Effects of rotation and magnetic field on the revival of a stalled shock in supernova explosions

Kotaro Fujisawa (Waseda)

Collaborators:

Hirotada Okawa (YITP / Waseda)

Yu Yamamoto (Waseda)

Shoichi Yamada (Waseda)

Fujisawa, Okawa et al. (2019) ApJ 872:155

Magnetic field structures of compact objects and their activities

Kotaro Fujisawa
Waseda University

Joint meeting @ RESCEU

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Self-introduction

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Fujisawa　Kotaro

藤澤　幸太郎

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幸:Happiness

Self-introduction

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Self-introduction

Fujisawa　Kotaro

藤澤　幸太郎

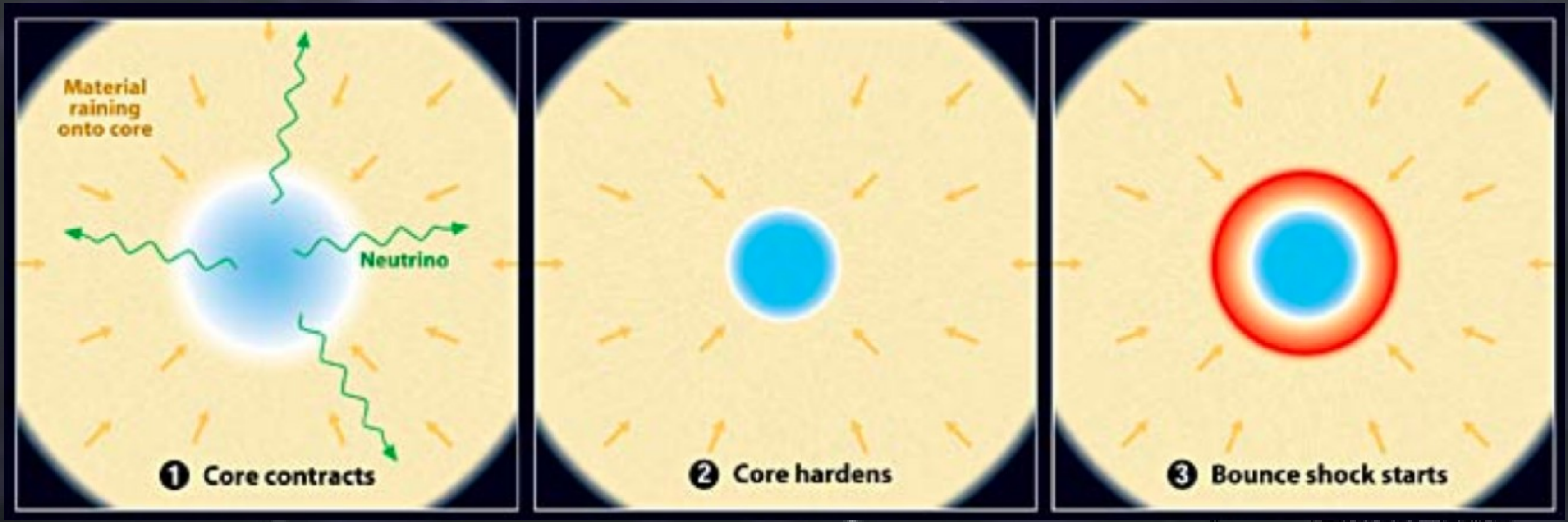
Two Happiness！

Core-collapse supernovae(CCSNe)

1. CCSNe begin with the gravitational collapse of the core of massive star.

2. Central density is increasing.

3. When the central density reaches the nuclear saturation density (∼10^{14} g/cm^3), core bounce occurs and a shock wave is generated.



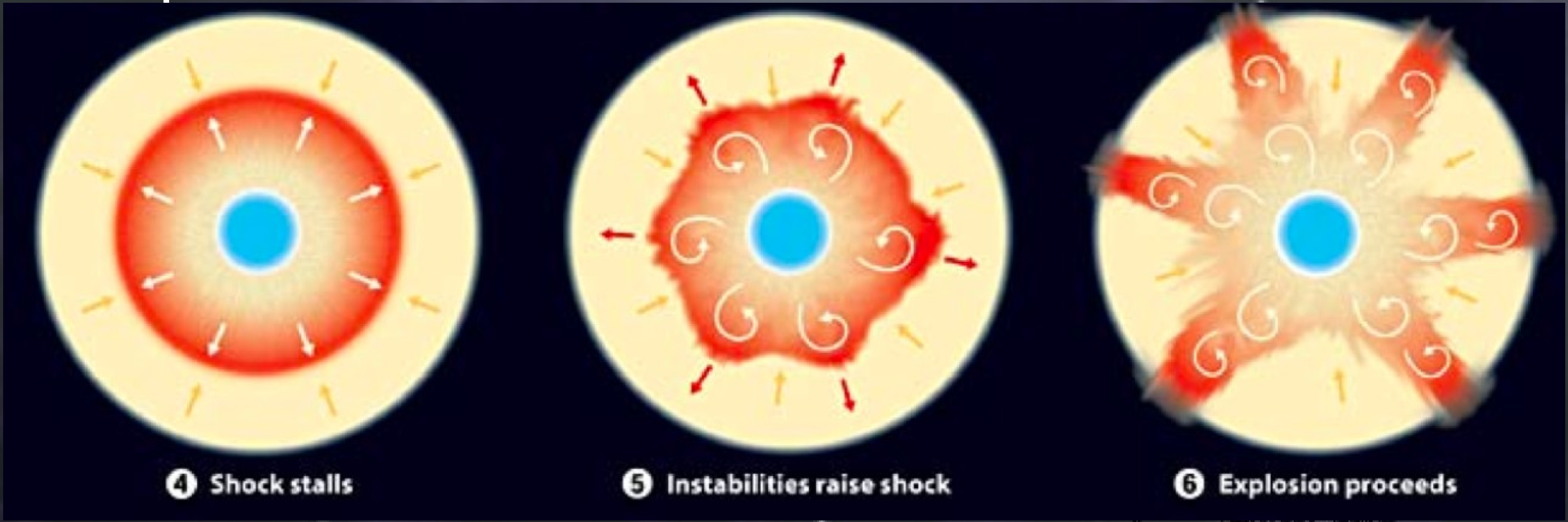
by StevenSimpson.

Core-collapse supernovae(CCSNe)

4. The shock loses energy via photo-dissociations of nuclei and neutrino cooling. The shock is stalled by the rap pressure of accretion.

5. The shock is revived by something.

6. Explosion is successful.



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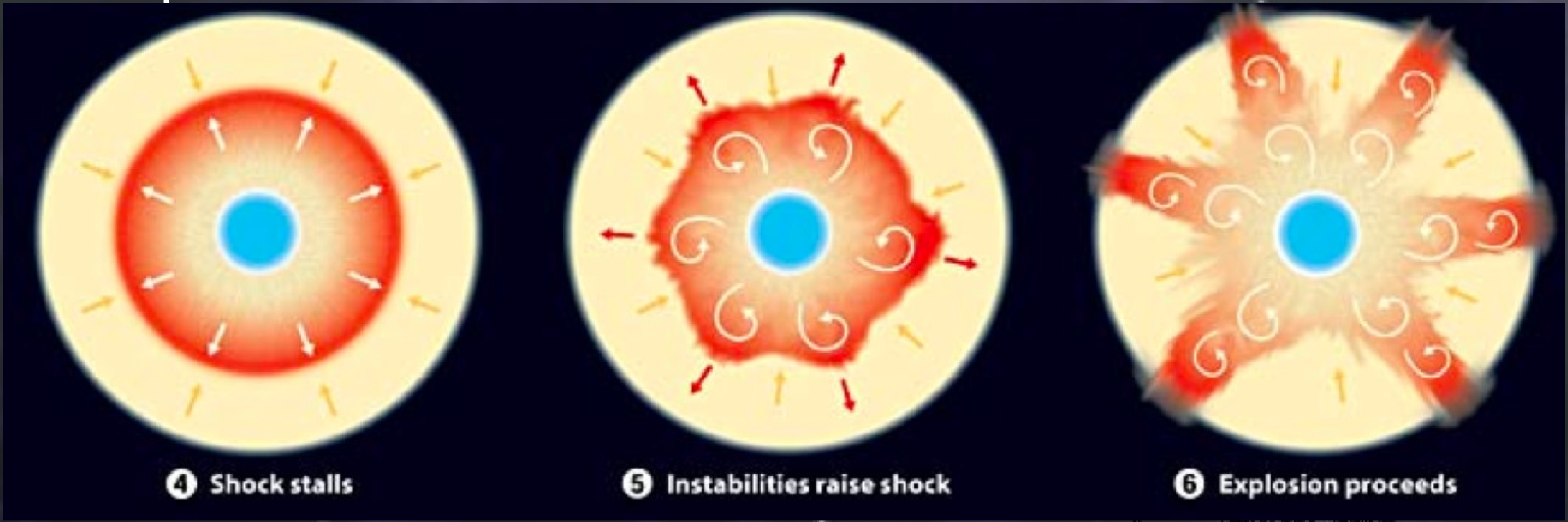
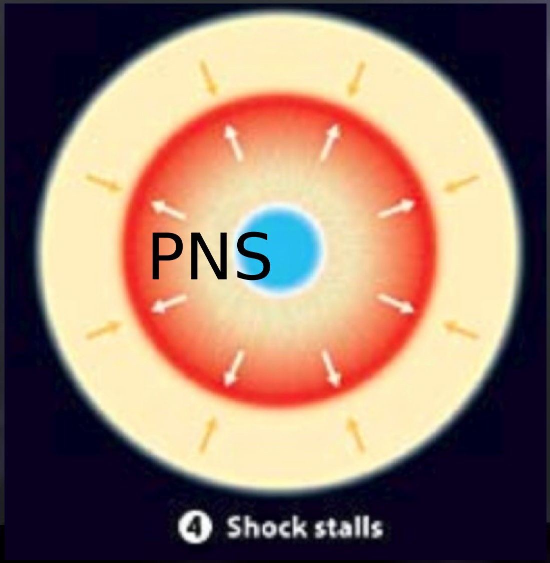


by Steven Simpson.

