International Symposium on
Cosmology and Particle Astrophysics

CosPA2017

Teleparallel Conformal Invariant Models Induced by Kaluza-Klein Reduction

References: Class. Quant. Grav. **34** 185004 (2017),
Phys. Lett. B **737**, 248 (2014)

Ling-Wei Luo

Department of Physics, National Tsing Hua University (NTHU)

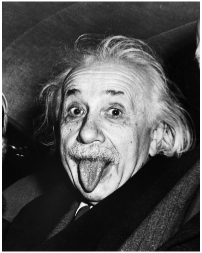
Collaborators: **Chao-Qiang Geng** (*NTHU/NCTS*)

December 14, 2017@Kyoto, Japan

Outline

Outline

Standard Gravity Theory



- General Relativity
  - Einstein equation

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \quad \text{with} \quad G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

(*Einstein, Nov. 25, 1915*)

  - Hilbert action

$$\frac{-1}{2\kappa} \int d^4x \sqrt{-g} R + S_{\mathrm{m}}$$

(*Hilbert, Nov. 20, 1915*)

- *"Spacetime tells matter how to move; matter tells spacetime how to curve."* — John Wheeler.

Absolute Parallelism

- Introducing the orthonormal frame $e_i = e_i{}^{\mu}\partial_{\mu}$ in Weitzenböck geometry T_4:

$$g(e_i, e_j) = \eta_{ij} \quad \text{with} \quad \eta_{ij} = \text{diag}(+1, -1, -1, -1)$$

$$\Longrightarrow \partial_{\mu} \to e_i = e_i{}^{\mu}\partial_{\mu} \quad \text{and} \quad \Gamma^{\rho}{}_{\mu\nu} \to \omega^i{}_{j\nu} = e^i{}_{\rho}\Gamma^{\rho}{}_{\mu\nu}e_j{}^{\mu} + e^i{}_{\sigma}\,\partial_{\nu}\,e_j{}^{\sigma}\,.$$

- Parallel vectors (absolute parallelism) (*Cartan, 1922/Eisenhart, 1925*)

$$\boxed{\nabla e_i = dx^{\nu}(\partial_{\nu}e_i{}^{\rho} + e_i{}^{\mu}\overset{\text{w}}{\Gamma}{}^{\rho}{}_{\mu\nu})\partial_{\rho} := dx^{\nu}(\nabla_{\nu}\,e_i{}^{\rho})\partial_{\rho} = 0\,.}$$

$$\Longrightarrow \text{Weitzenböck connection: } \overset{\text{w}}{\Gamma}{}^{\rho}{}_{\mu\nu} = e_i{}^{\rho}\partial_{\nu}e^i{}_{\mu} \quad \longleftarrow \quad \omega_{ij\mu} = 0.$$

- Curvature-free $\quad R^{\sigma}{}_{\rho\mu\nu}(\Gamma) = e_i{}^{\sigma}e^j{}_{\rho}R^i{}_{j\mu\nu}(\omega) = 0.$
- Torsion tensor $T^i{}_{\mu\nu} \equiv \overset{\text{w}}{\Gamma}{}^i{}_{\nu\mu} - \overset{\text{w}}{\Gamma}{}^i{}_{\mu\nu} = \partial_{\mu}e^i{}_{\nu} - \partial_{\nu}e^i{}_{\mu}.$

$$\begin{cases} \mathcal{T}^i = K^i{}_j \wedge \vartheta^j \quad \text{with} \quad K^i{}_j := K^i{}_{jk}\vartheta^k\,, \\ \widetilde{\omega}^i{}_j := \omega^i{}_j - K^i{}_j \text{ the torsion-free connection form (Decomposition)}\,. \end{cases}$$

- Contorsion tensor $K^{\rho}{}_{\mu\nu} = -\frac{1}{2}(T^{\rho}{}_{\mu\nu} - T_{\mu}{}^{\rho}{}_{\nu} - T_{\nu}{}^{\rho}{}_{\mu}) = -K_{\mu}{}^{\rho}{}_{\nu}.$

Teleparallel Equivalent to GR in T_4

- Weitzenböck connection $\overset{\text{w}}{\Gamma}{}^{\rho}{}_{\mu\nu} = \left\{ {}^{\rho}_{\mu\nu} \right\} + K^{\rho}{}_{\mu\nu}$.
- Teleparallel Equivalent to GR ($\text{GR}_{\|}$ or TEGR):

$$R(\Gamma) = \tilde{R}(e) + T - 2\,\tilde{\nabla}_{\mu} T^{\mu} = 0 \quad \Longrightarrow \quad \boxed{-\tilde{R}(e) = T - 2\,\tilde{\nabla}_{\mu} T^{\mu}.} \quad (T_{\mu} := T^{\nu}{}_{\nu\mu})$$

Torsion Scalar (*Einstein, 1929*)

$$\begin{aligned} T \equiv & K^{\nu}{}_{\mu\nu} K^{\mu\sigma}{}_{\sigma} - K^{\rho}{}_{\mu\nu} K^{\mu\nu}{}_{\rho} \\ = & \frac{1}{4} T^{\rho}{}_{\mu\nu} T_{\rho}{}^{\mu\nu} + \frac{1}{2} T^{\rho}{}_{\mu\nu} T^{\nu\mu}{}_{\rho} - 1\, T^{\nu}{}_{\mu\nu} T^{\sigma\mu}{}_{\sigma} = \frac{1}{2} T^{i}{}_{\mu\nu} S_{i}{}^{\mu\nu} \end{aligned}$$

$S_{\rho}{}^{\mu\nu} \equiv K^{\mu\nu}{}_{\rho} + \delta^{\mu}_{\rho} T^{\sigma\nu}{}_{\sigma} - \delta^{\nu}_{\rho} T^{\sigma\mu}{}_{\sigma} = -S_{\rho}{}^{\nu\mu}$ is superpotential .

- TEGR action

$$S_{\text{TEGR}} = \frac{1}{2\kappa} \int d^4x\, e\, T \qquad (e = \sqrt{-g})\,.$$

- *New General Relativity* (NGR): $\left(\frac{1}{4}, \frac{1}{2}, -1\right) \longrightarrow (a, b, c)$

(*Hayashi & Shirafuji, 1979*)

Motivation

- Fundamental fields in GR:
  - Metric tensor $g_{\mu\nu}$

    $\Longrightarrow$ Levi-Civita connection $\left\{ {}^{\rho}_{\mu\nu} \right\} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right)$
- Fundamental field in Teleparallelism:
  - Veierbein fields $e^i{}_\mu$

    $\Longrightarrow$ Weitzenböck connection $\Gamma^\rho{}_{\mu\nu} = e_i{}^\rho \partial_\nu e^i{}_\mu$.

What will be arised from the *extra dimensions*?

(i) Any new interaction different from GR?
(ii) New classes of gravity theory in Teleparallelism?

Outline

5D Teleparallelism

- Embedding: $T_4 \longrightarrow T_5$.
- 5D metric is given by

$$\bar{g}_{MN} = \begin{pmatrix} g_{\mu\nu} - \kappa^2 \phi^4 A_\mu A_\nu & \kappa\,\phi^2 A_\mu \\ \kappa\,\phi^2 A_\nu & -\phi^2 \end{pmatrix} .$$

- The coframes in T_5 are

$$\begin{cases} \theta^i = e^i{}_\mu\, dx^\mu \,, \\ \theta^{\hat{5}} = e^{\hat{5}}{}_\mu\, dx^\mu + e^{\hat{5}}{}_5\, dx^5 = -\kappa\,\phi\, A_\mu\, dx^\mu + \phi\, dy \,. \end{cases} \implies \quad dy = \kappa\, A_\mu\, e_i{}^\mu \theta^i + \frac{1}{\phi}\,\theta^{\hat{5}} \,.$$

The torsion components in T_5

$$\bar{T}^i{}_{jk} = T^i{}_{jk} + \kappa\, A_\mu (\partial_5 e^i{}_\nu)(e_j{}^\mu e_k{}^\nu - e_k{}^\mu e_j{}^\nu) \,,$$

$$\bar{T}^i{}_{\hat{5}j} = \frac{1}{\phi}(\partial_5 e^i{}_\mu) e_j{}^\mu \,,$$

$$\bar{T}^{\hat{5}}{}_{ij} = -\frac{\kappa}{2}\,\phi\, e_i{}^\mu e_j{}^\nu\, F_{\mu\nu} + \kappa^2\,\phi\, A_\mu (\partial_5 A_\nu)(e_i{}^\nu e_j{}^\mu - e_j{}^\nu e_i{}^\mu),$$

$$\bar{T}^{\hat{5}}{}_{i\hat{5}} = \frac{1}{\phi} e_i{}^\mu (\partial_\mu \phi) + \frac{1}{\phi}\kappa\, A_\mu e_i{}^\mu (\partial_5 \phi) + \kappa\, e_i{}^\mu (\partial_5 A_\mu) \,.$$

Extending to 5D

- NGR torsion scalar in 4D is given by

$$T_{\mathsf{NGR}} = a\,T_{ijk}\,T^{ijk} + b\,T_{ijk}\,T^{kji} + c\,T^{j}{}_{ji}\,T^{k}{}_{k}{}^{i} := \frac{1}{2}\,T^{i}{}_{jk}\,\Sigma_{i}{}^{jk}\,,$$

where

$$\Sigma_{i}{}^{jk} = 2a\,T_{i}{}^{jk} + b\,(T^{kj}{}_{i} - T^{jk}{}_{i}) + c\,(\delta_{i}^{k}\,T^{lj}{}_{l} - \delta_{i}^{j}\,T^{lk}{}_{l})\,.$$

- Extending to a 5D torsion scalar theory

Extended torsion scalar

$${}^{(5)}T^{(\mathsf{ext})} = a\,\bar{T}_{LMN}\bar{T}^{LMN} + b\,\bar{T}_{LMN}\bar{T}^{NML} + c\,\bar{T}^{L}{}_{LM}\bar{T}^{N}{}_{N}{}^{M}\,.$$

