Non-Gaussianity and finite length inflation

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Abstract

In the present paper, certain inflation models are shown to have large non-Gaussianity in special cases. Namely, finite length inflation models with an effective higher derivative interaction, in which slow-roll inflation is adopted as inflation and a scalar-matter-dominated period or power inflation is adopted as pre-inflation, are considered. Using Holman and Tolley’s formula of the nonlinearity parameter $f_{NL}^{\text{flattened}}$, we calculate the value of $f_{NL}^{\text{flattened}}$. A large value of $f_{NL}^{\text{flattened}}$ ($f_{NL}^{\text{flattened}} > 100$) can be obtained for all of the models considered herein when the length of inflation is 60-63 $e$-folds and $f_{NL}$ has strong dependence on the length of inflation. Interestingly, this length is similar to that for the case in which the suppression of the CMB angular power spectrum of $\ell = 2$ was derived using the inflation models described in our previous papers.

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1. Introduction

Non-Gaussianity of primordial perturbations is one of the most interesting problems implied by the WMAP data [1, 2]. The observational limits on the nonlinearity parameter from WMAP seven-year data [2] are $-10 < f_{\text{NL}}^{\text{local}} < 74$ (95% CL), $-214 < f_{\text{NL}}^{\text{equil}} < 266$ (95% CL) and $-410 < f_{\text{NL}}^{\text{orthog}} < 6$ (95% CL). However, the standard simple inflation model predicts approximately Gaussian fluctuation, the deviation from Gaussian of which is very small. Several studies have attempted to achieve such large non-Gaussianity. Holman and Tolley [3] showed that if the effective action for the inflaton contains a higher-derivative interaction, which is derived, for example, from k-inflation [4] or DBI inflation [5], and the initial state of inflation is not the Bunch-Davies vacuum, then enhanced non-Gaussianity is derived in the "flattened" triangle configurations, the contribution of which is also discussed in [6]. In their paper, the initial state of the curvature perturbation in inflation was assumed not to be the Bunch-Davies vacuum, i.e., squeezed states, but they did not report a concrete value or the physical mechanism that generates the initial state in inflation, although the value of the coefficient of the initial state in inflation has a very important effect on the non-Gaussianity.

On the other hand, the effect of the initial condition in inflation on the power spectrum of curvature perturbations has been considered [7] and the effect of the length of inflation and pre-inflation physics on the power spectrum and the angular power spectrum of scalar and tensor perturbations has been examined by the present authors. [8-9]. The suppression of the spectrum at $l = 2$ as indicated by Wilkinson Microwave Anisotropy Probe (WMAP) data [1] may be explained to a certain extent by the finite length of inflation for an inflation of 50–60 $e$-folds [9]. Of course, there are many attempts [10] to derive this suppression. Based on the physical conditions before inflation, we have shown that the initial state of scalar perturbations
inflation is not simply the Bunch-Davies state, but rather a more general state (a squeezed state), where a scalar-matter-dominated period, a radiation-dominated period, or another inflation is considered as pre-inflation, and the general initial states in inflation were calculated analytically. In the present paper, we demonstrate a new property of the proposed inflation model. Using Holman and Talley’s formula for the nonlinearity parameter $f_{NL}^{\text{flattened}}$, we calculate the value of $f_{NL}^{\text{flattened}}$ for the case in which the proposed finite inflation models have effective higher-derivative interactions, where slow-roll inflation is adopted as inflation and a scalar-matter-dominated period or power-law inflation period is adopted as pre-inflation. The obtained results are very interesting.