How to revive the stalled shock

- **Neutrino-heating mechanism (most favored)**
  - Irradiation by neutrino diffusing out of a PNS (Wilson 1985 etc.)
- Acoustic mechanism

  (Burrows et al. 2006;Harada et al. 2017)



But,,,

- 1D detailed and realistic dynamical simulations with neutrino-heating failed to explode.
  - Libendorfer et al. (2001,2005);

    Sumiyoshi et al. (2005) etc.
- Multi-dimensional effect (e.g. rotation, fluid-instability, magnetic field) are important.
  - Burrows et al. (2006); Marek & Janka (2009);

    Suwa et al. (2010); Takiwaki et al (2012;2016).

multidimensional effects are crucially important (Burrows et al. 2006; Bruenn et al. 2009; Marek & Janka 2009; Suwa et al. 2010; Müller et al. 2012; Takiwaki et al. 2012; Couch 2013; Couch & Ott 2013; Hanke et al. 2013; Murphy et al. 2013; Lentz et al. 2015; Melson et al. 2015; Nakamura et al. 2015; Bruenn et al. 2016; Roberts et al. 2016; O'Connor & Couch 2018). Among them are rotation (Fryer & Heger 2000; Kotake et al. 2003; Thompson et al. 2005; Marek & Janka 2009; Iwakami et al. 2014a; Nakamura et al. 2014; Takiwaki et al. 2016; Summa et al. 2018), a magnetic field (Akiyama et al. 2003; Kotake et al. 2004; Yamada & Sawai 2004; Sawai et al. 2005; Obergaulinger et al. 2006, 2014, 2018; Burrows et al. 2007; Takiwaki et al. 2009; Sawai & Yamada 2014, 2016; Mösta et al. 2015), non-spherical structures of the progenitor (Couch & Ott 2013; Takahashi & Yamada 2014; Couch et al. 2015; Takahashi et al. 2016), turbulence (Murphy & Burrows 2008; Murphy & Meakin 2011; Murphy et al. 2013; Couch & Ott 2015; Mabanta & Murphy 2018), (magneto)hydrodynamical instabilities (Blondin et al. 2003; Scheck et al. 2006; Blondin & Mezzacappa 2007; Iwakami et al. 2008; Guilet et al. 2010; Wongwathanarat et al. 2010; Fernández et al. 2014; Takiwaki et al. 2014; Fernández 2015), general relativistic gravity (Dimmelmeier et al. 2002; Shibata & Sekiguchi 2004, 2005; Kuroda et al. 2012, 2016; Müller et al. 2012; Ott et al. 2012), and neutrino transport (Nagakura et al. 2014, 2017, 2018; Dolence et al. 2015; Pan et al. 2016). It is true that large-scale

Stationary solution of stalled shock

- Stalled shock is realized after the core bounce.

Steady solutions are obtained by solving stationary equations.

(Burrows & Goshy 1993)

Stationary solution of stalled shock

$$4\pi r^2 \rho u_r = \dot{M},$$

$$u_r \frac{du_r}{dr} + \frac{1}{\rho}\frac{dp}{dr} + \frac{GM}{r^2} = 0,$$

$$u_r \frac{d\epsilon}{dr} - \frac{p}{\rho^2} u_r \frac{d\rho}{dr} = \dot{q},$$

+ EOS

$$\dot{q} = 4.8 \times 10^{32} \left[1 - \sqrt{1 - (r_\nu / r)^2}\right] \frac{L_{\nu_e}(\text{foe s}^{-1})}{2\pi r_\nu^2} T_\nu^2 - 2.0 \times 10^{18} T^6 \quad (\text{ergs s}^{-1}\ \text{g}^{-1}),$$

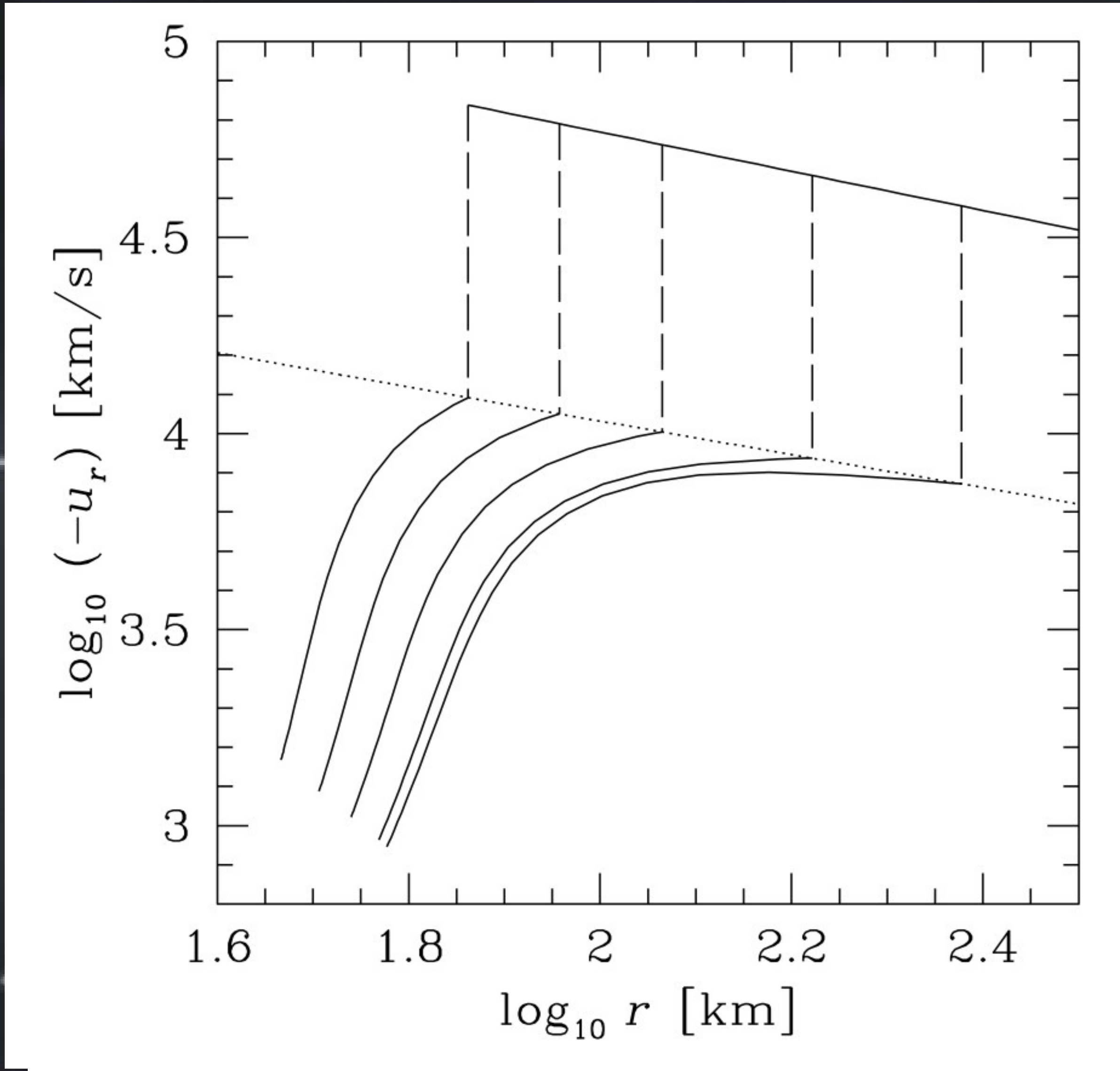
1. Set Mdot and Lnu
2. Impose boundary conditions.
3. Change the shock radius to satisfy the BCs.

Burrows & Goshy (1993)
Yamasaki & Yamada (2005)

Keshet & Balberg (2012)

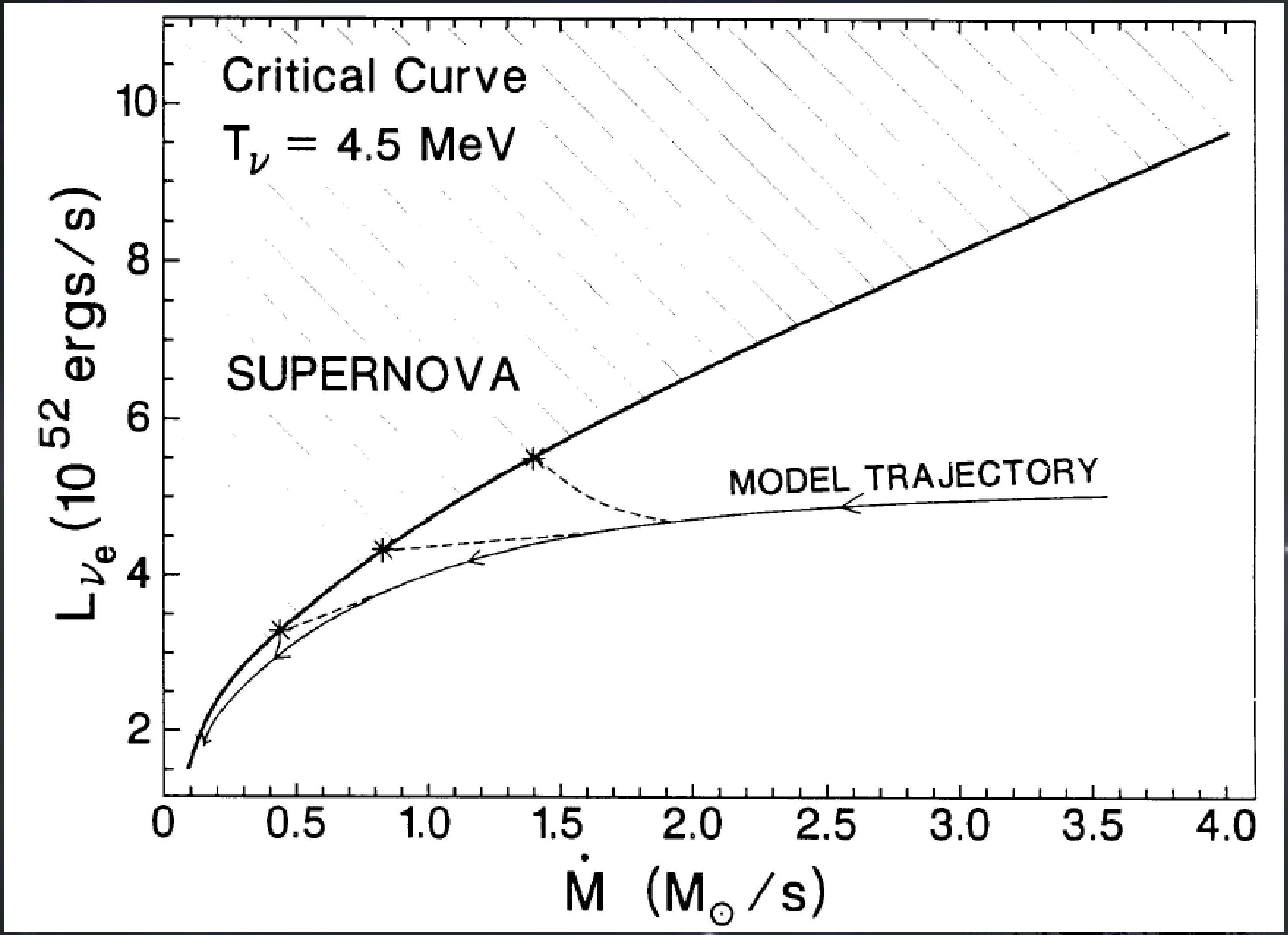
Stationary solution of stalled shock

Yamasaki & Yamada (2005)