- Decomposition ($\eta_{\hat{5}\hat{5}} = \eta^{\hat{5}\hat{5}} = -1$)

$$\begin{aligned}{}^{(5)}T^{(\mathsf{ext})} =\,& \bar{T}_{\mathsf{NGR}} + 2a\,\bar{T}_{i\hat{5}j}\bar{T}^{i\hat{5}j} + a\,\bar{T}_{\hat{5}ij}\bar{T}^{\hat{5}ij} + b\,\bar{T}_{i\hat{5}j}\bar{T}^{j\hat{5}i} + 2b\,\bar{T}_{\hat{5}ij}\bar{T}^{ji\hat{5}} \\ &+ (2a+b+c)\,\bar{T}_{\hat{5}i\hat{5}}\bar{T}^{\hat{5}i\hat{5}} + 2c\,\bar{T}^{j}{}_{j}{}^{i}\bar{T}^{\hat{5}}{}_{\hat{5}}{}^{i} + c\,\bar{T}^{i}{}_{i\hat{5}}\bar{T}^{j}{}_{j}{}^{\hat{5}}\,.\end{aligned}$$

$$
\begin{aligned}
\bar{T}_{\mathrm{NGR}} = T_{\mathrm{NGR}} &+ 4a\kappa T_l{}^{\rho\sigma} A_\rho (\partial_5 e^l{}_\sigma) - 2a\kappa^2 (g^{\mu\rho} A_\mu A_\rho)(g^{\nu\sigma}\eta_{il}(\partial_5 e^i{}_\nu)(\partial_5 e^l{}_\sigma)) \\
&- 2a\kappa^2 \eta_{il}(g^{\nu\rho} A_\rho (\partial_5 e^i{}_\nu))(g^{\mu\sigma} A_\mu (\partial_5 e^l{}_\sigma)) + 2b\kappa T^{\sigma\rho}{}_k A_\rho (\partial_5 e^k{}_\sigma) \\
&- 2b\kappa T^{\rho\sigma}{}_k A_\rho (\partial_5 e^k{}_\sigma) + b\kappa^2 (g^{\mu\rho} A_\mu A_\rho)(\partial_5 e^i{}_\nu)(\partial_5 e^k{}_\sigma) e_k{}^\nu e_i{}^\sigma \\
&- 2b\kappa^2 (g^{\mu\sigma} A_\mu (\partial_5 e^k{}_\sigma))(A_\rho e_i{}^\rho)(\partial_5 e^i{}_\nu) e_k{}^\nu \\
&+ b\kappa^2 (A_\mu e_k{}^\mu)(A_\rho e_i{}^\rho)(g^{\nu\sigma}(\partial_5 e^i{}_\nu)(\partial_5 e^k{}_\sigma)) \\
&+ 2c\kappa T^j{}_j{}^\sigma (A_\rho e_k{}^\rho)(\partial_5 e^k{}_\sigma) - 2c\kappa T^j{}_j{}^\rho A_\rho e_k{}^\sigma (\partial_5 e^k{}_\sigma) \\
&+ c\kappa^2 (A_\mu e_j{}^\mu)(A_\rho e_k{}^\rho)(g^{\nu\sigma}(\partial_5 e^j{}_\nu)(\partial_5 e^k{}_\sigma)) \\
&- 2c\kappa^2 (A_\mu e_j{}^\mu)(\partial_5 e^k{}_\sigma) e_k{}^\sigma (g^{\nu\rho} A_\rho (\partial_5 e^j{}_\nu)) \\
&+ c\kappa^2 (g^{\mu\rho} A_\mu A_\rho)(\partial_5 e^j{}_\nu) e_j{}^\nu (\partial_5 e^k{}_\sigma) e_k{}^\sigma \,,
\end{aligned}
$$

$$\bar{T}_{i\hat{5}j}\bar{T}^{i\hat{5}j} = \frac{1}{\phi^2}\,\eta^{\hat{5}\hat{5}}\eta_{ik}g^{\mu\nu}(\partial_5 e^i{}_\mu)(\partial_5 e^k{}_\nu)\,,$$

$$\begin{aligned}\bar{T}_{\hat{5}ij}\bar{T}^{\hat{5}ij} = {} & \boxed{\frac{\kappa^2}{4}\phi^2\eta_{\hat{5}\hat{5}}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}} + 2\kappa^3\phi^2\eta_{\hat{5}\hat{5}}g^{\mu\sigma}g^{\nu\rho}A_\mu(\partial_5 A_\nu)F_{\sigma\rho} \\ & + 2\kappa^4\phi^2\eta_{\hat{5}\hat{5}}(g^{\mu\rho}A_\mu A_\rho)(g^{\nu\sigma}(\partial_5 A_\nu)(\partial_5 A_\sigma)) \\ & - 2\kappa^4\phi^2\eta_{\hat{5}\hat{5}}(g^{\mu\sigma}A_\mu(\partial_5 A_\sigma))(g^{\nu\rho}(\partial_5 A_\nu)A_\rho)\,,\end{aligned}$$

$$\bar{T}_{i\hat{5}j}\bar{T}^{j\hat{5}i} = \frac{1}{\phi^2}\eta^{\hat{5}\hat{5}}(\partial_5 e^i{}_\mu)(\partial_5 e^j{}_\nu)e_j{}^\mu e_i{}^\nu\,,$$

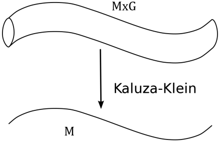
$$\begin{aligned}\bar{T}_{\hat{5}ij}\bar{T}^{ji\hat{5}} = {} & \frac{\kappa}{2}e_j{}^\nu g^{\mu\rho}F_{\mu\nu}(\partial_5 e^j{}_\rho) - \kappa^2(A_\mu e_j{}^\mu)(g^{\nu\rho}(\partial_5 A_\nu)(\partial_5 e^j{}_\rho)) \\ & + \kappa^2(A_\mu g^{\mu\rho})(\partial_5 A_\nu)(\partial_5 e^j{}_\rho)e_j{}^\nu\,,\end{aligned}$$

$$
\begin{aligned}
\bar{T}_{\hat{5}i\hat{5}}\bar{T}^{\hat{5}i\hat{5}} =& \frac{1}{\phi^2}\eta_{\hat{5}\hat{5}}\eta^{\hat{5}\hat{5}}(g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi)) + \frac{2\kappa}{\phi^2}\eta_{\hat{5}\hat{5}}\eta^{\hat{5}\hat{5}}(\partial_5\phi)(g^{\mu\nu}(\partial_\mu\phi)A_\nu) \\
&+ \frac{2\kappa}{\phi}\eta_{\hat{5}\hat{5}}\eta^{\hat{5}\hat{5}}(g^{\mu\nu}(\partial_\mu\phi)(\partial_5 A_\nu)) + \frac{\kappa^2}{\phi^2}\eta_{\hat{5}\hat{5}}\eta^{\hat{5}\hat{5}}(\partial_5\phi)^2(g^{\mu\nu}A_\mu A_\nu) \\
&+ \frac{2\kappa^2}{\phi}\eta_{\hat{5}\hat{5}}\eta^{\hat{5}\hat{5}}(\partial_5\phi)(g^{\mu\nu}A_\mu(\partial_5 A_\nu)) + \kappa^2\eta_{\hat{5}\hat{5}}\eta^{\hat{5}\hat{5}}(g^{\mu\nu}(\partial_5 A_\mu)(\partial_5 A_\nu)) \,, \\
\bar{T}^j{}_{ji}\bar{T}^{\hat{5}}{}_{\hat{5}}{}^i =& -\frac{1}{\phi}T^\rho(\partial_\rho\phi) - \frac{1}{\phi}T^\rho A_\rho(\partial_5\phi) - \kappa T^\rho(\partial_5 A_\rho) \\
&- \frac{\kappa}{\phi}(A_\mu e_j{}^\mu)(g^{\nu\rho}(\partial_5 e^j{}_\nu)(\partial_\rho\phi)) + \frac{\kappa}{\phi}(e_j{}^\nu(\partial_5 e^j{}_\nu))(g^{\mu\rho}A_\mu(\partial_\rho\phi)) \\
&- \frac{\kappa^2}{\phi}(\partial_5\phi)(A_\mu e_j{}^\mu)(g^{\nu\rho}A_\rho(\partial_5 e^j{}_\nu)) + \frac{\kappa^2}{\phi^2}(\partial_5\phi)(\partial_5 e^j{}_\nu)e_j{}^\nu(g^{\mu\rho}A_\mu A_\rho) \\
&- \kappa^2(A_\mu e_j{}^\mu)(g^{\nu\rho}(\partial_5 e^j{}_\nu)(\partial_5 A_\rho)) + \kappa^2(\partial_5 e^j{}_\nu)e_j{}^\nu(g^{\mu\rho}A_\mu(\partial_5 A_\rho)) \,, \\
\bar{T}^i{}_{i\hat{5}}\bar{T}^j{}_j{}^{\hat{5}} =& \frac{1}{\phi^2}\eta^{\hat{5}\hat{5}}(\partial_5 e^i{}_\mu)e_i{}^\mu(\partial_5 e^j{}_\nu)e_j{}^\nu \,.
\end{aligned}
$$

Outline

Kaluza-Klein (KK) Theory

- KK ansatz:
  - Cylindrical condition (no y dependency)
  - Compactification: $G = S^1$ and only consider zero KK mode



- The manifold is $M_4 \times S^1$ $(y = r\,\theta)$
- Harmonic expansions

$$e^i{}_\mu(x,y) = \sum_n e^{i\,(n)}_{\mu}(x)\, e^{iny/2r} \quad \Longrightarrow \quad g_{\mu\nu}(x,y) = \sum_n g^{(n)}_{\mu\nu}(x)\, e^{iny/r}\,,$$

$$A_\mu(x,y) = \sum_n A^{(n)}_\mu(x)\, e^{iny/r}\,, \quad \phi(x,y) = \sum_n \phi^{(n)}(x)\, e^{iny/r}\,.$$

