CosPA2017@YITP 2017/12/11—12/15

Degenerate Higher-Order Multi-Scalar-Tensor theories

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Motivation

Ostrogradsky stable scalar-tensor theories
(EOMs are at most $\ddot{\phi}$) ↖ ϕ **(single field)**

$$\mathcal{L} = \mathcal{L}(\nabla_\mu \nabla_\nu \phi, \nabla_\mu \phi, \phi; \partial_\rho \partial_\sigma g_{\mu\nu}, \partial_\rho g_{\mu\nu}, g_{\mu\nu})$$

⟶ generally, EOMs are higher derivatives

But some theories' EOMs $\sim \ddot{\phi}$: **degenerate theory**

classify degenerate theories into 2 types:

- "**trivially degenerate**" : EOMs $\sim \nabla\nabla\phi$
  e.g.) Horndeski [Horndeski 1970] [Kobayashi, et al. 2011]
- "**nontrivially degenerate**" : EOMs are higher, but at most $\ddot{\phi}$
  e.g.) GLPV [Gleyzes, et al. 2014]
  DHOST [Langlois, Noui 2015]

…we have been talking about single-scalar theories.

Can we construct some degenerate multi-scalar-tensor theories?

Setup

fields' number

$$\mathcal{L} = \mathcal{L}(\nabla_\mu \nabla_\nu \phi^I, \nabla_\mu \phi^I, \phi^I; \partial_\rho \partial_\sigma g_{\mu\nu}, \partial_\rho g_{\mu\nu}, g_{\mu\nu}) \quad (I = 1, ..., N)$$

we consider Lagrangians which contain

$\nabla\nabla\phi^I, (\nabla\nabla\phi^I)^2, (\nabla\nabla\phi^I)^3,$ ...

the most general Lagrangian (+ Einstein-Hilbert term) is:

$$\mathcal{L} = \sqrt{-g}\left[\frac{{}^{(4)}R}{2} - \mathcal{A}_{(IJ)K}(\phi^L, X^{MN})\nabla_\mu \phi^I \nabla^\nu \phi^J \nabla_\nu \nabla^\mu \phi^K\right]$$

arbitrary function: $\mathcal{A}_{(IJ)K} = \mathcal{A}_{(JI)K}$

kinetic term: $X^{IJ} := -\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi^I \nabla_\nu \phi^J$

if you don't do anything, this EOMs $\sim \nabla\nabla\nabla\phi^I$

⟶ **find degeneracy conditions**
(restrict $\mathcal{A}_{(IJ)K}$ and make EOMs degenerate)

after that, we see how these conditions are appeared in the EOMs

Degeneracy conditions

For single-scalar theories: $\det(\text{kinetic matrix}) = 0$

For multi-scalar theories [Motohashi, et al. 2016] [Crisostomi, et al. 2017]

$$L(\ddot{\phi}^I, \dot{\phi}^I, \phi^I, \dot{q}^\alpha, q^\alpha) = L(\dot{A}^I, A^I, \phi^I, \dot{q}^\alpha, q^\alpha) + \lambda_I(\dot{\phi}^I - A^I)$$

$\dddot{\phi}^I$ can be removed(degenerate) $\longleftrightarrow$ $S_{[IJ]} = 0$

where
$$S_{[IJ]} := \partial_i L_{\partial_i A^I \dot{A}^J} + V_I^\alpha \partial_i L_{\partial_i q^\alpha \dot{A}^J} + \partial_i L_{\partial_i A^I \dot{q}^\beta} V_J^\beta + V_I^\alpha \partial_i L_{\partial_i q^\alpha \dot{q}^\beta} V_J^\beta$$
$$+\partial_i V_J^\beta \left(L_{\partial_i A^I \dot{q}^\beta} + L_{\dot{A}^I \partial_i \dot{q}^\beta} + 2V_I^\alpha L_{\dot{q}^{(\alpha} \partial_i q^{\beta)}} \right)$$
$$+ (L_{\dot{A}^I A^J} - L_{A^I \dot{A}^J}) + V_I^\alpha \left(L_{\dot{q}^\alpha A^J} - L_{q^\alpha \dot{A}^J} \right)$$
$$+ \left(L_{\dot{A}^I q^\beta} - L_{A^I \dot{q}^\beta} \right) V_J^\beta + V_I^\alpha \left(L_{\dot{q}^\alpha q^\beta} - L_{q^\alpha \dot{q}^\beta} \right) V_J^\beta$$
$$\left(L_{\dot{A}^I \dot{A}^J} := \frac{\partial L}{\partial \dot{A}^I \partial \dot{A}^J}, V_I^\alpha := -L_{\dot{A}_I \dot{q}_\beta} L^{-1}_{\dot{q}_\beta \dot{q}_\alpha} \right)$$