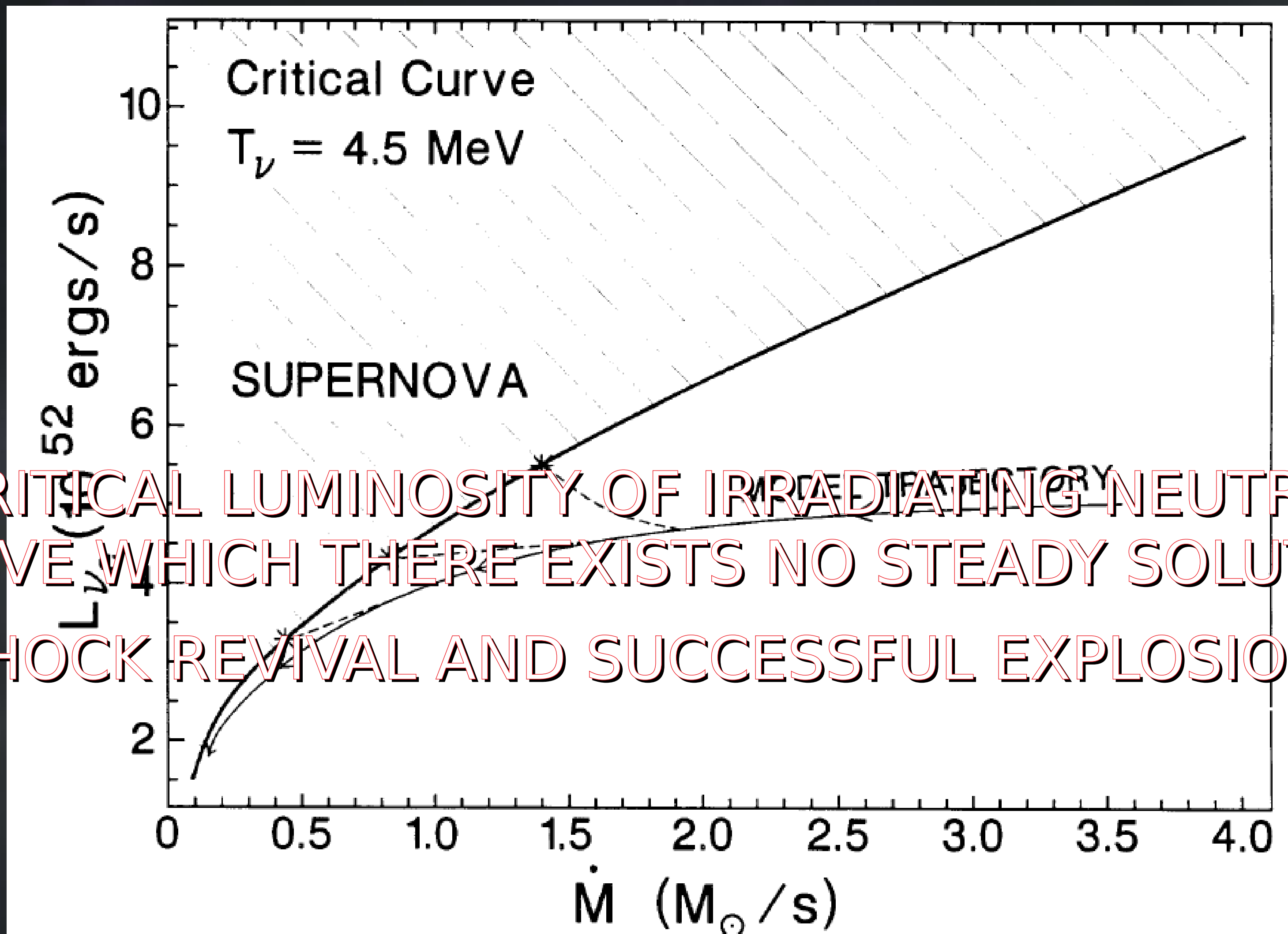
$\dot{M} = 2.0\ M_\odot\ s^{-1}$ and L_{ν_e} = (5, 6, 7, 8) $\times 10^{52}$ and 8.3167×10^{52} ergs s^{-1}

Critical neutrino luminosity



Burrows & Goshy (1993)

Critical neutrino luminosity



A CRITICAL LUMINOSITY OF IRRADIATING NEUTRINOS, ABOVE WHICH THERE EXISTS NO STEADY SOLUTION

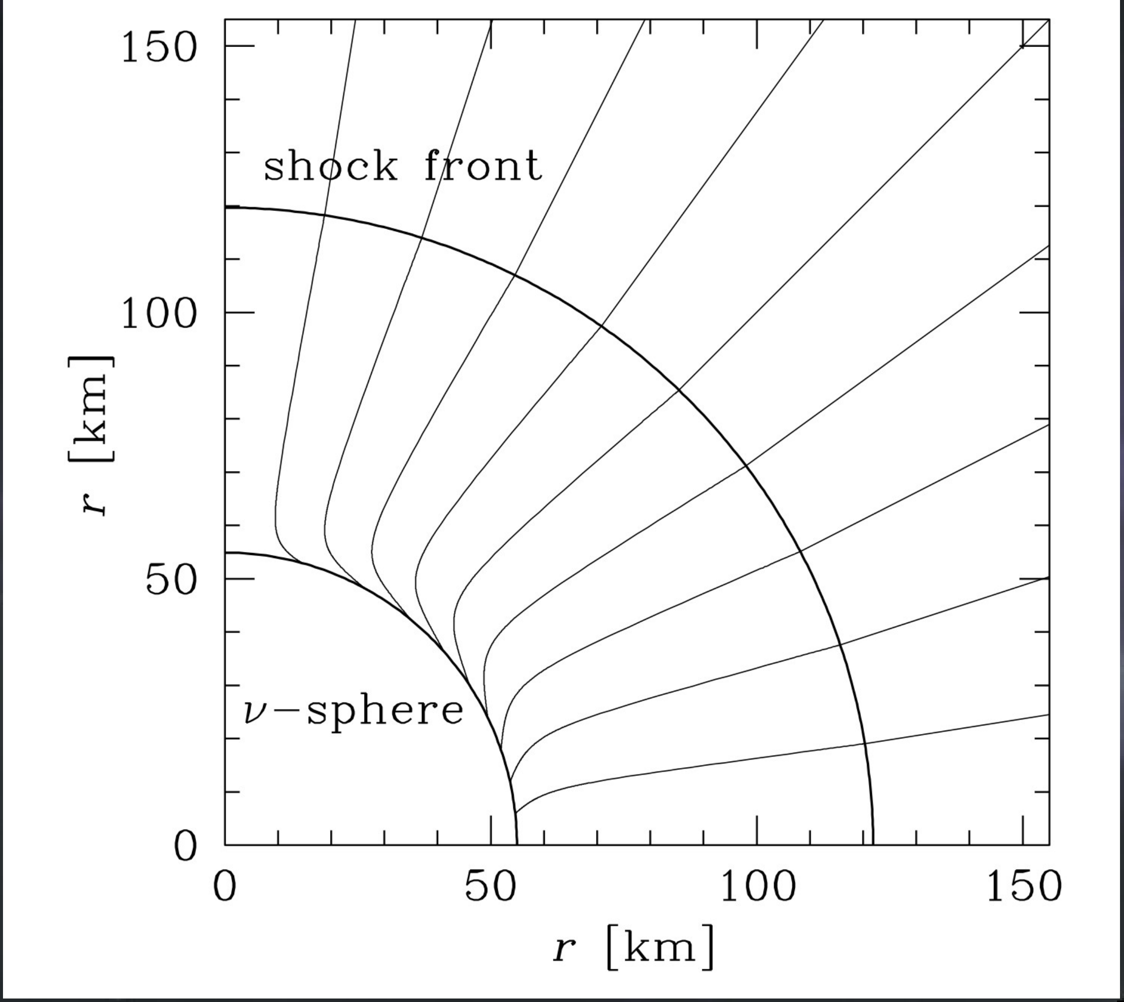
→ SHOCK REVIVAL AND SUCCESSFUL EXPLOSION!

Burrows & Goshy (1993)

How to reduce the critical luminosity

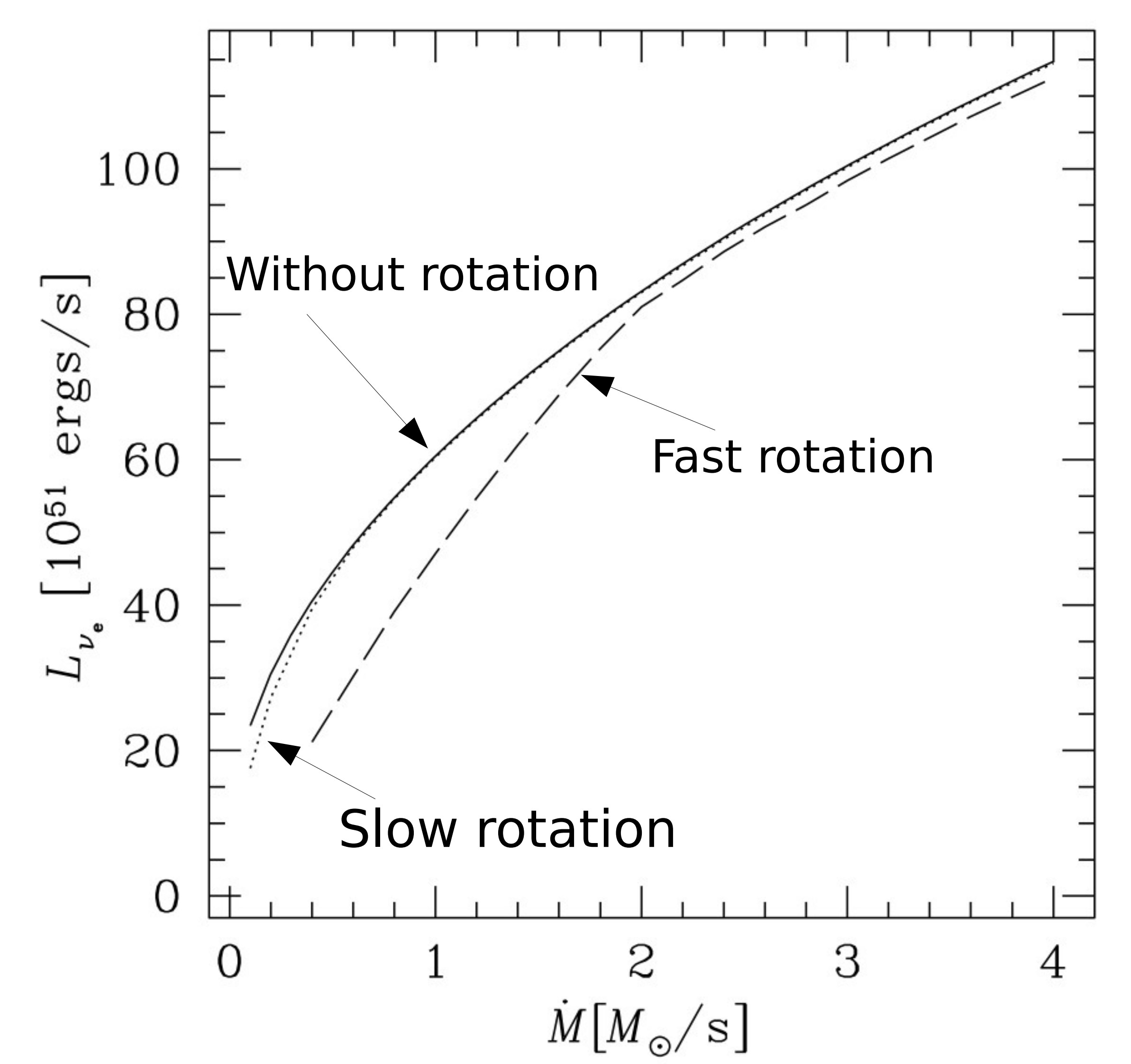
- Lowered by turbulence
  - Murphy & Burrows (2008); Murphy & Meakin(2011); Murphy et al. (2013).
  - Reduced by about 30 % due to the turbulence (Mabanta & Murphy 2018).
- Multi-dimensional effects
  - Rotation (Yamasaki & Yamada 2005)