KK Zero mode

- $n = 0$ mode

$$\begin{cases} \bar{T}_{\mathsf{NGR}} = T^{(0)}_{\mathsf{NGR}} \,, \\ \bar{T}_{\hat{5}ij}\bar{T}^{\hat{5}ij} = \dfrac{\kappa^2}{4}\phi^{(0)\,2}\,\eta_{\hat{5}\hat{5}}g^{\mu\rho}g^{\nu\sigma}F^{(0)}_{\mu\nu}F^{(0)}_{\rho\sigma} \,, \\ \bar{T}_{\hat{5}i\hat{5}}\bar{T}^{\hat{5}i\hat{5}} = \dfrac{1}{\phi^{(0)\,2}}\eta_{\hat{5}\hat{5}}\eta^{\hat{5}\hat{5}}(g^{\mu\nu}(\partial_\mu\phi^{(0)})(\partial_\nu\phi^{(0)})) \,, \\ \bar{T}^{j}{}_{j}{}^{i}\bar{T}^{\hat{5}}{}_{\hat{5}}{}^{i} = -\dfrac{1}{\phi^{(0)}}\,T^{(0)\,\rho}(\partial_\rho\phi^{(0)}) \,. \end{cases}$$

- $\bar{T}_{i\hat{5}j}\bar{T}^{i\hat{5}j} = \bar{T}_{i\hat{5}j}\bar{T}^{j\hat{5}i} = \bar{T}_{\hat{5}ij}\bar{T}^{ji\hat{5}} = \bar{T}^{i}{}_{i\hat{5}}\bar{T}^{j}{}_{j}{}^{\hat{5}} = 0$

Zero KK mode in 4D effective extended gravity in teleparallism

$$S_{\mathsf{eff}} = \int d^4x\, e\left(\frac{1}{2\kappa_4}\phi\, T_{\mathsf{NGR}} - \frac{a}{8}\phi^3\, g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} + \frac{2a+b+c}{2\kappa_4}\frac{1}{\phi}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{c}{\kappa_4}\,T^{\mu}\,\partial_\mu\phi\right).$$

Conformal 2 Weyl

- By considering 5D matter Lagrangian (Universal Extra Dimensions, UED) ${}^{(5)}\mathcal{L}_{\mathsf{m}} = {}^{(5)}e\, L_{\mathsf{m}}(e^I{}_M, \Psi, \mathcal{D}_M\Psi)$
  - The n-mode harmonic expansion of Ψ is given by

$$\Psi(x^\mu, y) = \sum_n \Psi(x^\mu)\, e^{iny/r}\,,$$

  - For the massless zero mode, the matter field $\Psi^{(0)}$ is assumed to be localized on the T_4 hypersurface, the resulting effective matter Lagrangian is

$$\mathcal{L}_{\mathsf{m,\,eff}} = e\lambda\phi L_{\mathsf{m}}(e^i{}_\mu, A_\mu, \phi, \Psi, \mathcal{D}_\mu\Psi)\,.$$

Note:

For 4D localized matter, the Lagrangian can be identified as

$${}^{(4)}\mathcal{L}_{\mathsf{m}} = \mathcal{L}_{\mathsf{m,\,eff}}\Big|_{\lambda=1,\, A_\mu=0,\, \phi=1} = eL_{\mathsf{m}}(e^i{}_\mu, \Psi, \mathcal{D}_\mu\Psi)\,.$$

Outline

Conformal Transformations

- Considering the conformal transformations Ω^λ with weight λ

$$\widetilde{e}^i{}_\mu = \Omega\, e^i{}_\mu \quad (\lambda_e = 1) \quad \Longrightarrow \quad \widetilde{g}_{\mu\nu} = \Omega^2(x)\, g_{\mu\nu} \quad (\lambda_g = 2)\,,$$

- Transformation of the torsion tensor and vector

$$\widetilde{T}^i{}_{\mu\nu} = \Omega\, T^i{}_{\mu\nu} + (\partial_\mu \Omega)\, e^i{}_\nu - e^i{}_\mu\, (\partial_\nu \Omega)\,,$$
$$\widetilde{T}_\mu = T_\mu - 3\,\Omega^{-1}\,\partial_\mu \Omega\,,$$

- Torsion scalar of NGR,

$$T_{\mathsf{NGR}} = \Omega^2\, \widetilde{T}_{\mathsf{NGR}} + \Big(4\,a + 2\,b + 6\,c\Big)\widetilde{g}^{\mu\nu}\widetilde{T}_\mu(\Omega\,\partial_\nu\Omega) + \Big(6\,a + 3\,b + 9\,c\Big)\widetilde{g}^{\mu\nu}\partial_\mu\Omega\,\partial_\nu\Omega\,.$$

- Scalar ϕ: $\widetilde{\phi} = \Omega^{\lambda_\phi}\,\phi$ with $\lambda_\phi = -2$
- Vector A_μ: $\widetilde{A}_\mu = A_\mu$ and $\widetilde{F}_{\mu\nu} = F_{\mu\nu}$

Transformed effective Lagrangian density

$$\mathcal{L}_g = \widetilde{e}\frac{1}{2\kappa_4}\Big\{\widetilde{\phi}\,\widetilde{T}_{\mathsf{NGR}} + \Big(4\,a + 2\,b + 2\,c\Big)\,\widetilde{\phi}\,\widetilde{g}^{\mu\nu}\,\widetilde{T}_\mu\,\partial_\nu\omega$$
$$+ \Big(14\,a + 7\,b + c\Big)\widetilde{\phi}\,\widetilde{g}^{\mu\nu}\,\partial_\mu\omega\,\partial_\nu\omega$$
$$- e^{6\omega}\,\frac{a\kappa^2}{4}\widetilde{\phi}^{\,3}\,\widetilde{g}^{\mu\rho}\widetilde{g}^{\nu\sigma}\,\widetilde{F}_{\mu\nu}\widetilde{F}_{\rho\sigma} + \Big(2a + b + c\Big)\frac{1}{\widetilde{\phi}}\,\widetilde{g}^{\mu\nu}\partial_\mu\widetilde{\phi}\,\partial_\nu\widetilde{\phi}$$
$$- 2\,c\,\widetilde{g}^{\mu\nu}\,\widetilde{T}_\mu\,\partial_\nu\widetilde{\phi} + \Big(8\,a + 4\,b - 2\,c\Big)\,\widetilde{g}^{\mu\nu}\,\partial_\mu\omega\,\partial_\nu\widetilde{\phi}\Big\}\,, \qquad (*)$$

with the conformal scalar $\omega := \ln\Omega$.

◂ Conformal 1

The Existence Einstein-Frame

- In general, the Einstein-frame does not exist for the non-minimal torsion scalar ϕT_{NGR}.
- The non-minimal coupling $\widetilde{g}^{\mu\nu}\widetilde{T}_{\mu}(\Omega\,\partial_{\nu}\Omega)$ will be always generated.
- The Einstein-frame is obtained by eliminating the term $\widetilde{g}^{\mu\nu}\widetilde{T}_{\mu}\partial_{\nu}\omega$ in the transformed Lagrangian.

Necessary condition: $2\,a + b + c = 0\,.$

- The Lagrangian is reduced to

$$\mathcal{L}_g = e\,\frac{1}{2\kappa_4}\left(\phi\,T_{\mathsf{NGR}} - 2\,c\,g^{\mu\nu}T_{\mu}\partial_{\nu}\phi - \frac{a\kappa^2}{4}\,\phi^3\,g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}\right).$$

- The corresponding Einstein-frame

$$\mathcal{L}_g^{(\mathsf{E})} = \widetilde{e}\left(\frac{1}{2\kappa_4}\widetilde{T}_{\mathsf{NGR}} - \frac{c}{2}\,\widetilde{g}^{\mu\nu}\,\partial_\mu\varphi\,\partial_\nu\varphi - \frac{a\kappa^2}{8\kappa_4}\,e^{6\omega}\,\widetilde{g}^{\mu\rho}\widetilde{g}^{\nu\sigma}\,\widetilde{F}_{\mu\nu}\widetilde{F}_{\rho\sigma}\right),$$

where where $\varphi := \sqrt{6/\kappa_4}\,\omega$ and the ghost-free condition is $c \leq 0$.

Einstein-frame condition

$$2\,a + b + c = 0 \quad \text{and} \quad c \leq 0\,.$$

- For a simple choice of $c = -1$, one gets the minimal coupled one-parameter family model with $2a + b = 1$ in teleparallelism.

Conformal Invariant Gravity

- Only keeping the terms of $\widetilde{\phi}^{-1}\,\widetilde{g}^{\mu\nu}\partial_\mu\widetilde{\phi}\,\partial_\nu\widetilde{\phi}$ and $\widetilde{g}^{\mu\nu}\,\widetilde{T}_\mu\,\partial_\nu\widetilde{\phi}$ ▸ Lagrangian

$$
\begin{aligned}
4a + 2b + 2c &= 0\,,\\
14a + 7b + c &= 0\,,\\
8a + 4b - 2c &= 0\,.
\end{aligned}
$$

Conformal Invariant condition

$$2a + b = 0 \qquad \text{and} \qquad c = 0$$

- Corresponding to the simple one-parameter conformal invariant gravity in teleparallelism

$$S_g^{(\mathsf{c})} = \int d^4x \left\{ e\frac{a}{2\kappa_4}\phi\left(T_{ijk}\,T^{ijk} - 2\,T_{ijk}\,T^{kji}\right)\right\}.$$

- The gravitational equation of motion

$$\phi\left\{\frac{1}{2}\,e_i{}^{\mu}\,T^j{}_{\rho\nu}\left(T_j{}^{\rho\nu}-2\,T^{\nu\rho}{}_j\right)-4\,e_i{}^{\rho}\,T^j{}_{\rho\nu}\,K^{\mu\nu}{}_j\right\}$$
$$+\frac{2\phi}{e}\,\partial_\nu\left\{e\left(T_i{}^{\mu\nu}-2\,T^{\nu\mu}{}_i\right)\right\}+2\left(T_i{}^{\mu\nu}-2\,T^{\nu\mu}{}_i\right)\partial_\nu\phi=0\,.$$

- The equation of motion of ϕ

$$e\,\frac{a}{2\kappa_4}\,T_{ijk}(T^{ijk}-2\,T^{kji})=0\,.$$

  - $T_{ijk}=0 \Longrightarrow$ no gravity $\Longrightarrow$ forbidden!!
  - $T^{ijk}=2\,T^{kji} \Longrightarrow$ it implies $T^{iik}=2T^{kii}=0$, no torsion vector!! $\Longrightarrow$ NO new interaction!!

