Gabadadze (2011)

Metric ansatz:

$g_{\mu\nu} : \text{Flat FLRW}$

$$g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$$

$\Sigma_{\mu\nu} : \text{Flat FLRW chart of Minkowski space time}$

$$\Sigma_{\mu\nu}dx^\mu dx^\nu = -f^2(t)dt^2 + \delta_{ij}dx^i dx^j$$

$$\text{Equation of motion obtained from the variation of}$$

$$f$$

$$\begin{align*}
a &= \text{const}
\end{align*}$$

Non trivial flat FLRW solution is not exist !!
In this analysis, we impose both metric are homogeneous and isotropic.

Fiducial metric $\Sigma_{\mu\nu}$ is not need to have the same symmetry with physical metric.

If we abandon homogeneity or isotropy of fiducial metric $\Sigma_{\mu\nu}$, flat FLRW solution can be exist!!

Actually, de Sitter solution was found!

Berezhiani, Chkareuli, de Rham, Gabadadze, Tolley (2012)
DE SITTER SOLUTION (1)
Berezhiani, Chkareuli, de Rham, Gabadadze, Tolley (2012)

They described de Sitter space time by Painleve-Gullstrand coordinate.

\[ g_{\mu\nu} dx^\mu dx^\nu = -\kappa^2 dt^2 + \tilde{\alpha}^2 (dr \pm \kappa m r dt)^2 + \tilde{\alpha}^2 r^2 d\Omega^2 \]

What is Painleve-Gullstrand coordinate?

This coordinate was used by Painleve and Gullstrand to describe Schwarzschild space time.

\[ g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + \left( dr + \sqrt{\frac{2M}{r}} dt \right)^2 + r^2 d\Omega^2 \]

Extending to r depended Schwarzschild mass, de sitter space time can express this coordinate.

\[ M \rightarrow M(r) = m^2 r^3 / 2 \quad t \rightarrow \kappa t, \quad r \rightarrow \tilde{\alpha} r \]

\[ g_{\mu\nu} dx^\mu dx^\nu = -\kappa^2 dt^2 + \tilde{\alpha}^2 (dr \pm \kappa m r dt)^2 + \tilde{\alpha}^2 r^2 d\Omega^2 \]
\( g_{\mu\nu} \): De Sitter Space time in Painleve-Gullstrand coordinate

\[ g_{\mu\nu} dx^\mu dx^\nu = -\kappa^2 dt^2 + \tilde{\alpha}^2 (dr \pm m r dt)^2 + \tilde{\alpha}^2 r^2 d\Omega^2 \]

\[ \Sigma_{\mu\nu} = \eta_{\mu\nu} \]

Particular parameters choice:

\[ \alpha_3 = \frac{1}{3} (\alpha - 1), \alpha_4 = \frac{1}{12} (\alpha^2 - \alpha + 1), \alpha = \frac{\tilde{\alpha}}{1 - \tilde{\alpha}} \]

\[ m_g^2 X_{\mu\nu} = \frac{m_g^2}{\alpha} g_{\mu\nu} \]

\[ \begin{pmatrix} \end{pmatrix} \]

Equation of motion

\[ G_{\mu\nu} + \Lambda g_{\mu\nu} + m_g^2 X_{\mu\nu} = \frac{1}{M_{PL}^2} T_{\mu\nu} \]

Mass term behaves as cosmological constant!

De sitter space time is exact solution in massive gravity!
FROM DE SITTER TO FLRW

key observation:

Not only de Sitter space time, but also FLRW space time can be expressed by Painleve-Gullstrand coordinate

\[ g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + \left( dr + \sqrt{\frac{2M}{r}} dt \right)^2 + r^2 d\Omega^2 \]

Kanai, Siino, Hosoya (2011)

Schwarzschild

\[ M \rightarrow M(t, r) = \left( \frac{\dot{a}}{\kappa a} \right)^2 \frac{r^3}{2} \rightarrow \kappa t, r \rightarrow \tilde{a} r \]

flat FLRW

\[ g_{\mu\nu}dx^\mu dx^\nu = -\kappa^2 dt^2 + \tilde{\alpha}^2 \left( dr - \frac{\dot{a}}{a} r dt \right)^2 + \tilde{\alpha}^2 r^2 d\Omega^2 \]

open, flat, closed FLRW (K=-1,0,+1)
CALCULATION OF MASS TERM

As expected, mass term behaves as cosmological constant!!

Open, flat, closed FLRW metric is the exact solution of massive gravity.
WHY FLAT FLRW EXISTS?