Steady solution with rotation in 2D



Yamasaki & Yamada (2005)

Critical luminosities



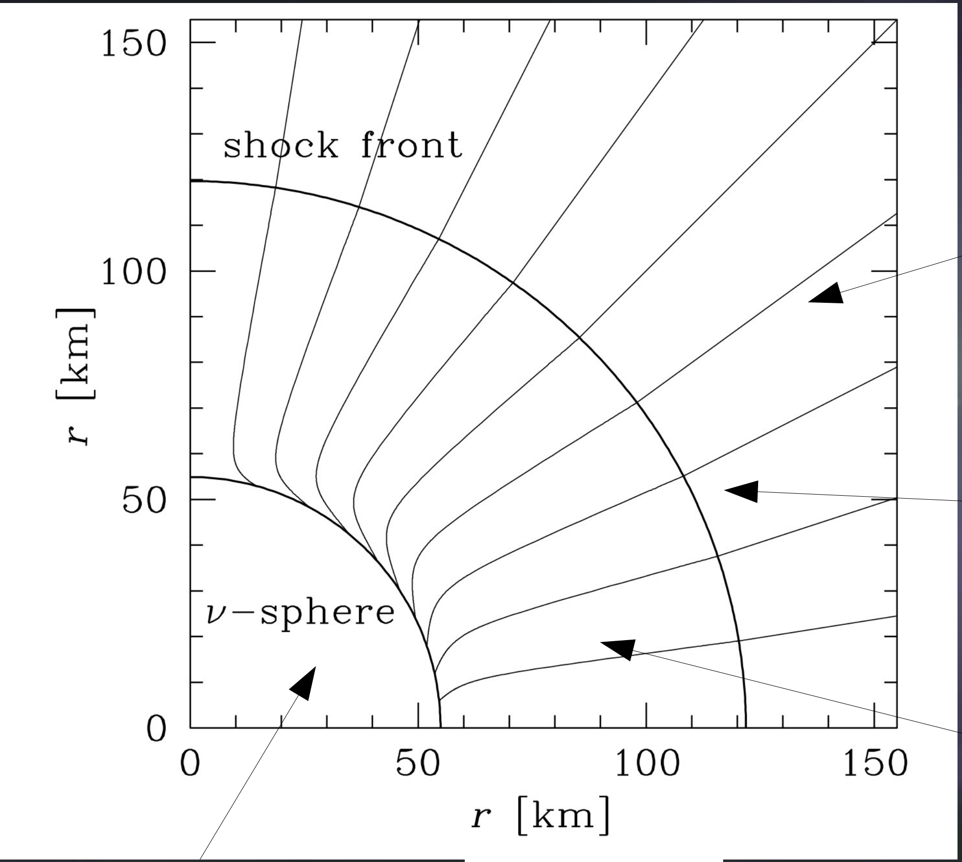
Yamasaki & Yamada (2005)

Our goal

- Stalled shock is realized after the core bounce.
- A critical luminosity, above which there exist no steady solutions.
- We systematically study effects of rotation and magnetic field on the revival of a stalled shock in supernova explosion.
  - Develop a new numerical method to solve equations.

Numerical model

Give uniform rotation and magnetic field at r=1000km $f_{1000\mathrm{km}}$ $B_{1000\mathrm{km}}$



Spherical accretion flow

Solve Rankine-Hugoniot relation on the stalled shock surface.

Accretion flow.

Spherical PNS $1.3\ \mathrm{M}_{\odot}$.

(ρ∼10^{11}g cm^{-3} on the surface)

Neutrino temperature $T_\nu = 4.5\ \mathrm{MeV}$

Parameter settings

- Rotation and magnetic field in progenitors are unclear.
  - Angular momentum contained between 1.3 $M_\odot$ - 2 $M_\odot$ (Heger et al. 2005)
    - Without rotation

$j \sim 10^{16} - 10^{17}\ \mathrm{cm^2\ s^{-1}}$ → $f_{1000\mathrm{km}} \sim 0.1 - 1\ \mathrm{s^{-1}}$

    - With rotation

$j \sim 10^{14} - 10^{15}\ \mathrm{cm^2\ s^{-1}}$ → $f_{1000\mathrm{km}} \sim 0.001 - 0.01\ \mathrm{s^{-1}}$

    - Toroidal magnetic field might be larger than poloidal one due to the differential winding inside the massive star

(Heger et al. 2005)

Parameter settings

- Rotational frequency at 1000km

$$f_{1000\mathrm{km}} \sim 0 - 0.45\ \mathrm{s}^{-1}$$

- Toroidal magnetic field at 1000km

$$B_{1000\mathrm{km}} \sim 0 - 3.0 \times 10^{12}\ \mathrm{G}$$

- Poloidal magnetic field at 1000km

$$B_0\ \sim\ 0 - 10^{11}\ \mathrm{G.}$$

Rotational frequency at 10 km (NS surface)

$$f \sim 0 - 5\,\mathrm{ms}^{-1}$$

Magnetic field at 10 km

$$B\ \sim\ 0 - 5 \times 10^{14}\ \mathrm{G}$$

Basic equations

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\rho u_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\rho u_\theta\right)=0,$$

$$u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta^2+u_\varphi^2}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}-\frac{GM}{r^2}-\frac{1}{4\pi\rho}\left[B_\theta\frac{\partial B_\theta}{\partial r}+B_\varphi\frac{\partial B_\varphi}{\partial r}-\frac{1}{r}\frac{\partial B_r}{\partial\theta}+\frac{B_\theta^2+B_\varphi^2}{r}\right],$$

$$u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r u_\theta}{r}-\frac{u_\varphi^2\cot\theta}{r}=-\frac{1}{\rho r}\frac{\partial p}{\partial\theta}+\frac{1}{4\pi\rho}\left[B_r\frac{\partial B_\theta}{\partial r}-\frac{B_r}{r}\frac{\partial B_r}{\partial\theta}-\frac{B_\varphi}{r}\frac{\partial B_\varphi}{\partial\theta}+\frac{B_r B_\theta-B_\varphi^2\cot\theta}{r}\right],$$

$$u_r\frac{\partial u_\varphi}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\varphi}{\partial\theta}+\frac{u_\varphi u_r}{r}+\frac{u_\theta u_\varphi\cot\theta}{r}=\frac{1}{4\pi\rho}\left[B_r\frac{\partial B_\varphi}{\partial r}+\frac{B_\theta}{r}\frac{\partial B_\varphi}{\partial\theta}+\frac{B_r B_\varphi-B_\theta B_\varphi\cot\theta}{r}\right],$$

$$u_r\left(\frac{\partial\varepsilon}{\partial r}-\frac{p}{\rho^2}\frac{\partial\rho}{\partial r}\right)+\frac{u_\theta}{r}\left(\frac{\partial\varepsilon}{\partial\theta}-\frac{p}{\rho^2}\frac{\partial\rho}{\partial\theta}\right)=\dot{q},$$

$$\nabla\cdot\mathbf{B}=0,\quad \nabla\times\mathbf{E}=0,\quad \mathbf{E}=-\mathbf{u}\times\mathbf{B}$$

- Simplified analytical EOS (ideal gas + radiation pressure)
- Simplified Neutrino cooling + heating (Herant et al. 1992)

$$\dot{q}=4.8\times10^{32}\left[1-\sqrt{1-\frac{r_\nu^2}{r^2}}\right]\frac{L_\nu}{2\pi r_\nu^2}T_\nu^2-2.0\times10^{18}T^6\quad(\mathrm{ergs\,s^{-1}\,g^{-1}}),$$

$$L_\nu=\frac{7}{4}\pi r_\nu^2\sigma T_\nu^4,$$

Numerical method

Basic equations are as follow:

$$\mathcal{A}\left(\boldsymbol{Q}\right)\frac{\partial \boldsymbol{Q}}{\partial q}+\mathcal{B}\left(\boldsymbol{Q}\right)\frac{\partial \boldsymbol{Q}}{\partial \theta}+\mathcal{C}\left(\boldsymbol{Q}\right)=0,$$

Discretized as

$$\boldsymbol{F}_{j-1,k}\equiv\mathcal{A}\left(\boldsymbol{Q}_{j-\frac{1}{2},k}\right)\frac{\boldsymbol{Q}_{j,k}-\boldsymbol{Q}_{j-1,k}}{q_j-q_{j-1}}+\mathcal{B}\left(\boldsymbol{Q}_{j-\frac{1}{2},k}\right)\frac{\boldsymbol{Q}_{j-\frac{1}{2},k+1}-\boldsymbol{Q}_{j-\frac{1}{2},k-1}}{2\Delta\theta}+\mathcal{C}\left(\boldsymbol{Q}_{j-\frac{1}{2},k}\right)=0,$$

$$\boldsymbol{Q}_{j-\frac{1}{2},k}=\frac{1}{2}\left(\boldsymbol{Q}_{j-1,k}+\boldsymbol{Q}_{j,k}\right).$$

Imposing boundary conditions and solving this nonlinear algebraic equations implicitly.