- The gravitational equation is reduced to

$$e_i^{\rho}\,T^j{}_{\rho\nu}\,T^{\nu\mu}{}_j=0\,.$$

Weyl Gauge Invariance

- The torsion vector T_μ is identified as the gauge field.
- Rewriting the effective Lagrangian

$$\mathcal{L}_g = e\frac{1}{2\kappa_4}\left\{\phi\left(T_{\mathsf{NGR}} - kc\, g^{\mu\nu}\, T_\mu T_\nu\right) + \frac{c}{k\phi}\left(g^{\mu\nu}(\partial_\mu - kT_\mu)\phi(\partial_\nu - kT_\nu)\phi\right)\right\},$$

where $k = \dfrac{c}{2a+b+c}$ is a fixed ratio.

Lagrangian

- We need *ghost-free* $2a+b+c>0$ and $c\neq 0$
- Conformal transformation

$$g_{\mu\nu} \longrightarrow e^{2\omega} g_{\mu\nu}\,, \qquad T_\mu \longrightarrow T_\mu - 3\,\partial_\mu\omega\,, \qquad \phi \longrightarrow e^{-2\omega}\,\phi\,.$$

- Under the conformal transformation, the effective Lagrangian becomes as

$$\mathcal{L}_g = \underbrace{\widetilde{e}}_{e\,e^{4\omega}} \frac{1}{2\kappa_4} \left\{ \widetilde{\phi}\,\widetilde{T}_{\text{NGR}} - kc\,\widetilde{\phi}\,\widetilde{g}^{\mu\nu}\,\widetilde{T}_\mu \widetilde{T}_\nu + \left(\frac{1}{k} + 2 - 3k\right) 2c\,\widetilde{\phi}\,\widetilde{g}^{\mu\nu}\,\widetilde{T}_\mu\,\partial_\nu \omega \right.$$

$$+ \left(\frac{1}{k} + 2 - 3k\right) 3c\,\widetilde{\phi}\,\widetilde{g}^{\mu\nu}\,\partial_\mu \omega\,\partial_\nu \omega - e^{6\omega}\,\frac{\alpha\kappa^2}{4}\widetilde{\phi}^{\,3}\,\widetilde{g}^{\mu\rho}\widetilde{g}^{\nu\sigma}\,\widetilde{F}_{\mu\nu}\widetilde{F}_{\rho\sigma}$$

$$\left. + \underbrace{\frac{c}{k\widetilde{\phi}} \left[\widetilde{g}^{\mu\nu} \left(\partial_\mu - k\widetilde{T}_\mu + (2-3k)\partial_\mu \omega\right) \widetilde{\phi} \left(\partial_\nu - k\widetilde{T}_\nu + (2-3k)\partial_\nu \omega\right) \widetilde{\phi} \right]}_{e^{-4\omega}\frac{c}{k\phi}\left(g^{\mu\nu}(\partial_\mu - kT_\mu)\phi\,(\partial_\nu - kT_\nu)\phi\right)} \right\}.$$

- $\frac{1}{k} + 2 - 3k = 0 \Longrightarrow$ conformal invariance!!

Conformal Invariant condition

$$\begin{cases} 2a + b + 4c = 0 & \text{for } k = -\frac{1}{3} \\ \qquad\qquad \text{or} & \\ 2a + b = 0 & \text{for } k = 1 \end{cases} \qquad \text{with } 2a + b + c > 0 \text{ and } c \neq 0$$

- Weyl derivative for general field ψ:

$$
{}^{*}\partial_{\mu}^{(\psi)} = \partial_{\mu} + \frac{\lambda_{\psi} k}{2} T_{\mu}
$$

- Weyl derivative for ϕ, $e^{i}{}_{\nu}$ and $g_{\nu\rho}$

$$
\begin{cases}
{}^{*}\partial_{\mu}^{(\phi)} \phi = \left(\partial_{\mu} + \frac{\lambda_{\phi} k}{2} T_{\mu} \right) \phi & \text{with } \lambda_{\phi} = -2 \\
{}^{*}\partial_{\mu}^{(e)} e^{i}{}_{\nu} = \left(\partial_{\mu} + \frac{\lambda_{e} k}{2} T_{\mu} \right) e^{i}{}_{\nu} & \text{with } \lambda_{e} = 1 \\
{}^{*}\partial_{\mu}^{(g)} g_{\nu\rho} = \left(\partial_{\mu} + \frac{\lambda_{g} k}{2} T_{\mu} \right) g_{\nu\rho} & \text{with } \lambda_{g} = 2
\end{cases}
$$

- Define an invariant connection ${}^{*}\Gamma^{\rho}_{\nu\mu} = e_{i}{}^{\rho}\, {}^{*}\partial_{\mu}^{(e)} e^{i}{}_{\nu} = \Gamma^{\rho}_{\nu\mu} + \frac{k}{2} \delta^{\rho}_{\nu} T_{\mu}$.

$$
\Longrightarrow \quad {}^{*}\partial_{\mu}^{(g)} g_{\nu\rho} - \Gamma^{\sigma}_{\nu\mu} g_{\sigma\rho} - \Gamma^{\sigma}_{\rho\mu} g_{\nu\sigma} = \frac{\lambda_{g} k}{2} T_{\mu} g_{\nu\rho}
$$

$$
\Longrightarrow \quad {}^{*}R^{\rho}{}_{\sigma\mu\nu} = \frac{k}{2} \delta^{\rho}_{\sigma} \left(\partial_{\mu} T_{\nu} - \partial_{\nu} T_{\mu} \right) \neq 0
$$

$\Longrightarrow$ Weyl-Cartan geometry!!!

- The modified covariant derivative for ψ given by

$$^*\nabla^{(\psi)} = {}^*d^{(\psi)} + {}^*\Gamma \,.$$

- The nonmetricity vanishes

$$^*\nabla^{(g)}_{\mu} g_{\nu\rho} = {}^*\partial^{(g)}_{\mu} g_{\nu\rho} - {}^*\Gamma^{\sigma}{}_{\nu\mu} g_{\sigma\rho} - {}^*\Gamma^{\sigma}{}_{\rho\mu} g_{\nu\sigma} = \nabla_{\mu} g_{\nu\rho} = 0 \,.$$

- We define ${}^*\widetilde{T}^{\rho}{}_{\mu\nu} := \widetilde{({}^*T)}^{\rho}{}_{\mu\nu}$, we have

$$\begin{cases} {}^*T^{\rho}{}_{\mu\nu} = T^{\rho}{}_{\mu\nu} + \frac{k}{2}(\delta^{\rho}_{\nu} T_{\mu} - \delta^{\rho}_{\mu} T_{\nu}) \,, \\ {}^*T_{\mu} = (1 - \frac{3}{2}k)\, T_{\mu} \,, \end{cases} \quad \text{and} \quad \begin{cases} {}^*\widetilde{T}^{\rho}{}_{\mu\nu} = \widetilde{T}^{\rho}{}_{\mu\nu} + \frac{k}{2}(\delta^{\rho}_{\nu} \widetilde{T}_{\mu} - \delta^{\rho}_{\mu} \widetilde{T}_{\nu}) \,, \\ {}^*\widetilde{T}_{\mu} = (1 - \frac{3}{2}k)\, \widetilde{T}_{\mu} \,. \end{cases}$$

- ${}^*\partial^{(\psi)}_{\mu}$ in terms of ${}^*T_{\mu} \Longrightarrow {}^*\partial^{(\psi)}_{\mu} = \partial_{\mu} - \lambda_{\psi} k^2\, {}^*T_{\mu}$.

- Due to ${}^*\widetilde{T}^\rho{}_{\mu\nu} = {}^*T^\rho{}_{\mu\nu} - (1/2)(\delta^\rho_\nu \partial_\mu \omega - \delta^\rho_\mu \partial_\nu \omega)$, we have

$$\begin{aligned} & e\,\phi\left(T_{\mathsf{NGR}} - kc\,g^{\mu\nu}\,T_\mu T_\nu\right) \\ &= e\,\phi\left({}^*T_{\mathsf{NGR}} - kc\,g^{\mu\nu}\,{}^*T_\mu {}^*T_\nu\right) \\ &= \widetilde{e}\,\widetilde{\phi}\left({}^*\widetilde{T}_{\mathsf{NGR}} - kc\,\widetilde{g}^{\mu\nu}\,{}^*\widetilde{T}_\mu {}^*\widetilde{T}_\nu\right) \end{aligned}$$

and

$$\begin{aligned} {}^*\widetilde{\nabla}^{(g)}_\mu \widetilde{g}_{\nu\rho} &= e^{2\omega}\left({}^*\nabla^{(g)}_\mu g_{\nu\rho}\right) = 0\,, \\ {}^*\widetilde{\nabla}^{(\phi)}_\mu \widetilde{\phi} &= \left[{}^*\widetilde{\partial}^{(\phi)}_\mu + \left(\frac{3}{2}\,\lambda_\phi k^2 - \lambda_\phi\right)\partial_\mu \omega\right]\widetilde{\phi} = e^{-2\omega}\left({}^*\nabla^{(\phi)}_\mu \phi\right). \end{aligned}$$

Weyl gauge invariance

$$\mathcal{L}_g = \widetilde{e}\frac{1}{2\kappa_4}\left\{\widetilde{\phi}\left({}^*\widetilde{T}_{\mathsf{NGR}} - kc\,\widetilde{g}^{\mu\nu}\,{}^*\widetilde{T}_\mu {}^*\widetilde{T}_\nu\right) + \frac{c}{k\widetilde{\phi}}\,\widetilde{g}^{\mu\nu}\,{}^*\widetilde{\nabla}^{(\phi)}_\mu \widetilde{\phi}\,{}^*\widetilde{\nabla}^{(\phi)}_\nu \widetilde{\phi}\right\},$$

Outline

Weak Field Approximation

- Define a canonical field $\Phi := \sqrt{\phi/\kappa_4}$,

$$
\begin{aligned}
S &= S_g + S_{\mathsf{m}} \\
&= \int d^4x \left\{ e\left(\frac{1}{2}\,\Phi^2\, T_{\mathsf{NGR}} - \frac{a\kappa^2\kappa_4^2}{8}\,\Phi^6\, g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} \right.\right. \\
&\qquad \left.\left. + (4a+2b+2c)\, g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - 2c\, g^{\mu\nu}\, T_\mu\,\Phi\partial_\nu\Phi \right) + \kappa_4\lambda\Phi^2\mathcal{L}_{\mathsf{m}} \right\}.
\end{aligned}
$$

- In the weak field approximation $e^i{}_\mu = \delta^i_\mu + h^i{}_\mu$ $(e_i{}^\mu = \delta^\mu_i - h_i{}^\mu)$, the tensor $h_{\mu\nu}$ contains the anti-symmetric fluctuations:

$$
h_{\mu\nu} = \underbrace{\frac{1}{2}\gamma_{\mu\nu}}_{\text{symmetric}} + \underbrace{a_{\mu\nu}}_{\text{anti-symmetric}} \quad \text{and} \quad \left| h^i{}_\mu \right| \ll 1\,.
$$

- The metric tensor $g_{\mu\nu} = \eta_{ij} e^i{}_\mu e^j{}_\nu \approx \eta_{\mu\nu} + \gamma_{\mu\nu}$ contains no anti-symmetric part of $h_{\mu\nu}$.