in our case,

$$S_{[IJ]} = 2(\mathcal{A}_{(KJ)I} - \mathcal{A}_{(KI)J}) A_*^K + \left(\frac{\partial \mathcal{A}_{(KL)I}}{\partial X^{MJ}} - \frac{\partial \mathcal{A}_{(KL)J}}{\partial X^{MI}} \right) A_*^K A_*^L A_*^M = 0$$

($A_*^I \leftrightarrow \dot{\phi}^I$ **is arbitrary)**

$\longrightarrow$ **degeneracy conditions:**

$$\mathcal{A}_{(KJ)I} = \mathcal{A}_{(KI)J}, \quad \frac{\partial \mathcal{A}_{(KL)I}}{\partial X^{MJ}} = \frac{\partial \mathcal{A}_{(KL)J}}{\partial X^{MI}}$$

EOMs

$\frac{\delta \mathcal{L}}{\delta \phi^I}$ contains $\nabla\nabla\nabla\phi^I \cdots$

- $\sqrt{-g}\nabla_\sigma \phi^J \left[\left(2\mathcal{A}_{(IJ)K} - \mathcal{A}_{(JK)I}\right)\nabla_\mu \nabla^\sigma \nabla^\mu \phi^K - \mathcal{A}_{(JK)I}\nabla^\sigma \nabla_\mu \nabla^\mu \phi^K\right]$

  → $\mathcal{A}_{(JK)I}$ (for $\mathcal{A}_{(IJ)K} = \mathcal{A}_{(JK)I}$)

  $= \sqrt{-g}\nabla_\sigma \phi^J \mathcal{A}_{(JK)I}\left(\nabla_\mu \nabla^\sigma - \nabla^\sigma \nabla_\mu\right)\nabla^\mu \phi^K$

  $= \sqrt{-g}\nabla_\sigma \phi^J \mathcal{A}_{(JK)I} R_\rho{}^\sigma \nabla^\rho \phi^K$

- $\sqrt{-g}\nabla_\mu \phi^M \nabla_\sigma \phi^N \nabla_\delta \phi^J \left[\frac{\partial \mathcal{A}_{(MN)I}}{\partial X^{KJ}}\nabla^\sigma \nabla^\delta \nabla^\mu \phi^K - \frac{\partial \mathcal{A}_{(MN)K}}{\partial X^{IJ}}\nabla^\delta \nabla^\sigma \nabla^\mu \phi^K\right]$

  If $\frac{\partial \mathcal{A}_{(MN)I}}{\partial X^{KJ}} = \frac{\partial \mathcal{A}_{(MN)K}}{\partial X^{IJ}}$ → $\frac{\partial \mathcal{A}_{(MN)I}}{\partial X^{KJ}} R^\mu{}_\rho{}^{\sigma\delta} \nabla^\rho \phi^K$

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degeneracy conditions → all $\nabla\nabla\nabla\phi^I$ are removed

(EOMs $\sim \nabla\nabla\phi^I$)

"**trivially degenerate**"

There are no "nontrivially degenerate" case in linear order of $\nabla\nabla\phi^I$

Quadratic order?