2. Scalar perturbations

We consider curvature perturbations in inflation and a scalar-matter-dominated epoch. The background spectrum considered is a spatially flat Friedman-Robertson-Walker (FRW) universe described by metric perturbations. The line element for the background and perturbations is generally expressed as [11]

$$ds^2 = a^2(\eta)\left\{(1+2A)d\eta^2 - 2\partial_i B dx^i d\eta - \{(1-2\Psi)\delta_{ij} + 2\partial_i \partial_j E + h_{ij}\}dx^i dx^j\right\}, \quad (2-1)$$

where $\eta$ is the conformal time, the functions $A$, $B$, $\Psi$, and $E$ represent the scalar perturbations, and $h_{ij}$ represents tensor perturbations. The density perturbation in terms of the intrinsic curvature perturbation of comoving hypersurfaces is given by $\mathcal{R} = -\Psi - (H/\phi)\delta\phi$, where $\phi$ is the inflaton field, $\delta\phi$ is the fluctuation of the inflaton field, $H$ is the Hubble expansion parameter, and $\mathcal{R}$ is the curvature perturbation. Overdots represent derivatives with respect to time $t$, and primes represent derivatives with respect to the conformal time $\eta$. Introducing the gauge-invariant potential $u = a(\eta)(\delta\phi + (\phi/H)\Psi)$ allows the action for scalar perturbations to
be written as [12]

\[ S = \frac{1}{2} \int d^3 x \left\{ \left( \frac{\partial u}{\partial \eta} \right)^2 - c_s^2 (\nabla u)^2 + \frac{Z^*}{Z} u^2 \right\}, \]  

where \( c_s \) is the velocity of sound, and in inflation \( Z = a \dot{\phi} / H \), and \( u = -Z \dot{\eta} \). The field \( u(\eta, x) \) is expressed using annihilation and creation operators as follows:

\[ u(\eta, x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3 k}{k} \{ u_k(\eta) a_k + u_k^*(\eta) a_k^\dagger \} e^{ikx}. \]  

The field equation for \( u_k(\eta) \) is derived as

\[ \frac{d^2 u_k}{d\eta^2} + \left( c_s^2 k^2 - \frac{1}{Z} \frac{d^2 Z}{d\eta^2} \right) u_k = 0, \]  

where \( c_s^2 = 1 \) is assumed in inflation. The solution of \( u_k \) satisfies the normalization condition

\[ u_k d u_k^* / d\eta - u_k^* d u_k / d\eta = i. \]

First, slow-roll inflation is considered. The slow-roll parameters are defined as [13, 14]:

\[ \varepsilon = 3 \left( \frac{\dot{\phi}^2}{2} + \frac{\phi^2}{2} + V \right)^{-1} = 2M_p^2 \left( \frac{H'(?)}{H(\phi)} \right)^2, \]  

\[ \delta = 2M_p^2 \frac{H''(\phi)}{H(\phi)}, \]  

\[ \xi = 4M_p^4 \frac{H'(\phi)H''(\phi)}{(H(\phi))^2}. \]  

The quantity \( V(\phi) \) is the inflation potential, and \( M_p \) is the reduced Plank mass. Other slow-roll parameters \( (\varepsilon_V, \eta_V, \xi_V) \) can be written in terms of the slow-roll parameters \( \varepsilon, \delta, \) and \( \xi \) for first-order slow roll, i.e., \( \varepsilon = \varepsilon_V, \) \( \delta = \eta_V - \varepsilon_V, \) and \( \xi = \xi_V - 3\varepsilon_V \eta_V + 3\varepsilon_V^2, \) where \( \varepsilon_V = M_p^2 (V'' / V)^2, \) \( \eta_V = M_p^2 (V'' / V), \) and \( \xi_V = M_p^4 (V'' / V^2). \) Using the slow-roll
parameters, \((d^2 Z / d \eta^2)/Z\) is written exactly as

\[
\frac{1}{Z} \frac{d^2 Z}{d \eta^2} = 2a^2H^2 (1 + \varepsilon - \frac{3}{2} \delta + \varepsilon^2 - 2\alpha\delta + \frac{\delta^2}{2} + \frac{\xi}{2}),
\]