- The torsion tensor and torsion vector are

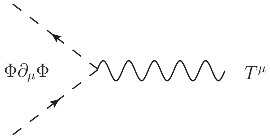
$$\begin{cases} T^{\rho}{}_{\mu\nu} = \delta^{\rho}_{i}(\partial_\mu h^{i}{}_{\nu} - \partial_\nu h^{i}{}_{\mu}) + \mathcal{O}(h^2_{\mu\nu})\,, \\ T_\nu = \partial_\mu h^{\mu}{}_{\nu} - \partial_\nu h + \mathcal{O}(h^2_{\mu\nu})\,. \end{cases}$$

- $e = 1 + h + \mathcal{O}(h^2_{\mu\nu})$ with $h \equiv \delta^{\mu}_{i}\, h^{i}{}_{\mu} = h^{\mu}{}_{\mu} = (1/2)\gamma$.

Lagrangian in the lowest order

$$\mathcal{L}_g \approx \frac{1}{2}\,\Phi^2\,T_{\text{NGR}} - \frac{a\kappa^2\kappa_4^2}{8}\,\Phi^6\eta^{\mu\rho}\eta^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} + (4a+2b+2c)\,\eta^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - 2c\,\eta^{\mu\nu}T_\mu\,\Phi\partial_\nu\Phi$$

- The current-vector interaction $\eta^{\mu\nu}T_\mu\,\Phi\partial_\nu\Phi$:



$$T_\mu \approx (1/2)\partial_\rho\gamma^{\rho}{}_{\mu} + \partial_\rho a^{\rho}{}_{\mu} - (1/2)\partial_\mu\gamma$$

$$\Longrightarrow \int d^4x - 2c\,\partial_\rho a^{\rho\nu}\,\Phi\partial_\nu\Phi = \int d^4x\,c\,\Phi^2\,\underbrace{\partial_\nu\partial_\rho}_{\text{symmetric}}\,a^{\rho\nu} \longrightarrow 0$$

$\Longrightarrow$ No contribution from $a_{\mu\nu}$

$$T_{\mathsf{NGR}} = \frac{1}{4}\Big((2a+b)\partial_\mu\gamma_{\nu\rho}\,\partial^\mu\gamma^{\nu\rho} - (2a+b)\partial_\mu\gamma_{\nu\rho}\,\partial^\rho\gamma^{\mu\nu} + c\,\partial^\rho\gamma_{\rho\mu}\,\partial_\sigma\gamma^{\sigma\mu} - 2c\,\partial_\mu\gamma\,\partial_\rho\gamma^{\rho\mu} + c\,\partial_\mu\gamma\,\partial^\mu\gamma \Big) + (2a+b)\partial_\mu\gamma_{\nu\rho}\,\partial^\nu a^{\mu\rho} + c\,\partial^\rho\gamma_{\rho\mu}\,\partial_\sigma a^{\sigma\mu} - c\,\partial_\mu\gamma\,\partial_\rho a^{\rho\mu} + (2a-b)\partial_\mu a_{\nu\rho}\,\partial^\mu a^{\nu\rho} + (2a-3b)\partial_\mu a_{\nu\rho}\,\partial^\rho a^{\mu\nu} + c\,\partial^\rho a_{\rho\mu}\,\partial_\sigma a^{\sigma\mu}$$

Note: T_{NGR} becomes the well-known Fierz-Pauli Lagrangian (*Fierz and Pauli, 1939*) by setting $(a,b,c) = (\frac{1}{4},\frac{1}{2},-1)$ and $a_{\mu\nu} = 0$.

- Gauge conditions

$$\begin{cases} \partial_\mu\gamma^\mu{}_\nu = 0 & \text{transversed condition,} \\ \partial_\mu a^\mu{}_\nu = 0 & \text{transversed condition,} \implies \text{torsion vector } T_\mu \text{ vanishes.} \\ \gamma = 0 & \text{traceless condition.} \end{cases}$$

- We define $j_\mu := \Phi\partial_\mu\Phi$

and
$$\begin{cases} T^{\mu\nu}_\gamma := \dfrac{1}{2}T^{(\mu\nu)} = -2\dfrac{\delta\mathcal{L}_{\mathsf{m}}}{\delta\gamma_{\mu\nu}}\,, \\ T^{\mu\nu}_a := T^{[\mu\nu]} = -2\dfrac{\delta\mathcal{L}_{\mathsf{m}}}{\delta a_{\mu\nu}}\,. \end{cases}$$

EoM

- EoM of $\gamma_{\mu\nu}$:

$$
\begin{aligned}
&-j_\rho\Big\{\frac{2a+b}{2}\partial^\rho\gamma^{\mu\nu}-\frac{2a+b}{4}\Big(\partial^\nu\gamma^{\rho\mu}+\partial^\mu\gamma^{\rho\nu}\Big)\\
&+\frac{c}{4}\Big(\eta^{\rho\mu}\partial_\sigma\gamma^{\sigma\nu}+\eta^{\rho\nu}\partial_\sigma\gamma^{\sigma\mu}\Big)\\
&-\frac{c}{2}\Big(\frac{1}{2}\eta^{\rho\mu}\partial^\nu\gamma+\frac{1}{2}\eta^{\rho\nu}\partial^\mu\gamma+\eta^{\mu\nu}\partial_\sigma\gamma^{\sigma\rho}\Big)\\
&+\frac{c}{2}\eta^{\mu\nu}\partial^\rho\gamma+\frac{2a+b}{2}\Big(\partial^\mu a^{\rho\nu}+\partial^\nu a^{\rho\mu}\Big)\\
&+\frac{c}{2}\Big(\eta^{\rho\mu}\partial_\sigma\alpha^{\sigma\nu}+\eta^{\rho\nu}\partial_\sigma\alpha^{\sigma\mu}\Big)-c\eta^{\mu\nu}\partial_\sigma a^{\sigma\rho}\Big\}\\
&-\Phi^2\Big\{\frac{2a+b}{4}\Box\gamma^{\mu\nu}-\frac{2a+b}{8}\Big(\partial_\rho\partial^\nu\gamma^{\mu\rho}+\partial_\rho\partial^\mu\gamma^{\nu\rho}\Big)\\
&+\frac{c}{8}\Big(\partial^\mu\partial_\sigma\gamma^{\sigma\nu}+\partial^\nu\partial_\sigma\gamma^{\sigma\mu}\Big)-\frac{c}{4}\Big(\partial^\mu\partial^\nu\gamma+\partial_\rho\partial_\sigma\gamma^{\sigma\rho}\eta^{\mu\nu}\Big)\\
&+\frac{c}{4}\eta^{\mu\nu}\Box\gamma+\frac{2a+b}{4}\Big(\partial_\rho\partial^\mu a^{\rho\nu}+\partial_\rho\partial^\nu a^{\rho\mu}\Big)\\
&+\frac{c}{4}\Big(\partial^\mu\partial_\sigma a^{\sigma\nu}+\partial^\nu\partial_\sigma a^{\sigma\mu}\Big)\Big\}+\frac{c}{2}\partial^\mu j^\nu+\frac{c}{2}\partial^\nu j^\mu-c\,\partial_\rho j^\rho\eta^{\mu\nu}=\frac{1}{2}\kappa_4\lambda\Phi^2T_\gamma^{\mu\nu}\,,
\end{aligned}
$$

where $\Box := \eta^{\mu\nu}\partial_\mu\partial_\nu$.