Okawa, KF et al. submitted
ArXiv:1809.04495
KF, Okawa et al. (2019)

A new method for solving nonlinear eqs.

$$F_i(x_1, x_2, \cdots, x_N) = 0 \quad i = 1, 2, \cdots, N.$$

Newton-Raphson

$$\boldsymbol{x}^{n+1} = \boldsymbol{x}^n + \delta \boldsymbol{x},$$

$$\delta x_i = -\sum_{j=1}^{N} J_{ij}^{-1} F_j,$$

W4

$$\frac{d^2 \boldsymbol{x}}{d\tau^2} + 2\frac{d\boldsymbol{x}}{d\tau} + M\boldsymbol{F} = 0,$$

$$\boldsymbol{x}^{n+1} = \boldsymbol{x} + \frac{1}{2} L^{-1} \boldsymbol{p}^n, \quad \boldsymbol{p}^{n+1} = \frac{1}{2} U^{-1} \boldsymbol{F}(\boldsymbol{x}$$

$$J \equiv UL$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \ell_{21} & 1 & 0 & 0 \\ \ell_{31} & \ell_{42} & 1 & 0 \\ \ell_{41} & \ell_{42} & \ell_{43} & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix}$$

Numerical scheme for nonlinear eqs.

- Newton-Raphson method is a most famous numerical method.
  - Initial guess is sufficiently close to the root,
    - Efficient
  - Initial guess is not close to the root,
    - Iteration usually unstable and we have problems with oscillations.

There are **no** good, general methods for solving systems of more than one nonlinear equations. Furthermore, it is not hard to see why (very likely) there **never will be** any good, general methods.

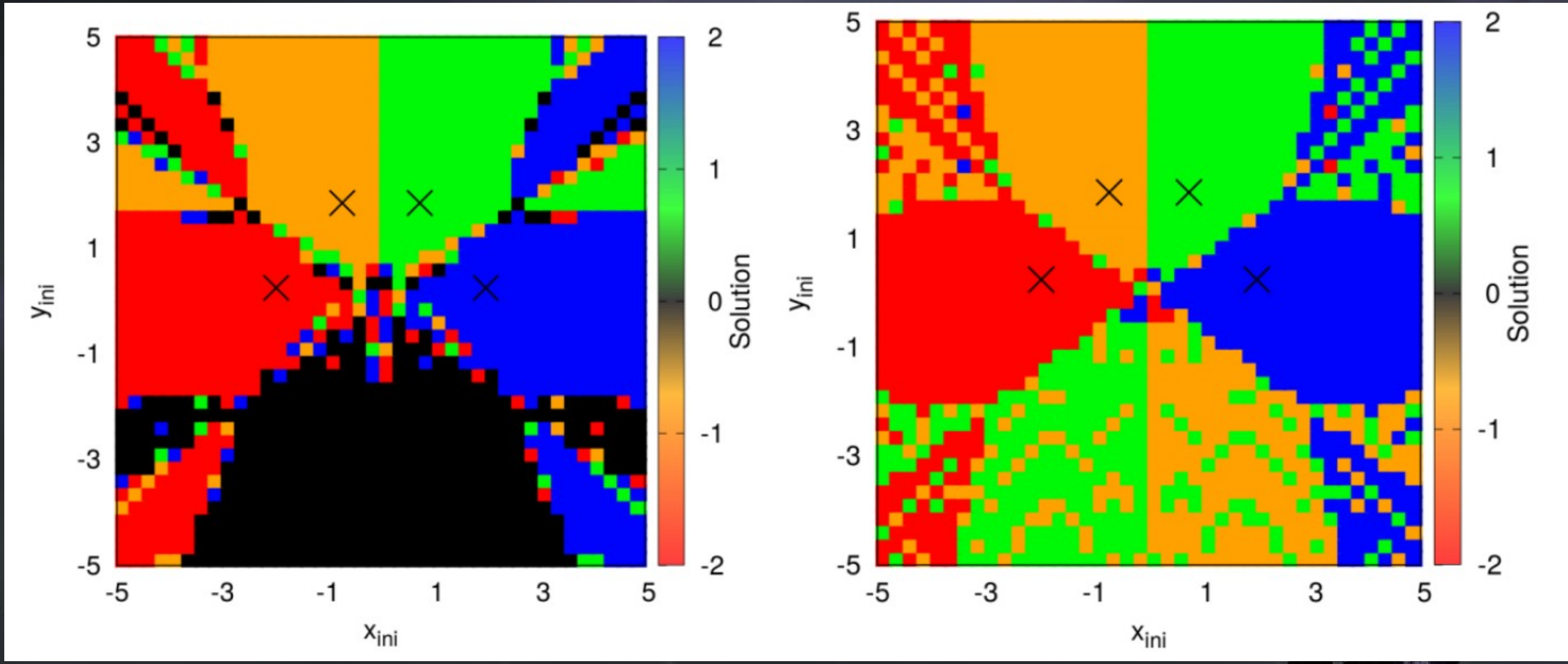
– Numerical recipe

Convergence region (Basin)

$$f_1(x,y) = x^2 + y^2 - 4 = 0, \; f_2(x,y) = x^2y - 1 = 0$$

NR

UL W4



Global convergent method!

Basic equations

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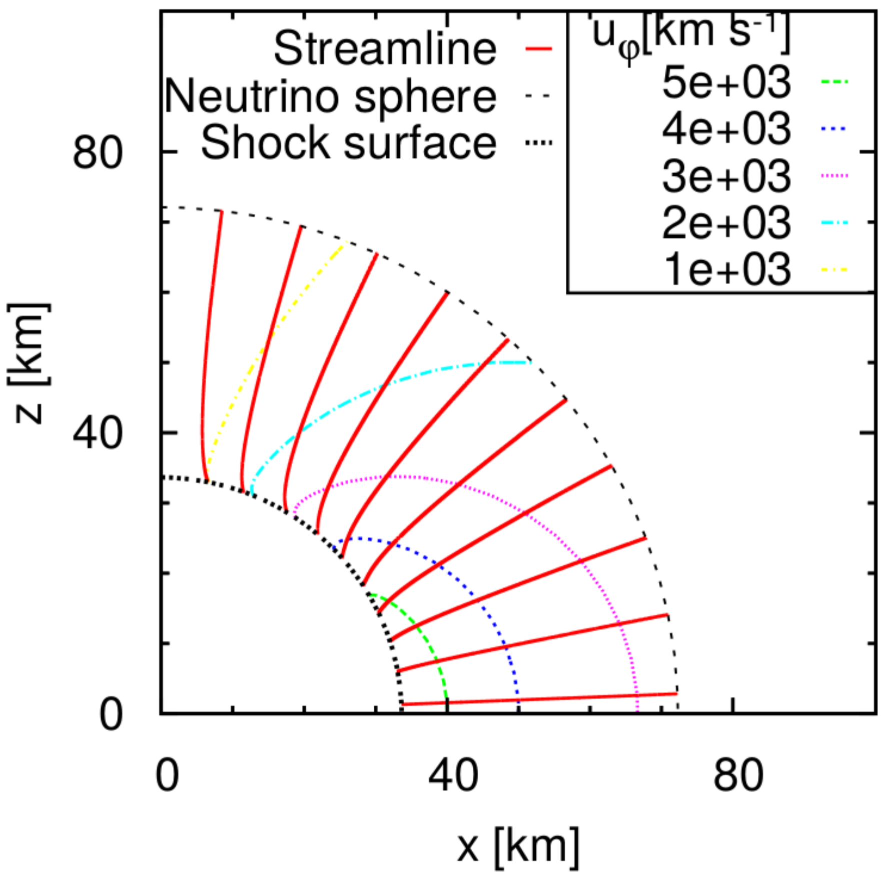
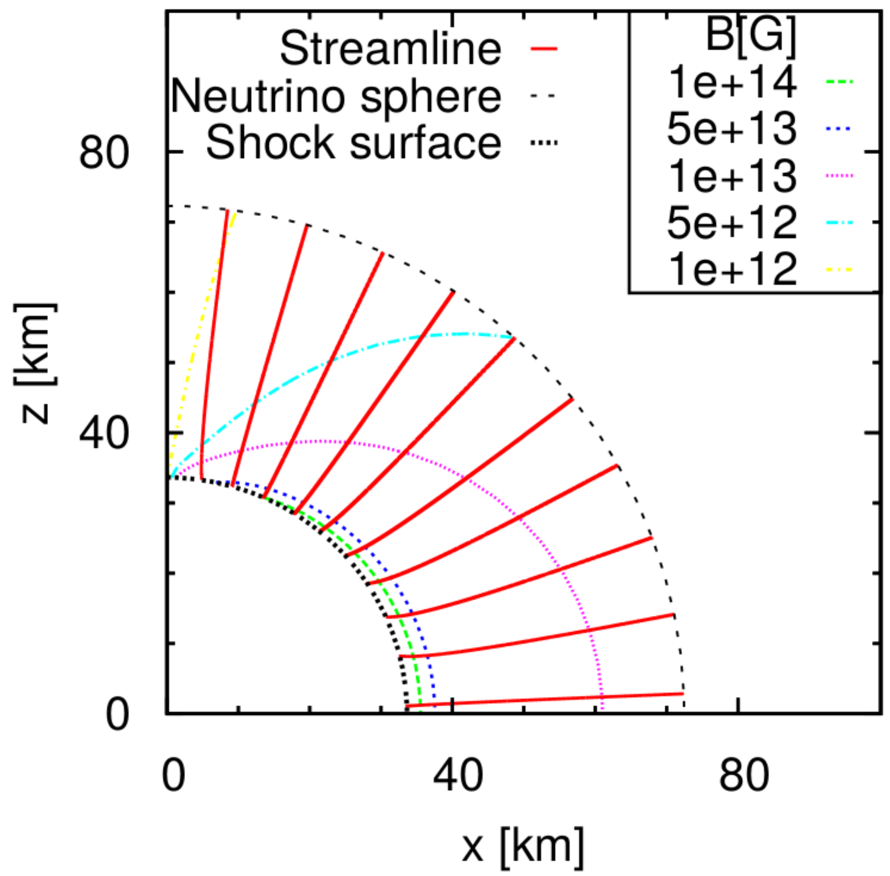
Imposing boundary conditions and solving this nonlinear algebraic equations implicitly.