and the scale factor is written as \(a(\eta) = -((1 - \varepsilon)\eta H)^{-1}\). Here, the slow-roll parameters are assumed to satisfy \(\varepsilon < 1, \delta < 1,\) and \(\xi < 1\). As only the leading-order terms of \(\varepsilon\) and \(\delta\) are adopted, \(\varepsilon\) and \(\delta\) may be considered to be constant, allowing the scale factor to be written as \(a(\eta) \approx (-\eta)^{1-\varepsilon}\) [14]. Equation (2-4) can be rewritten as

\[
\frac{d^2 u_k}{d \eta^2} + \left( k^2 - \frac{2 + 6\varepsilon - 3\delta}{\eta^2} \right) u_k = 0.
\]

(2-9)

The solution to Eq. (2-9) is written as [13]

\[
f_k^i(\eta) = \frac{\sqrt{\pi}}{2} e^{i(\nu+1/2)\pi/2} (-\eta)^{1/2} H^{(1)}_\nu (-k \eta),
\]

(2-10)

where \(\nu = 3/2 + 2\varepsilon - \delta\), and \(H^{(1)}_\nu\) is the Hankel function of the first kind of order \(\nu\). The mode functions \(u_k(\eta)\) of a general initial state in inflation are written as

\[
u_k(\eta) = c_1 f_k^i(\eta) + c_2 f_k^{i*}(\eta),
\]

(2-11)

where the coefficients \(c_1\) and \(c_2\) obey the relation \(|c_1|^2 - |c_2|^2 = 1\). The important point here is that the coefficients \(c_1\) and \(c_2\) do not change during inflation. In ordinary cases, the field \(u_k(\eta)\) is considered to be in the Bunch-Davies state, i.e., \(c_1 = 1\) and \(c_2 = 0\), because as \(\eta \to -\infty\), the field \(u_k(\eta)\) must approach plane waves \(e^{-i\kappa t} / \sqrt{2k}\). Second, in the case of power-law inflation, where \(a(t) \propto t^q\), a similar method can be used, and the solution is obtained as Eq. (2-10) with \(\nu = 3/2 + 1/(q-1)\). Third, the curvature perturbations in the scalar matter are calculated using a method similar to that used for inflation [12, 15, 7]. The field equation \(u_k\)
can be written in a form similar to Eq. (2-4) with a value of \( c_s^2 = 1 \) and with
\[
Z \propto a_p(\eta)[(H^2 - H')]^{1/2}/H, \quad \text{(where } H = a_p'/a_p \text{)}.
\]
The solution to Eq. (2-4) is then written as
\[
f_k^S(\eta) = (1 - i/k \eta) \exp[-ik \eta] \sqrt{2k}.
\]

### 3. Calculation of the nonlinearity parameter

Here, an inflation model is considered. Since we consider slow-roll inflation to have a finite length, we assume a pre-inflation period to be a scalar-matter-dominated period in which the scalar field is the inflaton field, or is power-law inflation, i.e., double inflation. A simple cosmological model is assumed, as defined by

**Pre-inflation:** \( a_p(\eta) = b_1(-\eta - \eta_j)^r \), (3-1)

**Slow-roll Inflation:** \( a_1(\eta) = b_2(-\eta)^{-1-\varepsilon} \), (3-2)

where
\[
\eta_j = -\left(\frac{r}{1+\varepsilon} + 1\right) \eta_h, \quad b_1 = \left(\frac{1-\varepsilon}{r}\right)^r (-\eta_h)^{-1+\varepsilon-r} b_2.
\]

The scale factor \( a_1(\eta) \) represents slow-roll inflation. Here, de-Sitter inflation (\( \varepsilon = 0 \)) is not considered. Slow-roll inflation is assumed to begin at \( \eta = \eta_1 \). In pre-inflation, for the case of \( r = 2 \), the scale factor \( a_p(\eta) \) indicates that pre-inflation is a scalar-matter-dominated period, and, for the case of \( r = -q/(q-1) \), the pre-inflation is power-law inflation, where the scale factor \( a_p(t) \propto t^q \).