- EoM of $a_{\mu\nu}$:

$$-j_\rho\Big\{\frac{2a+b}{2}\Big(\partial^\mu\gamma^{\rho\nu}-\partial^\nu\gamma^{\rho\mu}\Big)+\frac{c}{2}\Big(\eta^{\rho\mu}\partial_\sigma\gamma^{\sigma\nu}-\eta^{\rho\nu}\partial_\sigma\gamma^{\sigma\mu}\Big)$$
$$-\frac{c}{2}\Big(\eta^{\rho\mu}\partial^\nu\gamma-\eta^{\rho\nu}\partial^\mu\gamma\Big)+2\Big(2a-b\Big)\partial^\rho a^{\mu\nu}$$
$$+\Big(2a-3b\Big)\Big(\partial^\mu a^{\nu\rho}+\partial^\nu a^{\rho\mu}\Big)+c\Big(\eta^{\rho\mu}\partial_\sigma a^{\sigma\nu}-\eta^{\rho\nu}\partial_\sigma a^{\sigma\mu}\Big)\Big\}$$
$$-\Phi^2\Big\{\frac{2a+b}{4}\Big(\partial_\rho\partial^\mu\gamma^{\rho\nu}-\partial_\rho\partial^\nu\gamma^{\rho\mu}\Big)+\frac{c}{4}\Big(\partial^\mu\partial_\sigma\gamma^{\sigma\nu}-\partial^\nu\partial_\sigma\gamma^{\sigma\mu}\Big)$$
$$+\Big(2a-b\Big)\Box\, a^{\mu\nu}+\frac{2a-3b-c}{2}\Big(\partial_\rho\partial^\mu a^{\nu\rho}+\partial_\rho\partial^\nu a^{\rho\mu}\Big)\Big\}=\frac{1}{2}\kappa_4\lambda\Phi^2 T_a^{\mu\nu}\,.$$

- EoM of A_μ:

$$\kappa_4\lambda\Phi^2\frac{\delta\mathcal{L}_{\mathsf{m}}}{\delta A_\mu}+3a\kappa^2\kappa_4^2\Phi^5(\partial_\nu\Phi)F^{\mu\nu}+\frac{a\kappa^2\kappa_4^2}{2}\Phi^6\partial_\nu F^{\mu\nu}=0\,.$$

- EoM of Φ:

$$\Phi T_{\mathsf{NGR}}-\frac{3a\kappa^2\kappa_4^2}{4}\Phi^5 F_{\mu\nu}F^{\mu\nu}+2\lambda\kappa_4\Phi\mathcal{L}_{\mathsf{m}}$$
$$+\kappa_4\lambda\Phi^2\frac{\delta\mathcal{L}_{\mathsf{m}}}{\delta\Phi}-\Big(8a+4b+4c\Big)\Box\,\Phi+2c\,\Phi\partial_\mu T^\mu=0\,.$$

- We consider the case of Weyl gauge invariance

$$2a + b + 4c = 0 \quad \text{or} \quad 2a + b = 0$$

along with the gauge conditions $\partial_\mu \gamma^{\mu\nu} = 0$, $\partial_\mu a^{\mu\nu} = 0$ and $\gamma = 0$

- The EoM of $\gamma_{\mu\nu}$ and $a_{\mu\nu}$ for $2a + b + 4c = 0$

$$\begin{cases} c j_\rho \Big\{ 2\partial^\rho \gamma^{\mu\nu} - \partial^\mu \gamma^{\rho\nu} - \partial^\nu \gamma^{\rho\mu} + 2\partial^\mu a^{\rho\nu} + 2\partial^\nu a^{\rho\mu} \Big\} \\ \qquad + c\,\Phi^2 \Box\, \gamma^{\mu\nu} + \frac{c}{2}\,\partial^\mu j^\nu + \frac{c}{2}\,\partial^\nu j^\mu - c\,\partial_\rho j^\rho \eta^{\mu\nu} = \frac{1}{2}\kappa_4 \lambda \Phi^2 T_\gamma^{\mu\nu}\,, \\ \qquad j_\rho \Big\{ 2c \Big(\partial^\mu \gamma^{\rho\nu} - \partial^\nu \gamma^{\rho\mu} \Big) + 4(b+2c)\partial^\rho a^{\mu\nu} \\ +4\Big(b+c\Big)\Big(\partial^\mu a^{\nu\rho} + \partial^\nu a^{\rho\mu}\Big)\Big\} + 2\Big(b+2c\Big)\Phi^2 \Box\, a^{\mu\nu} = \frac{1}{2}\kappa_4 \lambda \Phi^2 T_a^{\mu\nu}\,. \end{cases}$$

  - Assuming that the scalar field varies slowly

$$\Phi \approx \Phi_c \text{ a constant field,} \quad \Longrightarrow \quad j_\mu \approx 0,$$

  - The eqs. reduce to

$$\begin{cases} \Box\, \gamma^{\mu\nu} = \dfrac{\kappa_4 \lambda}{2c}\, T_\gamma^{\mu\nu}\,, \\ \Box\, a^{\mu\nu} = \dfrac{\kappa_4 \lambda}{4(b+2c)}\, T_a^{\mu\nu}\,. \end{cases}$$

- The EoM of $\gamma_{\mu\nu}$ and $a_{\mu\nu}$ for $2a+b=0$

$$\begin{cases} \frac{c}{2}\,\partial^\mu j^\nu + \frac{c}{2}\,\partial^\nu j^\mu - c\,\partial_\rho j^\rho \eta^{\mu\nu} = \frac{1}{2}\kappa_4 \lambda \Phi^2 T_\gamma^{\mu\nu}\,, \\ -8a j_\rho f^{\rho\mu\nu} - 4a\,\Phi^2 \Box\, a^{\mu\nu} = \frac{1}{2}\kappa_4 \lambda \Phi^2 T_a^{\mu\nu}\,, \end{cases}$$

where $f^{\rho\mu\nu} := \partial^\rho a^{\mu\nu} + \partial^\mu a^{\nu\rho} + \partial^\nu a^{\rho\mu}$ is the field strength of $a^{\mu\nu}$.

- Only the anti-symmetric tensor $a^{\mu\nu}$ survives!!
  - For slowly varying scalar field, the eq. reduce to

$$\begin{cases} 0 = T_\gamma^{\mu\nu}\,, \\ \Box\, a^{\mu\nu} = -\frac{\kappa_4 \lambda}{8a}\, T_a^{\mu\nu}\,. \end{cases}$$

Outline

Summary

- We have summarized the possible choices of the coefficients (a, b, c) on the torsion scalar as shown by TABLE.
- The Einstein-frame can be achieved by taking $2a + b + c = 0$ with $c \leq 0$.
- We have obtained new classes of conformal invariant theories of gravity without the electromagnetic field A_μ.
- We provide a conformal invariant gravity in teleparallelism with the condition $2a + b = 0$ with $c = 0$, which gives rise to the existence of the Einstein-frame.
- The Weyl gauge theory under the ghost-free constraints $2a + b + c > 0$ and $c \neq 0$ can be obtained with the requirements either $2a + b + 4c = 0$ or $2a + b = 0$.
- For the conformal invariant models with $2a + b = 0$, we found that only the anti-symmetric tensor field is allowed rather than the symmetric one.

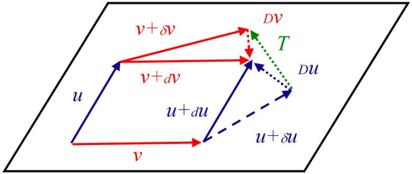
End

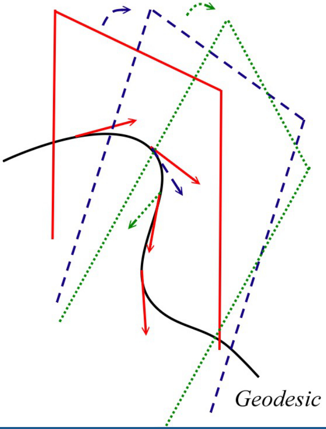
Thank You!!!

Outline

Geometrical Meaning of Torsion

- Torsion free: a tangent vector does not rotate when we parallel transport it. (*P.371, John Baez and Javier P. Muniain, "Gauge Fields, Knots and Gravity," 1994*)
- $T(u,v) = \nabla_u v - \nabla_v u - \underbrace{[u,v]}_{\text{vanished in coordinate space}}$





Notation in 5D

- In orthonormal frame, 5D metric is $\bar{g}_{MN} = \bar{\eta}_{IJ}\, e^{I}{}_{M}\, e^{J}{}_{N}$,
  $\bar{\eta}_{IJ} =$ diag$(+1, -1, -1, -1, \varepsilon)$ with $\varepsilon = -1$.

- Coordinate frame
  $M, N = 0, 1, 2, 3, 5,\quad \mu, \nu = 0, 1, 2, 3,\quad \alpha, \beta = 1, 2, 3.$

- Orthonormal frame (anholonomic frame)
  $A, B, I, J, K = \hat{0}, \hat{1}, \hat{2}, \hat{3}, \hat{5},\quad i, j, k = \hat{0}, \hat{1}, \hat{2}, \hat{3},\quad a, b = \hat{1}, \hat{2}, \hat{3}.$

Affine Connection and Lorentz Connection

- Consider noncoordinate basis (orthonormal frame)

$$\begin{aligned} e_i{}^\nu D_\mu V^i &= e_i{}^\nu(\partial_\mu V^i + \omega^i{}_{j\mu} V^j) \\ &= e_i{}^\nu\Big(\partial_\mu(e^i{}_\rho V^\rho) + \omega^i{}_{j\mu} V^j\Big) \\ &= e_i{}^\nu\Big((\partial_\mu e^i{}_\rho)V^\rho + e^i{}_\rho(\partial_\mu V^\rho) + \omega^i{}_{j\mu}\, e^j{}_\rho\, V^\rho\Big) \\ &= (e_i{}^\nu \partial_\mu e^i{}_\rho)V^\rho + \underbrace{\delta^\nu_\rho \partial_\mu V^\rho}_{\partial_\mu V^\nu} + e_i{}^\nu\, \omega^i{}_{j\mu}\, e^j{}_\rho\, V^\rho \\ &= \partial_\mu V^\nu + (e_i{}^\nu \partial_\mu e^i{}_\rho + e_i{}^\nu\, \omega^i{}_{j\mu}\, e^j{}_\rho)V^\rho \\ &\equiv \partial_\mu V^\nu + \Gamma^\nu{}_{\rho\mu} V^\rho = \nabla_\mu V^\nu. \end{aligned}$$

The relation between affine connection and Lorentz connection

$$\Gamma^\nu{}_{\rho\mu} \equiv e_i{}^\nu \partial_\mu e^i{}_\rho + e_i^\nu\, \omega^i{}_{j\mu}\, e^j_\rho$$

- We can define the total covariant derivative $\boldsymbol{\nabla}_\mu$

$$\partial_\mu e^i{}_\rho - \Gamma^\nu{}_{\rho\mu}\, e^i{}_\nu + \omega^i{}_{j\mu}\, e^j{}_\rho = 0$$

$$\Longrightarrow \boldsymbol{\nabla}_\mu e^i{}_\rho = 0 \text{ (vielbein postulate).}$$

◂ Absolute parallelism

Brief History of 5-Dimensional Theories

- Kaluza-Klein (KK) theory: to unify electromagnetism and gravity by gauge theory
  - Cylindrical condition (*Kaluza, 1921*)
  - Compactification to a small scale (*Klein, 1926*)
- Generalization of KK: induced-matter theory
  $\Longrightarrow$ matter from the 5th-dimension (*Wesson, 1998*)
- Large Extra dimension (*Arkani-Hamed, Dimopoulos and Dvali (ADD), 1998*)
  - Solving hierarchy problem
  - SM particles confined on the 3-brane