Numerical results

With rotation

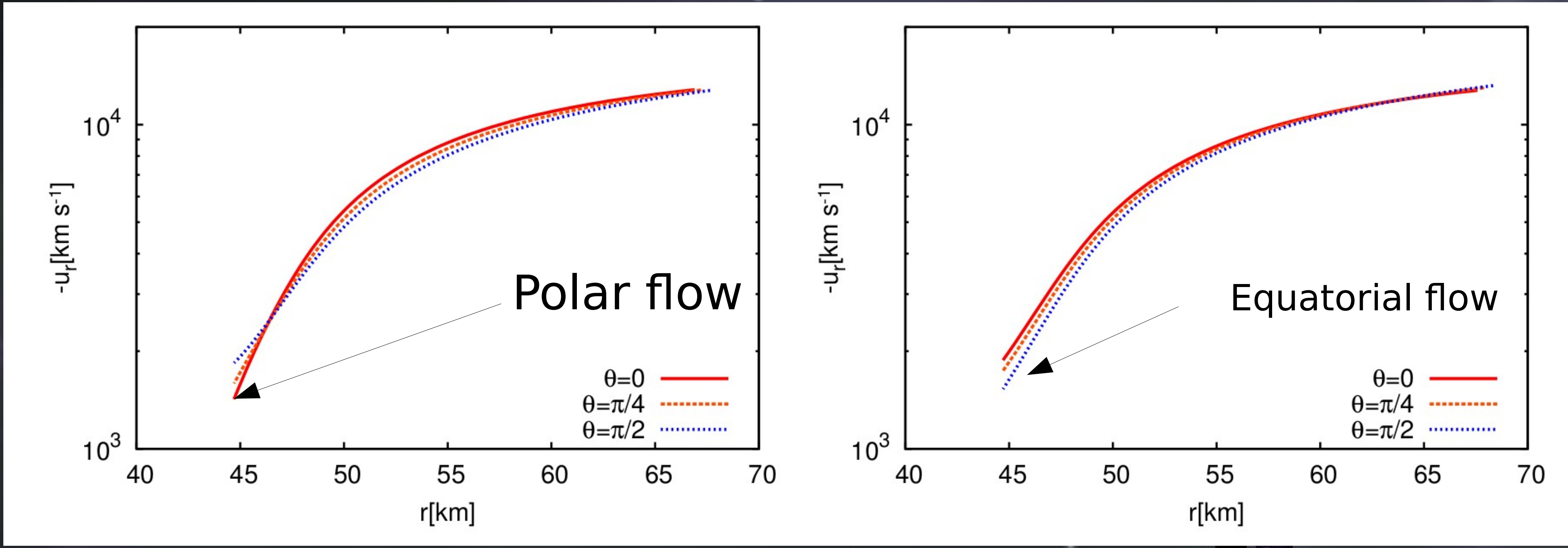


With toroidal field



Radial flow velocity with critical neutrino luminosity

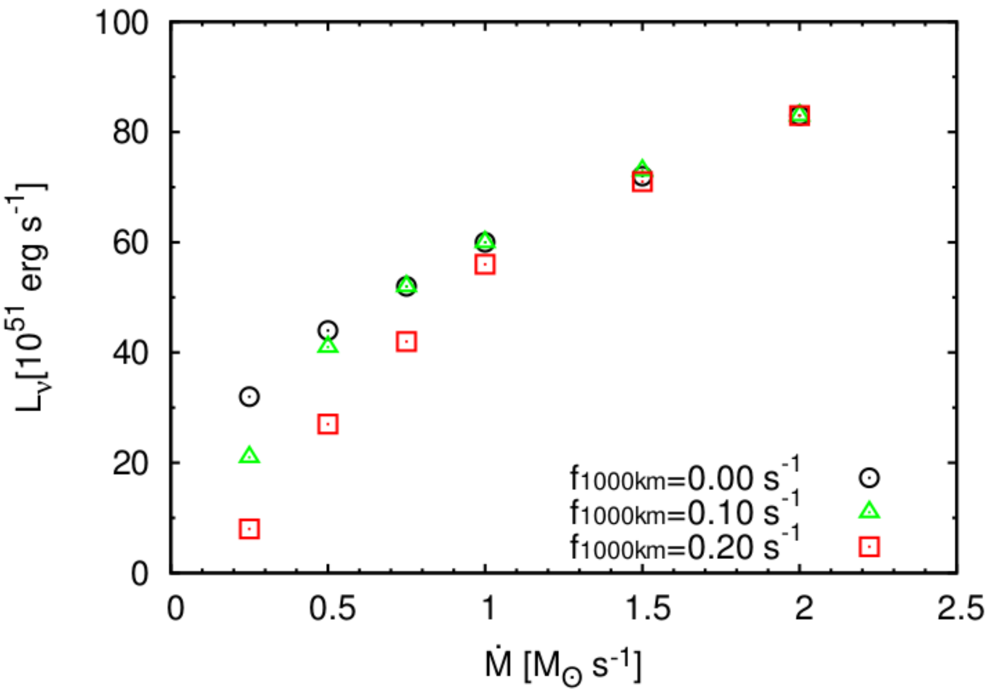
With rotation　　With toroidal magnetic field



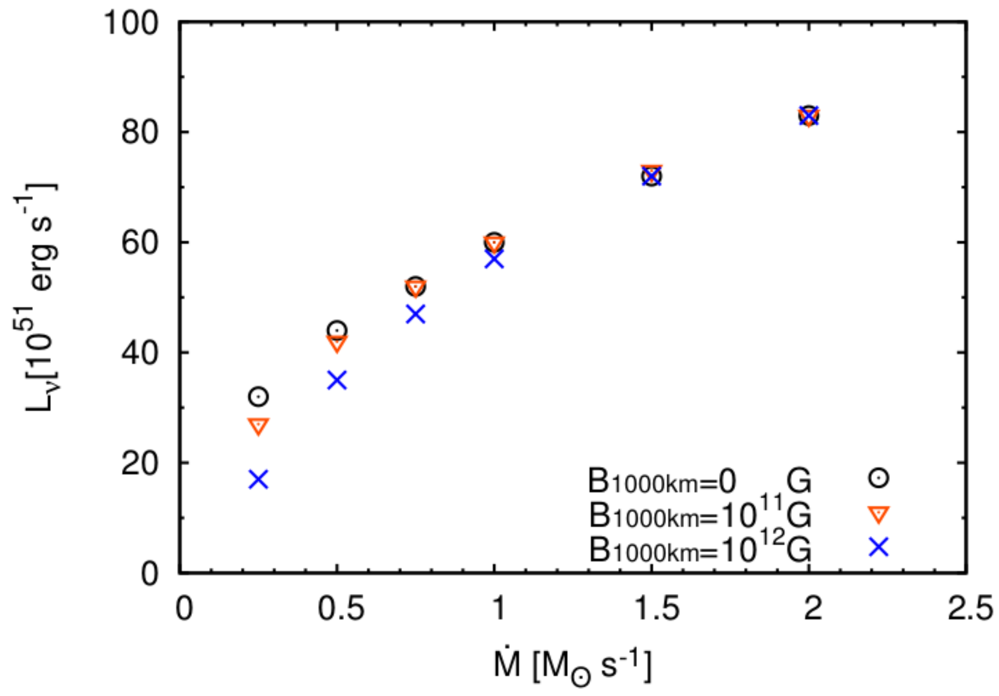
The polar / equatorial flow velocity is smaller than equatorial / polar velocity.

Critical neutrino luminosity 1

With rotation



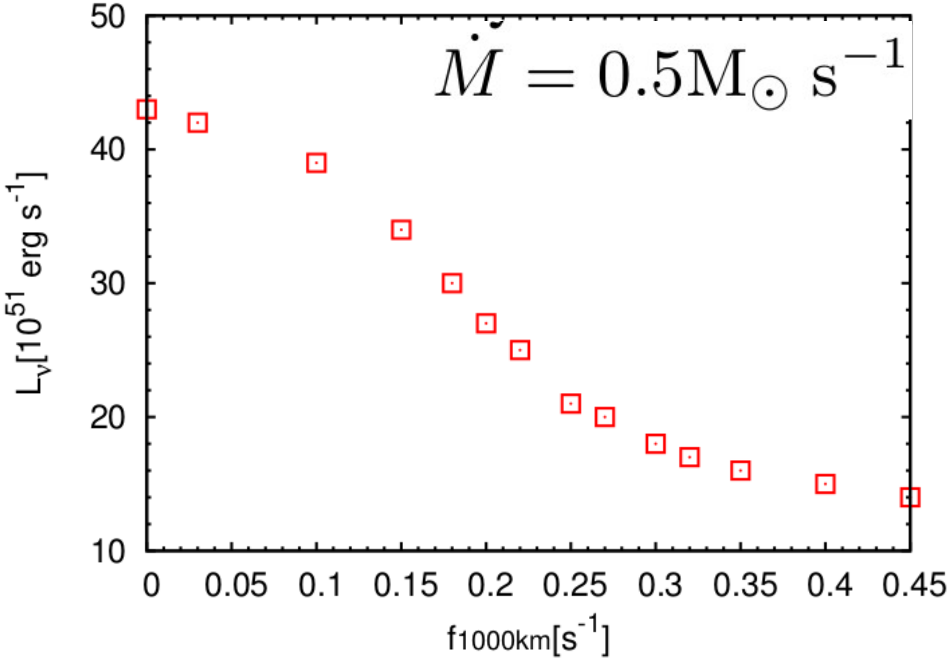
With toroidal field

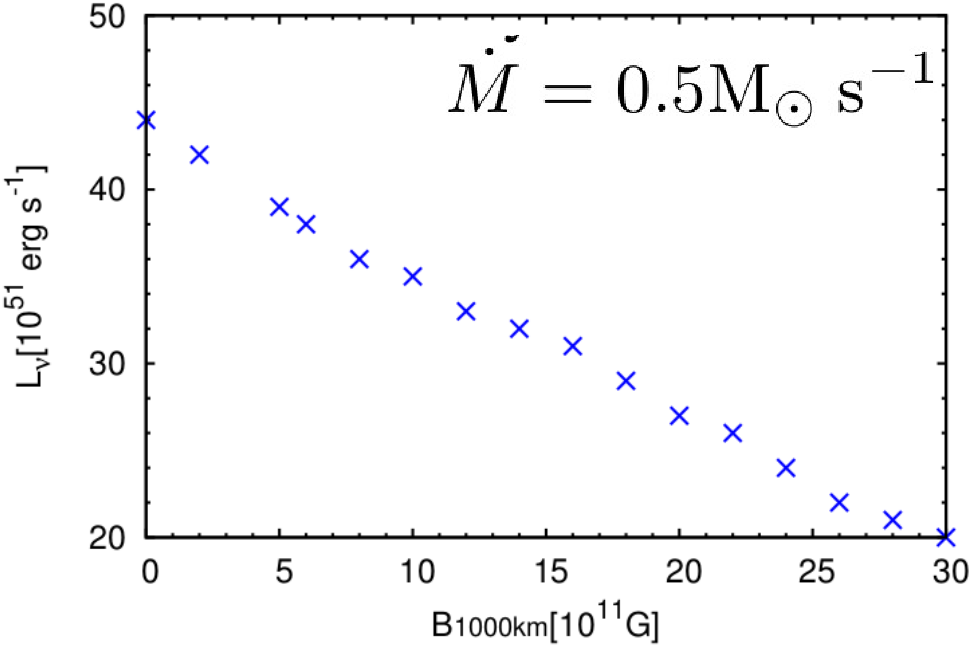


Critical luminosity is lowered by rotation and toroidal magnetic field.