Transforming to familiar coordinate 

\[ t \rightarrow T = \kappa t, \quad r \rightarrow R = \frac{\tilde{\alpha} r}{a} \]

\[
\begin{align*}
\Sigma_{\mu \nu} d\bar{x}^\mu d\bar{x}^\nu &= -dT^2 + a^2 \left( \frac{dR^2}{1 - KR^2} + R^2 d\Omega^2 \right) \\

\sum_{\mu \nu} d\bar{x}^\mu d\bar{x}^\nu &= -\left( \frac{1}{\kappa^2} - \frac{a^2 H^2 R^2}{\tilde{\alpha}^2} \right) dT^2 + 2 \frac{a^2 HR}{\tilde{\alpha}^2} dT dR + \frac{a^2}{\tilde{\alpha}^2} (dR^2 + R^2 d\Omega^2) 
\end{align*}
\]

\[ \Sigma_{\mu \nu} \text{ in this coordinate breaks homogeneity.} \]

\[ \Sigma_{\mu \nu} \text{ does not satisfy D’Amico’s metric ansatz. for this reason , our FLRW solutions can include flat one.} \]
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GENERALIZATION OF OUR SOLUTION (1)

Our analysis can be generalized wider class of solutions

Key observation:

Calculation of mass term dose not include derivations:

\[ X_{\mu \nu} = -\frac{1}{2} g_{\mu \nu} U + K_{\mu \nu} - [K] g_{\mu \nu} + [K] K_{\mu \nu} - K_{\mu \nu}^2 \]

\[ [K] = \text{Tr}K = K_{\mu}^\mu \]

\[ + \alpha_3 \left( -\frac{3}{2} ([K]^2 - [K]^2) (g_{\mu \nu} - K_{\mu \nu}) + 3 [K] (K_{\mu \nu} - K_{\mu \nu}^2) - 3 (K_{\mu \nu}^2 - K_{\mu \nu}^3) \right) \]

\[ + \alpha_4 \left( -2 ([K]^3 - 3 [K] [K]^2] + 2 [K]^3) (g_{\mu \nu} - K_{\mu \nu}) + 6 ([K]^2 - [K]^2) (K_{\mu \nu} - K_{\mu \nu}^2) \]

\[ - 12 [K] (K_{\mu \nu}^2 - K_{\mu \nu}^3) + 12 (K_{\mu \nu}^3 - K_{\mu \nu}^4) \right) \]

\[ K_{\nu}^\mu = \delta_{\nu}^\mu - \sqrt{g_{\mu \rho} \partial_{\rho} \phi^{a} \partial_{\nu} \phi^{b} \eta_{ab}} \equiv \delta_{\nu}^\mu - \sqrt{g_{\mu \rho} \Sigma_{\rho \nu}} \]

Analytic property of \( g_{\mu \nu} \) and \( \Sigma_{\mu \nu} \) is not so important.
GENERALIZATION
OF OUR SOLUTION (2)

\[ g_{\mu\nu}dx^\mu dx^\nu = -\kappa^2 dt^2 + \frac{\tilde{\alpha}^2}{1 - K\tilde{\alpha}^2 r^2 / a^2} \left( dr - \frac{\dot{a}}{a} r dt \right)^2 + \tilde{\alpha}^2 r^2 d\Omega^2 \]

\[ g_{\mu\nu}dx^\mu dx^\nu = -V^2(x) dt^2 + U^2(x)(dr - f(x) dt)^2 + \tilde{\alpha}^2 r^2 d\Omega^2 \]

\[ \Sigma_{\mu\nu} = \eta_{\mu\nu} \quad \alpha_3 = \frac{1}{3}(\alpha - 1), \quad \alpha_4 = \frac{1}{12}(\alpha^2 - \alpha + 1) \]

\[ m_g^2 X_{\mu\nu} = \frac{m_g^2}{\alpha} g_{\mu\nu} \]

As expected, Also in this class of metric, mass term behaves cosmological constant!!
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CONCLUSION

For the metric
\[ g_{\mu \nu} dx^\mu dx^\nu = -V^2(x) dt^2 + U^2(x) (dr + f(x) dt)^2 + \tilde{\alpha}^2 r^2 d\Omega^2, \]
the equation of motion in massive gravity can be the same equation in general relativity, except the value of cosmological constant.

So, exact solutions of this form in general relativity are also exact solutions in massive gravity.

Our series of solutions includes general isotropic metric, especially open, flat, and closed FLRW solutions.

In our cosmological solutions, fiducial metric \( \Sigma_{\mu \nu} \) breaks homogeneity.