As the next case, we focus on:

$$\mathcal{L} = \mathcal{L}(\nabla_\mu \nabla_\nu \phi^I, \nabla_\mu \phi^I, \phi^I; \partial_\rho \partial_\sigma g_{\mu\nu}, \partial_\rho g_{\mu\nu}, g_{\mu\nu})$$

$$\nabla\nabla\phi^I, (\nabla\nabla\phi^I)^2, (\nabla\nabla\phi^I)^3, \ \ldots$$

$$\mathcal{A}_{(IJ)(KL)} = \mathcal{A}_{(KL)(IJ)}$$

$$\delta^{\mu_1\mu_2\mu_3}_{\nu_1\nu_2\nu_3} = 3!\delta^{[\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2}\delta^{\mu_3]}_{\nu_3}$$

$$\mathcal{L} = \sqrt{-g}\left[\frac{^{(4)}R}{2} + \mathcal{A}_{(IJ)(KL)}(\phi^P, X^{MN})\delta^{\mu_1\mu_2\mu_3}_{\nu_1\nu_2\nu_3}\nabla_{\mu_1}\phi^I\nabla^{\nu_1}\phi^J\nabla_{\mu_2}\nabla^{\nu_2}\phi^K\nabla_{\mu_3}\nabla^{\nu_3}\phi^L\right]$$

$$= N\sqrt{\gamma}\left[2\mathcal{C}^{ij}K_{ij} + 2\mathcal{F}^{ij}_I V^I_* K_{ij} + \mathcal{K}^{ij,kl}K_{ij}K_{kl} + 2\mathcal{C}_I V^I_* - \mathcal{U}\right]$$

ADM form

$$+ \lambda_I(\dot{\phi}^I - NA^I_* - N^i D_i \phi^I) \qquad A^I_* \sim \dot{\phi}^I, \ V^I_* \sim \dot{A}^I_*$$

$$\left(\begin{aligned} \mathcal{K}^{ij,kl} = & \frac{1}{2}(\gamma^{ik}\gamma^{jl} - \gamma^{ij}\gamma^{kl}) \\ & + \mathcal{A}_{(IJ)(KL)}\Big[\left(A^I_m A^m_J - A^I_* A^J_*\right)\left\{A^K_* A^L_*\left(\gamma^{ij}\gamma^{kl} - \gamma^{i(k}\gamma^{l)j}\right) + A^j_K A^{(k}_L \gamma^{l)i} + A^i_K A^{(k}_L \gamma^{l)j}\right\} \\ & + \left\{\left(2A^I_* A^{(i}_J A^{j)}_K A^L_* \gamma^{kl} - A^{(i}_I A^{j)}_J A^K_* A^L_* \gamma^{kl}\right) + (ij) \leftrightarrow (kl)\right\} \\ & + \frac{1}{2}\left(A^I_* A^J_* A^j_K A^{(l}_L \gamma^{k)i} + (i \leftrightarrow j)\right) + \frac{1}{2}\left(A^K_* A^L_* A^j_I A^{(l}_J \gamma^{k)i} + (i \leftrightarrow j)\right) \\ & - \left(A^J_* A^L_* A^j_I A^{(l}_K \gamma^{k)i} + (i \leftrightarrow j)\right) - 2A^i_I A^k_J A^j_K A^l_L\Big] \end{aligned}\right)$$

degeneracy condition: $S_{[IJ]} \ni \left(\mathcal{K}^{ij,kl}\right)^{-1}$ **⋯ difficult to calculate**

→ **we are now trying another approach to get some restrictions**

Quadratic order?

→ cosmological perturbation (work in progress)

scalar perturbations

- spatially flat gauge: $N = 1 + \delta N, \quad N_i = \partial_i \chi, \quad \gamma_{ij} = a^2 \delta_{ij}$
- scalar fields: $\phi^I(t, \boldsymbol{x}) = \bar{\phi}^I(t) + Q^I(t, \boldsymbol{x})$

N+2 scalars

in order not to propagate extra scalar modes, we restrict the arbitrary functions

in the same way to the quadratic DHOST,

$$\mathcal{L} = \sqrt{-g}\left[f^{(4)} R + C_{IJ}^{\mu\nu\rho\sigma} \nabla_\mu \nabla_\nu \phi^I \nabla_\rho \nabla_\sigma \phi^J\right]$$