Critical neutrino luminosity 2

- With rotation   With toroidal field





- Reduced by 70% (rotation)/50% (toroidal field)
- There exist no solutions beyond the critical rotation and critical magnetic field

Discussion and summary

- Non-spherical structure produced by rotation / magnetic field.
- Critical neutrino luminosity is lowered by (magnetic field) / 70% (rapid rotation).
- There exist critical toroidal magnetic field and critical angular momentum, beyond which there exit no steady solutions.
  - Critical angular momentum

    (Iwakami et al. 2014)
- Rotation / magnetic field assist shock revivals.

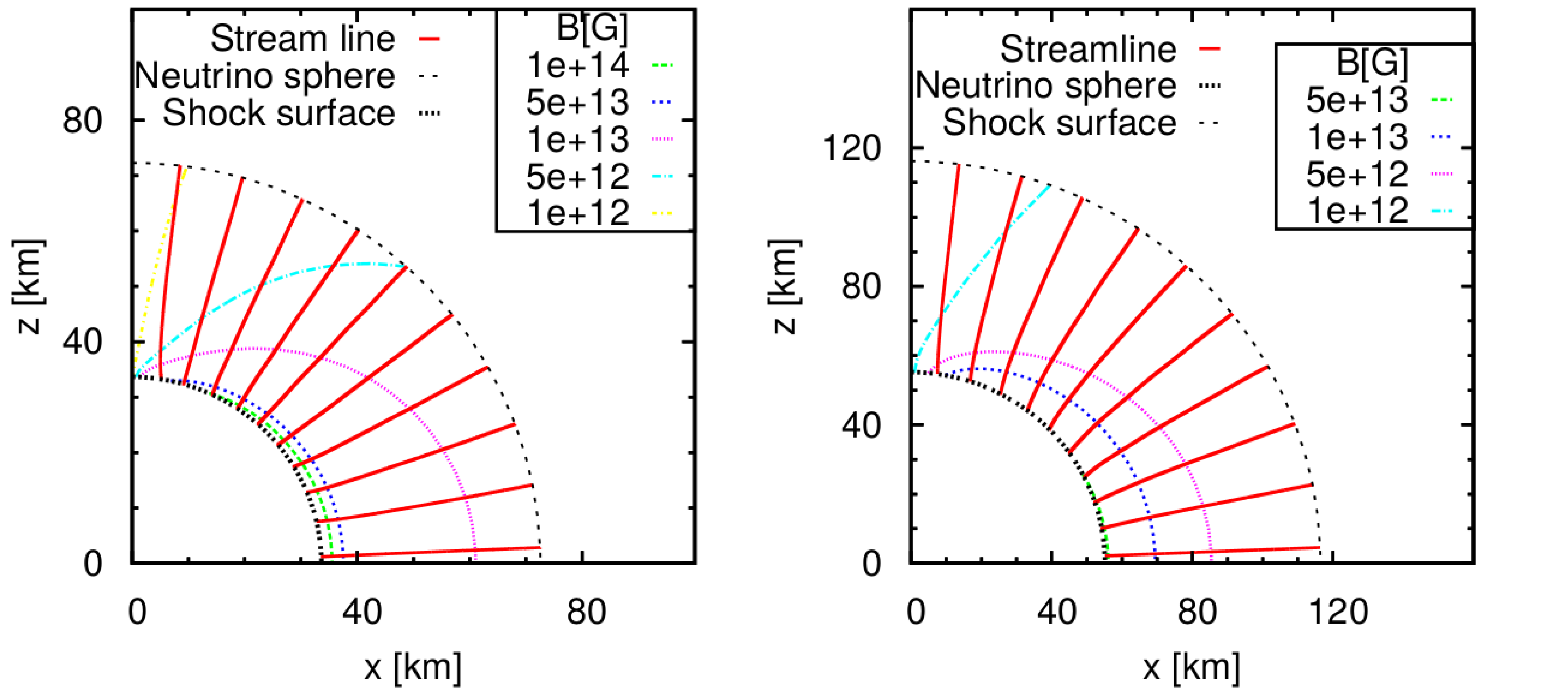
Future works

- Deformed PNS.
- Rotation and magnetic field of progenitors are unclear.

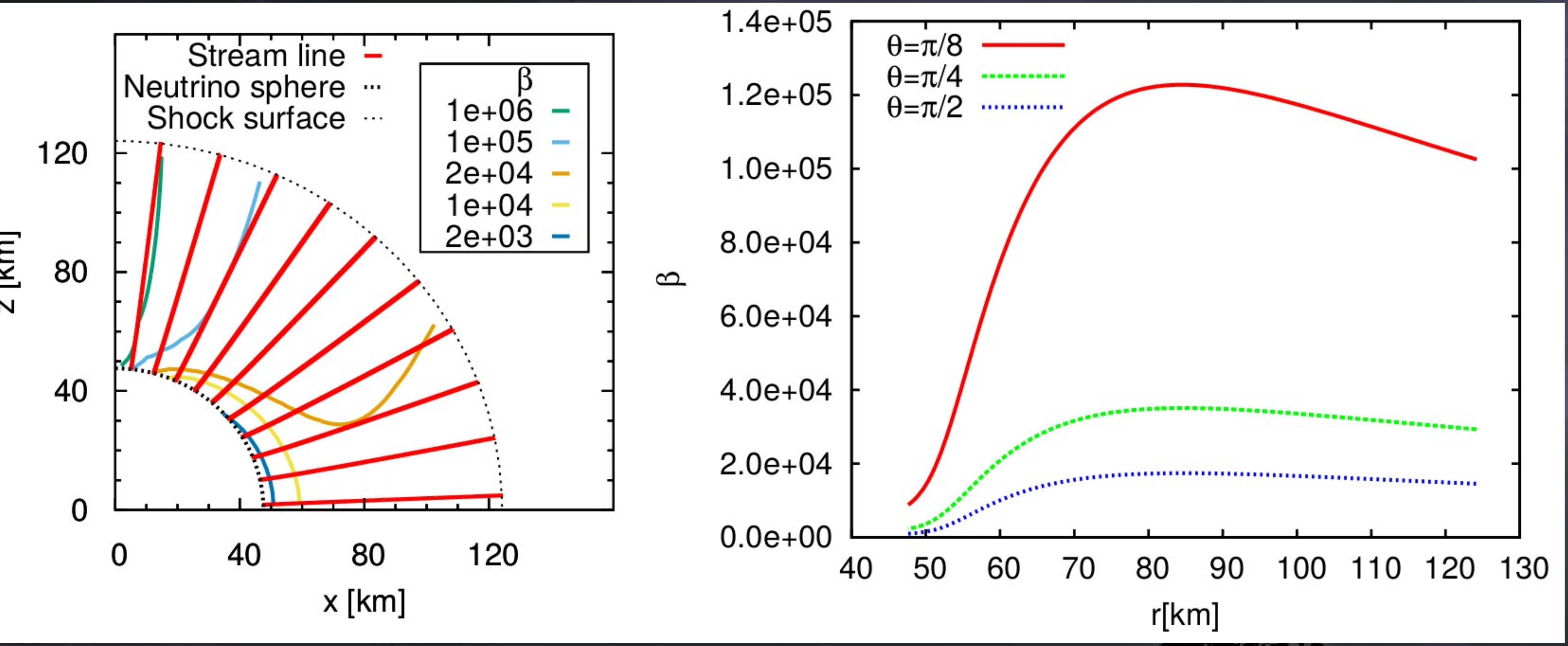
  → Multi-dimensional stellar evolution model (with rapid rotation and strong magnetic field)
- Anisotropic wind from rapidly rotating WR.

  (Callingham et al. 2018)

Rot. + tor. and pol. Magnetic field



Distribution of plasmaβ



Convergence of numerical result

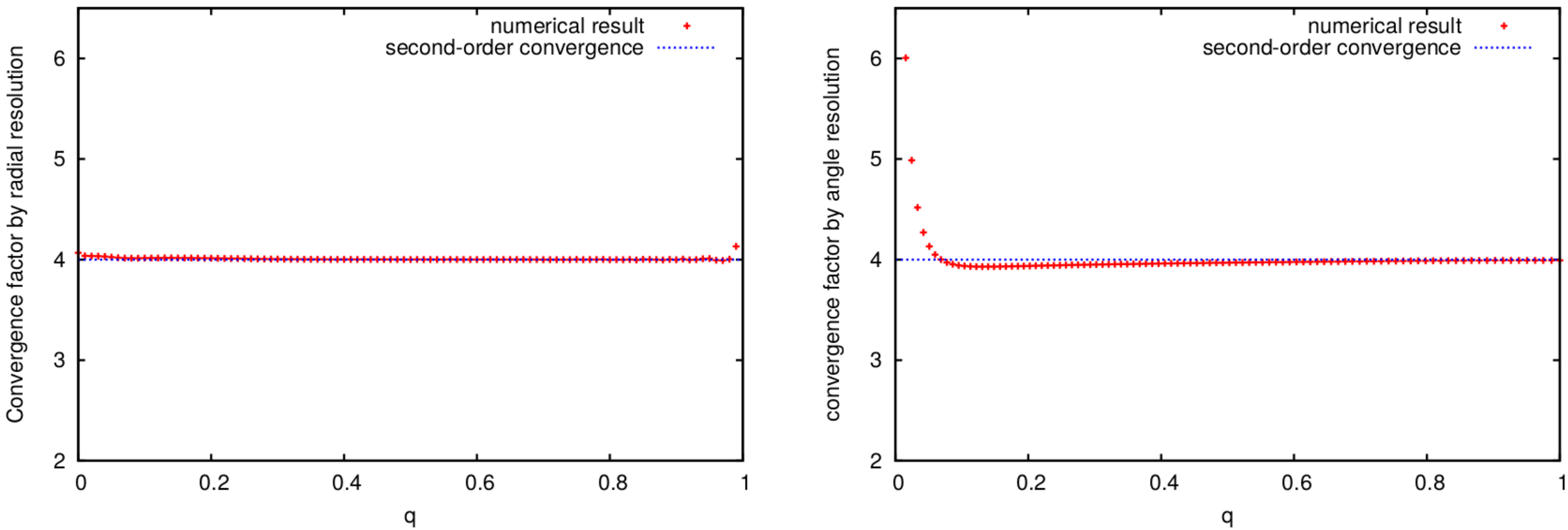


Figure 10. Convergence factors Q_q (left) and Q_θ (right) as functions of the q with $N_q = 200$ and $N_\theta = 20$. The mass accretion rate and neutrino luminosity are set to $\dot{M} = 2.0\ \mathrm{M}_\odot\ \mathrm{s}^{-1}$ and $L_\nu = 4.5 \times 10^{52}$ ergs s^{-1}, respectively. The rotational frequency is $f_{1000\mathrm{km}} = 0.03\ \mathrm{s}^{-1}$ and the strengths of the toroidal and poloidal magnetic fields are given as $B_0 = 10^6$ G and $B_{1000\mathrm{km}} = 10^6$ G at the outer boundary. The dotted blue line denotes the second-order convergence ($Q = 4$).