- Randall-Sundrum model in AdS_5 spacetime (*Randall and Sundrum, 1999*)
  - RS-I (UV-brane and SM particles confined on IR-brane)
    $\Longrightarrow$ solving hierarchy problem
  - RS-II (only one UV brane)
    $\Longrightarrow$ compactification to generate 4-dimensional gravity

- DGP brane model (*Dvali, Gabadadze and Porrati, 2000*)
  $\Longrightarrow$ accelerating universe

- Universal Extra Dimension (*Appelquist, Cheng and Dobrescu, 2001*)
  - Not only graviton but SM particles can propagate to the extra dimension $\Longrightarrow$ low compactification scale: reach to the electroweak scale

5D TEGR without Vector

- In Gauss normal coordinate

$$\bar{g}_{MN} = \begin{pmatrix} g_{\mu\nu}(x^\mu, y) & 0 \\ 0 & \varepsilon\phi^2(x^\mu, y) \end{pmatrix} .$$

- The 5D torsion scalar in the orthonormal frame

$$^{(5)}T = \underbrace{\bar{T}}_{\text{induced 4D torsion scalar}} + \frac{1}{2}\left(\bar{T}_{i\hat{5}j}\,\bar{T}^{i\hat{5}j} + \bar{T}_{i\hat{5}j}\,\bar{T}^{j\hat{5}i}\right) + 2\,\bar{T}^{j}{}_{j}{}^{i}\,\bar{T}^{\hat{5}}{}_{i\hat{5}} - \bar{T}^{j}{}_{\hat{5}j}\,\bar{T}^{k\hat{5}}{}_{k} .$$

- The non-vanishing components of vielbein are $e^{i}{}_{\mu}$ and $e^{\hat{5}}{}_{5}$

Projection of the torsion tensor

$\bar{T}^{\rho}{}_{\mu\nu} = T^{\rho}{}_{\mu\nu}$ (purely 4-dimensional object)

$$
\begin{array}{cccc}
i & \longrightarrow & \mu & (\bigcirc) \\
i & \longrightarrow & 5 & (\times) \\
\hat{5} & \longrightarrow & \mu & (\times) \\
\hat{5} & \longrightarrow & 5 & (\bigcirc)
\end{array}
$$

- The 5D torsion scalar in the coordinate frame

$$
{}^{(5)}T = \bar{T} + \frac{1}{2}\left(\bar{T}_{\rho 5\nu}\,\bar{T}^{\rho 5\nu} + \bar{T}_{\rho 5\nu}\,\bar{T}^{\nu 5\rho}\right) + 2\,\bar{T}^{\sigma}{}_{\sigma}{}^{\mu}\,\bar{T}^{5}{}_{\mu 5} - \bar{T}^{\nu}{}_{5\nu}\,\bar{T}^{\sigma 5}{}_{\sigma}\,.
$$

Note:
In general, the induced torsion is $\bar{T}^{\rho}{}_{\mu\nu} = T^{\rho}{}_{\mu\nu} + \bar{C}^{\rho}{}_{\mu\nu}$, where

$$
\bar{C}^{\rho}{}_{\mu\nu} = \bar{e}_{\hat{5}}{}^{\rho}\overbrace{(\partial_{\mu}e^{\hat{5}}{}_{\nu} - \partial_{\nu}e^{\hat{5}}{}_{\mu})}^{\bar{C}^{\hat{5}}{}_{\mu\nu}}\,.
$$

$\bar{C}^{\hat{5}}{}_{\mu\nu} = \Gamma^{\hat{5}}{}_{\nu\mu} - \Gamma^{\hat{5}}{}_{\mu\nu} = h^{M}_{\mu}\,h^{N}_{\nu}\,T^{\hat{5}}{}_{MN} \sim \omega_{\mu\nu}$ is related to the extrinsic torsion or twist $\omega_{\mu\nu}$.

KK Reduction

- The metric is reduced to

$$\bar{g}_{MN} = \begin{pmatrix} g_{\mu\nu}(x^\mu) & 0 \\ 0 & -\phi^2(x^\mu) \end{pmatrix} .$$

- The residual components are $T^\rho{}_{\mu\nu}$ and $\bar{T}^5{}_{\mu 5} = \frac{1}{\phi}\partial_\mu \phi$.
- The 5D torsion scalar with $\kappa_4 = \kappa_5/(2\pi r)$

$${}^{(5)}T = T + 2\, T^\sigma{}_\sigma{}^\mu\, \bar{T}^5{}_{\mu 5}$$

Effective action of 5D TEGR

$$S_{\text{eff}} = \frac{1}{2\kappa_4} \int d^4x\, e\, (\phi\, T + 2\, T^\mu\, \partial_\mu \phi)$$

(*C.Q. Geng, LWL, H.H. Tseng, 2014*)

Minimal and Non-Minimal Coupling

Minimal coupled case

$$T \sim -R \quad \text{(TEGR)}.$$

- TEGR in 5D KK scenario with the metric given by

$$\bar{g}_{MN} = \begin{pmatrix} g_{\mu\nu} - k^2 A_\mu A_\nu & k\, A_\mu \\ k\, A_\nu & -1 \end{pmatrix} \text{ with } k^2 = \kappa_4,$$

The effective Lagrangian is

$$\mathcal{L}_{\text{eff}} = e\left(\frac{1}{2\kappa_4} T - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right) \quad \text{(coincides with the form of GR).}$$

(*de Andrade, Guillen, Pereira, 2000*)

Non-minimal coupled case

$$\phi T \nsim -\phi R$$

Remark:
The curvature-torsion relation in TEGR: $-\tilde{R}(e) = T - 2\,\tilde{\nabla}_\mu T^\mu$.

5D GR vs. 5D TEGR in KK Scenario

The dimensional reduction of 5D GR

- Brans-Dicke theory with $\omega_{\text{BD}} = 0$

$$-\sqrt{-{}^{(5)}g}\,{}^{(5)}\tilde{R} \quad\longrightarrow\quad -\sqrt{-g}\Big(\phi\tilde{R}\underbrace{-\Box\phi}_{\text{surface term}}\Big).$$

Remark:

Brans-Dicke theory (*Brans & Dicke, 1961*):

$$\int d^4x\sqrt{-g}\left\{\frac{-1}{2\kappa}\phi\tilde{R}+\frac{\omega_{\text{BD}}}{\phi}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi\right\}.$$

The dimensional reduction of 5D TEGR

$$ {}^{(5)}e\,{}^{(5)}T \quad \longrightarrow \quad e\left(\phi T + \underbrace{2\,T^{\mu}\,\partial_{\mu}\phi}_{\text{no analogue}}\right). $$

- Substituting the relation $-\tilde{R}(e) = T - 2\,\tilde{\nabla}_{\mu}T^{\mu}$ into the 4D effective Lagrangian

Equivalence

$$ \frac{-1}{2\kappa_4}\int d^4x\, e\left(\phi\tilde{R}(e) \underbrace{-2\,\tilde{\nabla}_{\mu}(\phi\, T^{\mu})}_{\text{surface term}}\right). $$

Einstein-frame

- By conformal transformation ($\widetilde{g}_{\mu\nu} = \Omega^2(x)\, g_{\mu\nu}$):

$$\begin{aligned} T &= \Omega^2\, \widetilde{T} - 4\,\Omega\, \widetilde{g}^{\mu\nu}\, \widetilde{T}_\mu \partial_\nu \Omega + 6\, \widetilde{g}^{\mu\nu}\, \partial_\mu \Omega\, \partial_\nu \Omega\,, \\ T_\mu &= \widetilde{T}_\mu + 3\,\Omega^{-1}\, \partial_\mu \Omega\,. \end{aligned}$$

- Choosing $\phi = \Omega^2$, the action reads

$$S_{\text{eff}} = \int d^4x\, \widetilde{e}\, \left[\frac{1}{2\,\kappa_4}\, \widetilde{T} + \frac{1}{2}\, \widetilde{g}^{\mu\nu} \partial_\mu \psi\, \partial_\nu \psi \right]\,,$$

(*C.Q. Geng, Chang Lai, LWL, H.H. Tseng, 2014*)

where $\psi = (6/\kappa_4)^{1/2} \ln \Omega$ is the dilaton field.

- There exists an Einstein-frame for such a non-minimal coupled effective Lagrangian in teleparallel gravity.

Equation of Motion of the TEGR Effective Action

- The gravitational equation of motion

$$\frac{1}{2}\, e_i{}^{\mu}\left(\phi\, T + 2\, T^{\sigma}\,\partial_{\sigma}\phi\right) - e_i{}^{\rho}\left(\phi\, T^{j}{}_{\rho\nu}\, S_j{}^{\mu\nu}\right) - e_i{}^{\nu}\left(\partial_{\sigma}\phi\, T^{\mu}{}_{\nu}{}^{\sigma} + \partial_{\nu}\phi\, T^{\mu} + \partial^{\mu}\phi\, T_{\nu}\right) + \frac{1}{e}\,\partial_{\nu}\left(e\left(\phi\, S_i{}^{\mu\nu} + e_i{}^{\mu}\,\partial^{\nu}\phi - e_i{}^{\nu}\,\partial^{\mu}\phi\right)\right) = \kappa_4\,\Theta_i^{\mu}$$

with $\Theta^{\mu}_{\nu} = \mathrm{diag}(\rho, -P, -P, -P)$

- The modified Friedmann equations in flat FLRW universe are

$$3\,\phi\, H^2 + 3\, H\,\dot{\phi} = \kappa_4\,\rho\,,$$
$$3\,\phi\, H^2 + 2\,\dot{\phi}\, H + 2\,\phi\,\dot{H} + \ddot{\phi} = -\,\kappa_4\, P\,,$$

where $H = \dot{a}/a$ is the Hubble parameter (here $\rho = P = 0$ is assumed.)