$$\left(\begin{aligned} C_{IJ}^{\mu\nu\rho\sigma} = & \frac{1}{2}\alpha_{1,IJ}\left(g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right) + \alpha_{2,IJ}g^{\mu\nu}g^{\rho\sigma} + \frac{1}{2}\alpha_{3,IJKL}\left(\nabla^\mu\phi^K\nabla^\nu\phi^L g^{\rho\sigma} + \nabla^\rho\phi^K\nabla^\sigma\phi^L g^{\mu\nu}\right) \\ & + \frac{1}{8}\alpha_{4,IJKL}\left(\nabla^\mu\phi^K\nabla^\rho\phi^L g^{\nu\sigma} + \nabla^\nu\phi^K\nabla^\rho\phi^L g^{\mu\sigma} + \nabla^\mu\phi^K\nabla^\sigma\phi^L g^{\nu\rho} + \nabla^\nu\phi^K\nabla^\sigma\phi^L g^{\mu\rho} \right. \\ & \left. + (K \leftrightarrow L)\right) + \frac{1}{6}\alpha_{5,IJKLMN}\left(\nabla^\mu\phi^K\nabla^\nu\phi^L\nabla^\rho\phi^M\nabla^\sigma\phi^N + \nabla^\mu\phi^K\nabla^\nu\phi^M\nabla^\rho\phi^L\nabla^\sigma\phi^N \right. \\ & + \nabla^\mu\phi^K\nabla^\nu\phi^N\nabla^\rho\phi^M\nabla^\sigma\phi^L + \nabla^\mu\phi^M\nabla^\nu\phi^N\nabla^\rho\phi^K\nabla^\sigma\phi^L + \nabla^\mu\phi^L\nabla^\nu\phi^N\nabla^\rho\phi^K\nabla^\sigma\phi^M \\ & \left. + \nabla^\mu\phi^M\nabla^\nu\phi^L\nabla^\rho\phi^K\nabla^\sigma\phi^N\right) \\ & \qquad \alpha = \alpha(\phi^I, X^{JK}) \end{aligned}\right)$$

→ **some restrictions on α-s?**

Summary

Can we construct some degenerate multi-scalar-tensor theories?

① $\boxed{(\nabla\nabla\phi^I)^1}$ the most general degenerate Lagrangian is

$$\mathcal{L} = \sqrt{-g}\left[\frac{{}^{(4)}R}{2} - \mathcal{A}_{(IJ)K}(\phi^L, X^{MN})\nabla_\mu\phi^I\nabla^\nu\phi^J\nabla_\nu\nabla^\mu\phi^K\right]$$

with $\mathcal{A}_{(IJ)K} = \mathcal{A}_{(JK)I}, \frac{\partial\mathcal{A}_{(MN)I}}{\partial X^{KJ}} = \frac{\partial\mathcal{A}_{(MN)K}}{\partial X^{IJ}}$

(EOMs $\sim \nabla\nabla\phi^I$)

There are no "nontrivially degenerate" case in linear order of $\nabla\nabla\phi^I$

② $\boxed{(\nabla\nabla\phi^I)^2}$ difficult to calculate $S_{[IJ]} = 0$

⟶ cosmological perturbation

(work in progress)

back up

The linear Lagrangian

linear of $\nabla\nabla\phi^I$: $\nabla_\mu\nabla_\nu\phi^I$ or $\Box\phi^I$

Lagrangian is a scalar

$\bigcirc\,\Box\phi^I$

scalar

↓

$$G_{3I}(\phi^J, X^{KL})\Box\phi^I$$

integration by parts

$\bigcirc^{\mu\nu}\,\nabla_\mu\nabla_\nu\phi^I$

$\bigcirc^{\mu\nu}$ can be only $\nabla_\mu\phi^J\nabla_\nu\phi^K$

↓

$$\mathcal{A}_{(IJ)K}(\phi^L, X^{MN})\nabla_\mu\phi^I\nabla^\nu\phi^J\nabla_\nu\nabla^\mu\phi^K$$

the most general form

Hamiltonian analysis

physical DOFs = { (canonical variables) -2×(1st-class constraints) -(2nd-class constraints) }

	#
canonical variables: $(\gamma_{ij}, \pi^{ij}),\ (N, \pi_N),\ (N^i, \pi_i),\ (\phi^I, p_I),\ (A^I_*, B^*_I),\ (\lambda_I, \Lambda^I)$	10+3N
1st-class constraints: $\pi_N \approx 0,\ \pi_i \approx 0,\ \mathcal{H}_0 \approx 0,\ \mathcal{H}_i \approx 0$	8
2nd-class constraints: $\Lambda^I \approx 0,\ \Phi^{(1)}_I \equiv p_I - \lambda_I \approx 0,\ \Phi^{(2)}_I \equiv B^*_I - 2\sqrt{\gamma}\mathcal{C}_I \approx 0$ degeneracy conditions ⟶ another 2nd-class $\Psi_I \approx 0$	3N+N

physical DOFs = 2×(10+3N)-2×8-(3N+N)