Using above the pre-inflation model, the initial state of inflation given by Eq. (2-11) will be fixed as follows. The coefficients \( c_1 \) and \( c_2 \) are fixed using the matching condition in which the mode function and first \( \eta \)-derivative of the mode function are continuous at the transition time \( \eta = \eta_t \). (\( \eta_t \) is the time at which slow-roll inflation begins.) For simplicity pre-inflation
states are assumed to be the Bunch-Davies vacuum. The coefficients \( c_1 \) and \( c_2 \) can be calculated analytically in the case of the scalar-matter-dominated period:

\[
c_1 = \frac{i}{8\varepsilon^{3/2}} \sqrt{\frac{\pi}{2}} e^{i((-1+\delta-2\varepsilon)\pi/2-2qz/(1+\varepsilon))} [2z(-1+2iz-\varepsilon)H_{5/2+2\varepsilon-\delta}^{(2)}(z) + (4z^2 + (3-2\delta + 3\varepsilon)(1+\varepsilon + 2iz)H_{3/2+2\varepsilon-\delta}^{(2)}(z)],
\]

\[
c_2 = \frac{i}{8\varepsilon^{3/2}} \sqrt{\frac{\pi}{2}} e^{-i((-1+\delta-2\varepsilon)\pi/2+2z/(1+\varepsilon))} [2z(-1+2iz-\varepsilon)H_{5/2+2\varepsilon-\delta}^{(1)}(z) + (4z^2 + (3-2\delta + 3\varepsilon)(1+\varepsilon + 2iz)H_{3/2+2\varepsilon-\delta}^{(1)}(z)],
\]

and in the case of the double inflation model:

\[
c_1 = \frac{\pi}{4\sqrt{q(q-1)(1+\varepsilon)}} \{ e^{i\pi(6+2/(q-1)-2\delta+4\varepsilon)/4} (-q z H_{5/2+1/(q-1)}^{(1)}(zz) H_{3/2+2\varepsilon-\delta}^{(2)}(z) - H_{3/2+1/(q-1)}^{(1)}(zz) (-q z H_{5/2+2\varepsilon-\delta}^{(2)}(z) + (1+q(-2+\delta-4\varepsilon)+\varepsilon) H_{3/2+2\varepsilon-\delta}^{(2)}(z)))],
\]

\[
c_2 = \frac{\pi}{4\sqrt{q(q-1)(1+\varepsilon)}} \{ e^{-i\pi(q(1+\delta-2\varepsilon)2/(q-1))} (-q z H_{5/2+1/(q-1)}^{(1)}(zz) H_{3/2+2\varepsilon-\delta}^{(1)}(z) - H_{3/2+1/(q-1)}^{(1)}(zz) (-q z H_{5/2+2\varepsilon-\delta}^{(1)}(z) + (1+q(-2+\delta-4\varepsilon)+\varepsilon) H_{3/2+2\varepsilon-\delta}^{(1)}(z)))],
\]

where \( z = -k\eta_1 \) and \( zz = q z /((q-1)(1+\varepsilon)) \). The initial states of inflation can be written in terms of the slow-roll parameters, the start time of slow-roll inflation \( \eta_1 \), and the double inflation parameter \( q \). Here, three slow-roll inflation models are adopted: the new inflation model with the potential term given by \( V(\phi) = \lambda^2v^4(1-2(\phi/v)^p) \) ( \( p = 3,4 \), \( v \approx M_p \) ), the chaotic inflation model with the potential term given by \( V(\phi) = M^4/2(\phi/m)^a \) ( \( a = 2,4,6 \), \( m \approx M_p \) ), and the hybrid model \( V(\phi) = \alpha((v^2-\sigma^2)/2m^2/2\phi^2 + g^2\phi^2\sigma^2) \)
\[ \approx \alpha (v^4 + m^2/2\phi^2) , \text{ } (v \approx 10^{-2}M_p, m \approx 2 \times 10^{-5}M_p) \] [16]. Using the normalization value from the WMAP five-year data, we obtain the values of the spectral index and the slow-roll parameters, such as