- The equations of motion of scalar field ϕ in the

$$T - 2\,\partial_\mu T^\mu - 2T^\mu\Gamma^\nu_{\nu\mu} + e\,L_m = 0 \xrightarrow[\Gamma^\nu{}_{\nu\mu}=\Gamma^\alpha{}_{\alpha 0}=3\,\frac{\dot a}{a}]{\text{absence of matter}} a\,\ddot a + \dot a^2 = 0\,.$$

- Suppose the solution of $a(t)$ is proportional to t^m, the solution is

$$a(t) = a_s + b\sqrt{t}\,.$$

- The constraint of the coefficient: $a_s\,b = 0$
- $b = 0$ case:
  - $a(t) = a_s \Rightarrow$ the static universe.
- $a_s = 0$ case:
  - The Hubble parameter $H = 1/(2\,t) > 0$
  - The the acceleration of scale factor $\ddot a = -b/(4\,t^{2/3}) > 0$ for $b < 0$ $\Longrightarrow$ accelerated expanding universe.
- In general relativity, the equation of motion of ϕ is $\tilde R(e) = 0$ $\Longrightarrow$ the same solution for the scale factor in vacuum.

- Assuming that $\omega = \omega(\phi)$ under the conformal transformation
- Due to $d\widetilde{\phi}/d\phi = \omega' \exp(\lambda_\phi \omega)(1 + \lambda_\phi \omega' \phi)$ with $\omega' := d\omega/d\phi$

$$\widetilde{\phi} = \widetilde{\phi}(\phi) \quad \Longrightarrow \quad \omega(\widetilde{\phi})$$

- By setting $\partial_\mu \omega = \partial_\mu \ln \widetilde{\phi}$.
- Lagrangian density $(*)$ without A_μ becomes

$$\mathcal{L}_g = \widetilde{e} \frac{1}{2\kappa_4} \left\{ \widetilde{\phi}\, \widetilde{T}_{\mathsf{NGR}} + \Big(4\,a + 2\,b\Big) \widetilde{g}^{\mu\nu}\, \widetilde{T}_\mu\, \partial_\nu \widetilde{\phi} + \Big(24a + 12b\Big) \frac{1}{\widetilde{\phi}} \widetilde{g}^{\mu\nu} \partial_\mu \widetilde{\phi}\, \partial_\nu \widetilde{\phi} \right\}.$$

- Comparing with the effective Lagrangian

Lagrangian

$$\left.\begin{aligned} 2a + b &= -c\,, \\ 24a + 12b &= 2a + b + c\,, \end{aligned}\right\} \Longrightarrow \left\{\begin{aligned} 2a + b &= 0\,, \\ c &= 0\,. \end{aligned}\right.$$

Other Conformal Invariant Model

The conformal invariant model investigated by Maluf and Faria is

$$\mathcal{L} = ke\left[-\phi^2\left(\frac{1}{4}T^{abc}T_{abc} + \frac{1}{2}T^{abc}T_{cba} - \frac{1}{3}T^aT_a\right) + k' g^{\mu\nu}D_\mu\phi D_\nu\phi\right],$$

(*Maluf and Faria, 2012*)

where $k = 1/(16\pi G)$, $k' = 6$ and $\eta_{ab} = (-1, +1, +1, +1)$ as well as $D_\mu := \partial_\mu - \frac{1}{3}T_\mu$, which gives the conformal invariant condition

$$2a + b + 3c = 0\,.$$

In their discussion, a new arbitrary parameter k' for the scalar kinetic term is introduced, resulting in a four-parameters model.

Note:
In our model, the coefficient of the kinetic term of ϕ is $2a + b + c$ so that the conformal invariant theory is totally determined by three parameters only.

Comparison Table

- Minimal coupled models with $\frac{1}{2\kappa}T_{\text{NGR}}$

Class	Additional condition	Reference
$2a+b+c=0$,	-	*Einstein, 1929*
$(a,b,c)=(\frac{1}{4},\frac{1}{2},-1)$	-	*Cho, 1976*
$2a+b+c=0$,	-	*Hehl et al., 1978*
$c=-1$	-	*Nitsch and Hehl, 1980*
	-	*Hayashi and Shirafuji, 1979*
$2a+b+c=0$,	Static isotropic	*Hehl et al., 1978*
$(a,b,c)=(\frac{1}{2},0,-1)$	metric by Scherrer	*Nitsch and Hehl, 1980*
$2a+b+c=0$, $(a,b,c)=(\frac{1}{4},\frac{1}{2},-1)$	Einstein-frame	*Geng et al., 2014*
$2a+b+c=0$, $c\leq 0$	Einstein-frame	✓

- Non-minimal coupled models with $\frac{1}{2\kappa}\phi T_{\mathsf{NGR}}$ (conformal invarience)

Class	Additional condition	Reference
$2a+b+3c=0$	$k' g^{\mu\nu} D_\mu \phi D_\nu \phi$, where $D_\mu := \partial_\mu - \frac{1}{3} T_\mu$ with arbitrary k'	*Maluf and Faria, 2012*
$2a+b+c=0$, $c=0$	-	✓
$2a+b+4c=0$ $2a+b=0$	$2a+b+c>0$, $c\neq 0$	✓

Equations of Motion of the NGR Effective Action

- Varying the full action $S = S_g + S_{\mathsf{m}}$ with respect to $e^i{}_\mu$, A_μ and ϕ

$$\frac{1}{2}\, e_i{}^\mu \Big(\phi\, T_{\mathsf{NGR}} - \frac{a\kappa^2}{4} \phi^3\, g^{\lambda\rho} g^{\nu\sigma} F_{\lambda\nu}\, F_{\rho\sigma} + \frac{2a+b+c}{\phi}\, g^{\lambda\nu} \partial_\lambda \phi\, \partial_\nu \phi$$

$$-2\, c\, g^{\lambda\nu}\, T_\lambda\, \partial_\nu \phi \Big) - e_i{}^\rho \Big\{ \phi\, T^j{}_{\rho\nu}\, \Sigma_j{}^{\mu\nu} - \frac{a\kappa^2}{2} \phi^3\, g^{\mu\lambda} g^{\nu\sigma}\, F_{\lambda\nu}\, F_{\rho\sigma}$$

$$+ \frac{2a+b+c}{\phi}\, g^{\mu\lambda} \partial_\lambda \phi\, \partial_\rho \phi - c \Big(\partial_\sigma \phi\, T^\mu{}_\rho{}^\sigma + \partial_\rho \phi\, T^\mu + \partial^\mu \phi\, T_\rho \Big) \Big\}$$

$$+ \frac{1}{e}\, \partial_\nu \Big\{ e \Big(\phi\, \Sigma_i{}^{\mu\nu} - c\, e_i{}^\mu\, \partial^\nu \phi + c\, e_i{}^\nu\, \partial^\mu \phi \Big) \Big\} = \kappa_4 \lambda \phi\, \Theta_i{}^\mu\,,$$

$$\frac{\lambda}{e} \frac{\delta \mathcal{L}_{\mathsf{m}}}{\delta A_\nu} + \frac{1}{e} \frac{a\kappa^2}{2\kappa_4} \phi^2\, \partial_\mu \Big(e F^{\mu\nu} \Big) + \frac{3a\kappa^2}{2\kappa_4} \phi\, \partial_\mu \phi\, F^{\mu\nu} = 0\,,$$

$$T_{\mathsf{NGR}} - \frac{3a\kappa^2}{4} \phi^2\, g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} + 2\kappa_4 \lambda \Big(L_{\mathsf{m}} + \frac{\phi}{e} \frac{\delta \mathcal{L}_{\mathsf{m}}}{\delta \phi} \Big)$$

$$+ \frac{2a+b+c}{\phi^2} \Big(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 2\phi\, \Box \phi \Big) + \frac{2c}{e}\, \partial_\nu \Big(e g^{\mu\nu} T_\mu \Big) = 0\,,$$

where $\Box := \frac{1}{e} \partial_\nu (e g^{\mu\nu} \partial_\mu)$ and $\Theta_i{}^\mu := -\frac{1}{e} \frac{\delta\, \mathcal{L}_m}{\delta\, e^i{}_\mu}$ is the dynamical energy-momentum tensor with $\mathcal{L}_{\mathsf{m}} := e L_{\mathsf{m}}$.

Weyl Geometry

- Parallel transport of the vector V from point $\mathbf{P}(= x^\mu)$ to point $\mathbf{P}'(= x^\mu + dx^\mu)$

$$\nabla V = dx^\nu (\nabla_\nu V^\mu)\partial_\mu = (dV^\mu - \delta V^\mu)\partial_\mu = 0\,.$$

- Weyl Geometry (*Weyl, 1918*): Define the measure $l := g_{\mu\nu} V^\mu V^\nu$ of V^μ, the variation of the measure l is proportional to l with the 1-form factor $\varphi = \varphi_\mu dx^\mu$

$$dl = -\varphi l = -(\varphi_\rho dx^\rho) g_{\mu\nu} V^\mu V^\nu\,.$$

- However the variation of l can be written as

$$\begin{aligned} dl &= d\left(g_{\mu\nu} V^\mu V^\nu\right) \\ &= \partial_\rho g_{\mu\nu} dx^\rho V^\mu V^\nu dx^\rho - g_{\mu\nu} \Gamma^\mu{}_{\sigma\rho} dx^\rho V^\sigma V^\nu - g_{\mu\nu} \Gamma^\nu{}_{\sigma\rho} dx^\rho V^\mu V^\sigma \\ &= (\partial_\rho g_{\mu\nu} - g_{\sigma\nu} \Gamma^\sigma{}_{\mu\rho} - g_{\mu\sigma} \Gamma^\sigma{}_{\nu\rho}) dx^\rho V^\mu V^\mu\,, \end{aligned}$$

where $dV^\mu = \delta V^\mu = -\Gamma^\mu{}_{\sigma\rho} V^\sigma dx^\rho$.

Identity: $$\partial_\rho g_{\mu\nu} - g_{\sigma\nu} \Gamma^\sigma{}_{\mu\rho} - g_{\mu\sigma} \Gamma^\sigma{}_{\nu\rho} = -\varphi_\rho g_{\mu\nu} \neq 0\,.$$

◂ WGT