= N+2 **stable owing to** $\Psi_I \approx 0$

Hamiltonian analysis (1)

new variables:

$$\partial_\mu \phi^I \equiv A^I_\mu \longrightarrow A^I_* \equiv \frac{1}{N}(A^I_0 - N^i A^I_i) \sim \dot{\phi}^I$$

$$V^I_* \equiv \frac{1}{N}(\dot{A}^I_* - A^i_I D_i N - N^i D_i A^I_*) \sim \ddot{\phi}^I$$

rewriting Lagrangian:

$$\mathcal{L} = N\sqrt{\gamma}\left[2\mathcal{C}^{ij} \underbrace{K_{ij}}_{\dot{\gamma}_{ij}} + \mathcal{K}^{ij,kl} \underbrace{K_{ij}K_{kl}}_{\dot{\gamma}_{ij}\dot{\gamma}_{kl}} + 2\mathcal{C}_I \underbrace{V^I_*}_{\dot{A}^I_*} - \mathcal{U}\right] + \lambda_I(\dot{\phi}^I - NA^I_* - N^i D_i \phi^I)$$

$$\begin{cases} 2\mathcal{C}^{ij} := \mathcal{A}_{(IJ)K}\gamma^{ik}\gamma^{jl}(D_l\phi^I D_k\phi^J A^K_* - 2D_k\phi^J D_l\phi^K A^I_*) \\ \mathcal{K}^{ij,kl} := \frac{1}{2}(\gamma^{ik}\gamma^{jl} - \gamma^{ij}\gamma^{kl}) \\ 2\mathcal{C}_K := -\mathcal{A}_{(IJ)K}A^I_* A^J_* \\ \mathcal{U} := -\frac{R}{2} + \mathcal{A}_{(IJ)K}\left(\gamma^{ik}\gamma^{jl}D_k\phi^I D_l\phi^J D_i D_j \phi^K - 2\gamma^{ij}D_j\phi^J A^I_* D_i A^K_*\right) \end{cases}$$

canonical variables:

$$(\gamma_{ij}, \pi^{ij}), \ (N, \pi_N), \ (N^i, \pi_i), \ (\phi^I, p_I), \ (A^I_*, B^*_I), \ (\lambda_I, \Lambda^I)$$

Hamiltonian analysis (2)

primary constraints:

"satisfied on the constraint surface"

$$\pi_N \approx 0, \;\; \pi_i \approx 0, \;\; \Lambda^I \approx 0,$$

$$\Phi_I^{(1)} \equiv p_I - \lambda_I \approx 0, \quad \Phi_I^{(2)} \equiv B_I^* - 2\sqrt{\gamma}\mathcal{C}_I \approx 0$$

consistency conditions of $\Phi_I^{(2)} \approx 0$ **:** $\dot{\Phi}_I^{(2)} \approx 0$

$$\dot{\Phi}_I^{(2)} \ni \{\Phi_I^{(2)}, \Phi_J^{(2)}\}$$

$$\longrightarrow \{\Phi_I^{(2)}, \Phi_J^{(2)}\} = 2\sqrt{\gamma}\Big[2(\mathcal{A}_{(IK)J} - \mathcal{A}_{(JK)I})A_*^K + \left(\frac{\partial \mathcal{A}_{(LK)J}}{\partial X^{IM}} - \frac{\partial \mathcal{A}_{(LK)I}}{\partial X^{JM}}\right) A_*^K A_*^L A_*^M\Big] = 0$$

secondary constraints:

$$\mathcal{H}_0 \approx 0, \;\; \mathcal{H}_i \approx 0, \;\; \Psi_I \approx 0$$

physical DOFs = { (canonical variables) -2×(1st-class constraints) -(2nd-class constraints) } **=** $N+2$

stable owing to $\Psi_I \approx 0$