New inflation: \( n_s = 0.935, \varepsilon = 1.027 \times 10^{-9} \), \( \delta = -0.03228 \)

Hybrid inflation: \( n_s = 0.9816, \varepsilon = 0.00504, \delta = 0.000878 \)

Chaotic inflation model:

\( \phi^2 \) model: \( n_s = 0.967, \varepsilon = 0.00828, \delta = 0.000022 \)

\( \phi^4 \) model: \( n_s = 0.950, \varepsilon = 0.01655, \delta = 0.008298 \)

\( \phi^6 \) model: \( n_s = 0.9334, \varepsilon = 0.0248, \delta = 0.01657 \).

Now, we calculate the values of the nonlinearity parameter \( f_{NL}^{flattened} \). Holman and Tolley [3] showed that if the effective action for the inflaton contains the higher-derivative interaction [17] \( \xi = \sqrt{-g} \frac{\lambda}{8M^4}((\nabla \phi)^2)^2 \), which is derived, for example, from \( k \)-inflation or DBI inflation, and the initial state of inflaton is not the Bunch-Davies vacuum, then the enhanced non-Gaussianity is derived as follows:

\[ f_{NL}^{flattened} \approx \frac{\phi^2}{M^4} \left| c_2 \right| \left( \frac{k}{a(\eta_1)H} \right) = \frac{2\varepsilon M_p^2}{H^2 z^3} \left| c_2 \right| , \quad (3-8) \]

where \( M \) is the cutoff scale, which is the limit of effective theory, and we assume \( M \approx k / a(\eta_1) \) where \( \eta_1 \) is the beginning time of slow-roll inflation, and \( z = -k \eta_1 \). The present treatment considers the effect of the length of inflation, where \( z = 1 \) indicates that inflation starts at the time when the present-day size perturbation \( k = 0.002 \) (1/Mpc) exceeds the Hubble radius in inflation (i.e., inflation of close to 60 \( e \)-folds). Using the values of the above parameters we can

\[ n_s = 0.935, \varepsilon = 1.027 \times 10^{-9} \]

\[ n_s = 0.9816, \varepsilon = 0.00504, \delta = 0.000878 \]

\[ n_s = 0.967, \varepsilon = 0.00828, \delta = 0.000022 \]

\[ n_s = 0.950, \varepsilon = 0.01655, \delta = 0.008298 \]

\[ n_s = 0.9334, \varepsilon = 0.0248, \delta = 0.01657 \].
calculate the values of $|c_1|$, $|c_2|$, and $f_{NL}^{\text{flattened}}$ in terms of $z (=-k\eta)$. The values of $|c_2|$ change only slightly among the models, but vary with the value of $z$, as 0.0063 for $z=8$, 0.004 for $z=10$, and 0.001 for $z=20$, and $|c_1|\approx 1$. From all of the models except for the $\phi^4$ model, similar values of $f_{NL}^{\text{flattened}}$ are calculated, i.e., $f_{NL}^{\text{flattened}} \approx 120$ at $z=8$, and $f_{NL}^{\text{flattened}} \approx 40$ at $z=10$. Details are shown in Table 1. With respect to the other values of $z$, larger values of $f_{NL}^{\text{flattened}}$ can be derived at smaller $z$ ($z<8$), and small values of $f_{NL}^{\text{flattened}}$ can be derived at larger $z$ ($z>20$). Based on the above results, the value of $f_{NL}^{\text{flattened}}$ appears to depend strongly on the value of $z$, which represents the length of inflation, and the difference of the values of $f_{NL}^{\text{flattened}}$ among our three slow-roll inflation models is not large. Since the $z$-dependence of $f_{NL}^{\text{flattened}}$ is very steep, any value of $f_{NL}^{\text{flattened}}$ can be derived at some point of $z$. We next consider the case of double inflation, the value of $f_{NL}^{\text{flattened}}$ is 100 at $3<z<4$ in the chaotic inflation, at $4<z<5$ in the case of new inflation, and at $z \approx 3$ in the case of hybrid inflation. With respect to the $q$-dependence ($a(t) \propto t^q$), the values of $f_{NL}^{\text{flattened}}$ are similar at very large $q$ but change at $q \approx 100$. The details are shown in Tables 2-4.

4. Discussion

We have derived a new property of the proposed finite inflation model. The possibility of large non-Gaussianity is demonstrated. The proposed inflation model is a finite length inflation model with an effective higher derivative interaction, where slow-roll inflation is adopted as inflation and a scalar-matter-dominated period or power inflation is adopted as pre-inflation. Owing to the existence of pre-inflation, the initial state in inflation is not the Bunch-Davies
state, but is instead a more general state. The coefficients $c_1$ and $c_2$ can be analytically calculated. Using Holman and Tolley’s formula of the nonlinearity parameter $f_{NL}^{\text{flattened}}$, we calculated the value of $f_{NL}^{\text{flattened}}$. For the case in which the scalar-matter-dominated period is considered to be pre-inflation, large values of $f_{NL}^{\text{flattened}} (f_{NL}^{\text{flattened}} \approx 100)$ are obtained at $8 < z < 10$ in all the models considered herein, and similar results are derived for the case of double inflation at $3 < z < 4$. These ranges can be written as 60-63 e-folds. This length is similar to that obtained when the suppression of CMB angular power spectrum of $\ell = 2$ was derived using the inflation models described in previous papers [7], but such spectral suppression is not inconsistent when considering cosmic variance. On the experimental value of $f_{NL}^{\text{flattened}}$, the orthogonal shape ($f_{NL}^{\text{orthog}}$) is peaked both on equilateral-triangle configurations ($f_{NL}^{\text{equil}}$) and on flattened-triangle configurations ($f_{NL}^{\text{flattened}}$) [18], but we think we need further consideration to drive the constraint of $f_{NL}^{\text{flattened}}$ from the constraints of $f_{NL}^{\text{orthog}}$ and $f_{NL}^{\text{equil}}$. Therefore, we do not show it here. We assume such a high-derivative interaction in order to obtain non-linearity and effective interactions for slow-roll interaction. This high-derivative interaction appears to influence the parameters of slow-roll inflation. In order to clarify this problem, we must investigate a concrete inflation model such as $k$-inflation or DBI inflation. In the future, we would like to apply the proposed method to other inflation models and investigate the dependence of the length of inflation on $f_{NL}^{\text{flattened}}$.

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References


[17] Creminelli P. 2003 JCAP 0310,003


Table 1 Values of $f_{NL}^{\text{flattened}}$ for the case of the matter-dominated period as pre-inflation

<table>
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<th>New inflation</th>
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<th>Chaotic inflation</th>
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<td>p=4</td>
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Table 2 Values of $f_{NL}^{\text{flattened}}$ in the hybrid inflation for double inflation

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Table 3 Values of $f_{NL}^{\text{flattened}}$ for the new inflation case of $n = 3$ and for the new inflation case of $n = 4$ for double inflation

### $n = 3$

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<th>$q=10^4$</th>
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Table 4 Values of $f_{NL}^{\text{flattened}}$ for the Chaotic inflation case of $\phi^2$, $\phi^4$, and $\phi^6$ for double inflation

#### $\phi^2$ model

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\[ z=3.5 \quad 76.1 \quad 76.4 \quad 78.6 \quad 108.1 \quad 530.0 \\
\]
\[ z=4 \quad 36.1 \quad 36.2 \quad 37.5 \quad 53.9 \quad 274.4 \]

\( \phi^6 \) model

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<td>257.6</td>
</tr>
</tbody>
</table>