$$\phi(q) = \frac{1}{N_\theta} \sum_{k=1}^{N_\theta} u_\theta(q, \theta_k),$$

$$Q_q \equiv \left| \frac{\phi_{2N_q} - \phi_{N_q}}{\phi_{N_q} - \phi_{N_q/2}} \right|,$$

$$Q_\theta \equiv \left| \frac{\phi_{2N_\theta} - \phi_{N_\theta}}{\phi_{N_\theta} - \phi_{N_\theta/2}} \right|,$$

Newton-Raphson method

Eqs.

$$F_i(x_1, x_2, \cdots, x_N) = 0 \quad i = 1, 2, \cdots, N,$$

Taylor series

$$F_i(\boldsymbol{x} + \delta\boldsymbol{x}) = F_i(\boldsymbol{x}) + \sum_{j=1}^{N} \frac{\partial F_i}{\partial x_j} \delta x_j + \mathcal{O}(\delta\boldsymbol{x}^2).$$

$$F_i(\boldsymbol{x} + \delta\boldsymbol{x}) = F_i(\boldsymbol{x}) + \sum_{j=1}^{N} J_{ij} \delta x_j = 0,$$

$$J_{ij} \equiv \frac{\partial F_i}{\partial x_j}.$$

Renewal of guess

$$\boldsymbol{x}^{n+1} = \boldsymbol{x}^n + \delta\boldsymbol{x},$$

$$\delta x_i = -\sum_{j=1}^{N} J_{ij}^{-1} F_j,$$

W4 method

Consider NR method as differential eq.

$$\frac{\boldsymbol{x}^{n+1} - \boldsymbol{x}^n}{\Delta\tau} = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{f}(\boldsymbol{x}) \equiv \frac{\delta \boldsymbol{x}}{\Delta\tau} = -\frac{\mathbf{J}^{-1}\boldsymbol{F}}{\Delta\tau}.$$

$$\frac{d\boldsymbol{x}}{d\tau} = \boldsymbol{f}(\boldsymbol{x}).$$

Add a second time derivative term

$$\frac{d^2\boldsymbol{x}}{d\tau^2} + \mathbf{M}_1\frac{d\boldsymbol{x}}{d\tau} + \mathbf{M}_2\boldsymbol{F} = 0,$$

$$\frac{d\boldsymbol{x}}{d\tau} = \mathbf{X}\boldsymbol{p}, \quad \frac{d\boldsymbol{p}}{d\tau} = -2\boldsymbol{p} - \mathbf{Y}\boldsymbol{F},$$

Looks like damping term.

$$\mathbf{X}, \mathbf{Y} \equiv \mathbf{J}^{-1}$$

- Discretize and obtain equations as

$$\boldsymbol{x}^{n+1} = \boldsymbol{x}^n + \Delta\tau\mathbf{X}\boldsymbol{p}^n, \quad \boldsymbol{p}^{n+1} = (1 - 2\Delta\tau)\boldsymbol{p}^n - \Delta\tau\mathbf{Y}\boldsymbol{F}.$$

UL W4 method

$$\boldsymbol{x}^{n+1} = \boldsymbol{x}^n + \frac{1}{2}\mathbf{L}_n^{-1}\boldsymbol{p}^n, \quad \boldsymbol{p}^{n+1} = -\frac{1}{2}\mathbf{U}_n^{-1}\boldsymbol{F}(\boldsymbol{x}^n),$$

And finally obtain

$$\boldsymbol{x}^{n+1} = \boldsymbol{x}^n - \frac{1}{4}\mathbf{L}_n^{-1}\mathbf{U}_{n-1}^{-1}\boldsymbol{F}(\boldsymbol{x}^{n-1}),$$

$$J \equiv UL$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \ell_{21} & 1 & 0 & 0 \\ \ell_{31} & \ell_{42} & 1 & 0 \\ \ell_{41} & \ell_{42} & \ell_{43} & 1 \end{pmatrix} \quad U = \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix}$$

- c.f NR method

$$\boldsymbol{x}^{n+1} = \boldsymbol{x}^n + \delta\boldsymbol{x},$$

$$\delta x_i = -\sum_{j=1}^{N} J_{ij}^{-1} F_j,$$

- In general $\mathbf{L}_n^{-1}\mathbf{U}_{n-1}^{-1} \neq \mathbf{J}_n^{-1}$ and $\mathbf{L}_n^{-1}\mathbf{U}_{n-1}^{-1} \neq \mathbf{J}_{n-1}^{-1}$
- Near root $\mathbf{L}_n^{-1}\mathbf{U}_{n-1}^{-1} \simeq \mathbf{J}_{n-1}^{-1} \simeq \mathbf{J}_n^{-1}$

LH W 4 method 1

- QR decomposition as $\mathbf{J}^T \equiv \mathbf{QR}$

  $\mathbf{Q}$ diagonal $\mathbf{R}$ upper triangle matrices

$$\mathbf{J}^T = \mathbf{A}_{(0)} \equiv \begin{pmatrix} a_{11}^{(0)} & a_{12}^{(0)} & a_{13}^{(0)} \\ a_{21}^{(0)} & a_{22}^{(0)} & a_{23}^{(0)} \\ a_{31}^{(0)} & a_{32}^{(0)} & a_{33}^{(0)} \end{pmatrix}.$$

Let consider vector

$$\boldsymbol{a}_{(0)} \equiv \left[a_{11}^{(0)} \; a_{21}^{(0)} \; a_{31}^{(0)}\right]^T,$$

$$\boldsymbol{b}_{(0)} \equiv \left[-\mathrm{sign}\left(a_{11}^{(0)}\right) |\boldsymbol{a}_{(0)}| \; 0 \; 0\right]^T,$$

LH W 4 method 2

Make householder matrix $\mathbf{H}$ $\mathbf{H}^T = \mathbf{H}^{-1} = \mathbf{H}$

$$\mathbf{H}_{(0)} \equiv \mathbf{E} - 2\boldsymbol{c}_{(0)}\boldsymbol{c}_{(0)}^T, \quad \boldsymbol{c}_{(0)} \equiv \frac{\boldsymbol{a}_{(0)} - \boldsymbol{b}_{(0)}}{|\boldsymbol{a}_{(0)} - \boldsymbol{b}_{(0)}|},$$

$$\mathbf{H}_{(0)}\boldsymbol{a}_{(0)} = \boldsymbol{b}_{(0)} \text{ and } \mathbf{H}_{(0)}\boldsymbol{b}_{(0)} = \boldsymbol{a}_{(0)}.$$

$$\mathbf{H}_{(0)}\mathbf{A}_{(0)} \equiv \mathbf{A}_{(1)} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} \end{pmatrix},$$

Similarly,

$$\mathbf{H}_{(1)}\mathbf{A}_{(1)} \equiv \mathbf{A}_{(2)} = \mathbf{H}_{(1)}\mathbf{H}_{(0)}\mathbf{J}^T = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix} = \mathbf{R}.$$

LH W 4 method 3

Then we obtain

$$\mathbf{J}^T = \mathbf{H}_{(0)}\mathbf{H}_{(1)}\mathbf{R} \;\Rightarrow\; \mathbf{J} = (\mathbf{H}_{(0)}\mathbf{H}_{(1)}\mathbf{R})^T = \mathbf{L}\mathbf{H}_{(1)}\mathbf{H}_{(0)},$$

From $\mathbf{H}^T = \mathbf{H}^{-1} = \mathbf{H}$ we obtain following step equations.

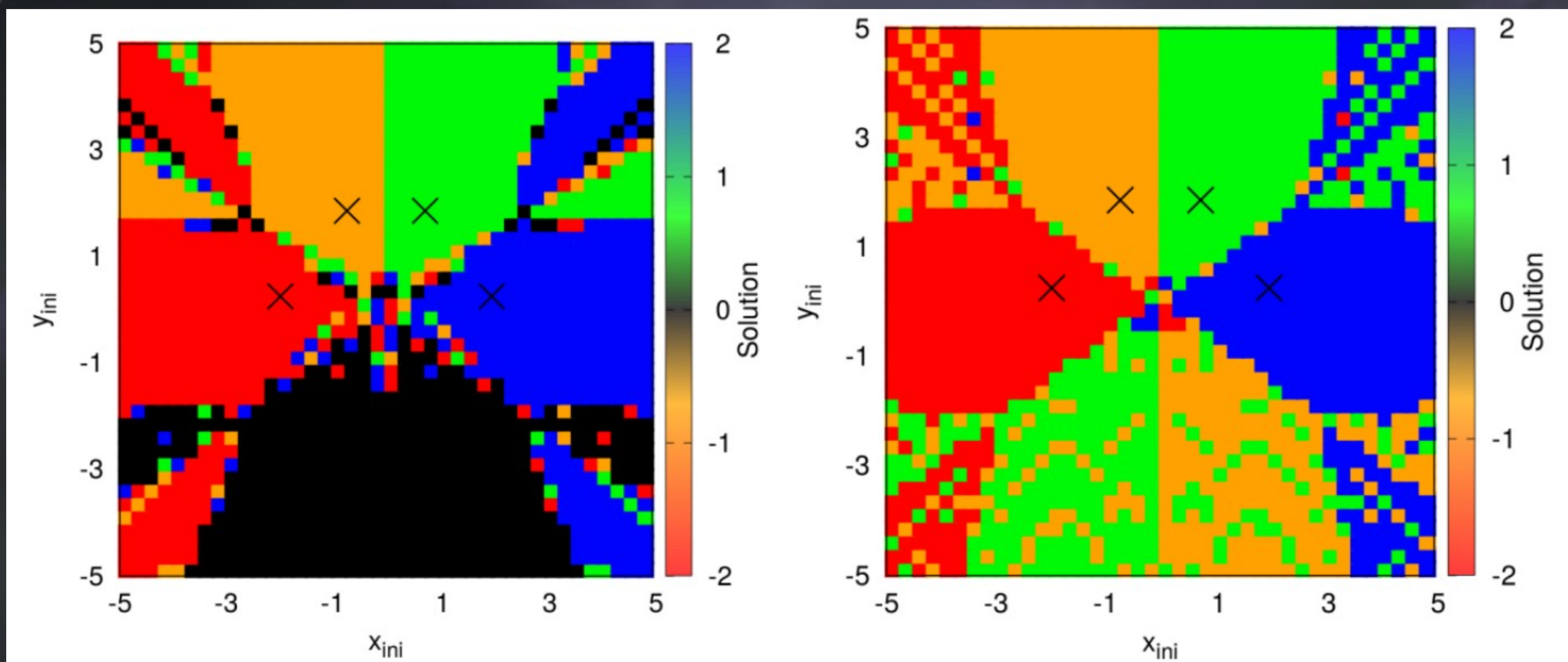
$$\boldsymbol{x}^{n+1} = \boldsymbol{x} + \frac{1}{2}\mathbf{H}_{(0)}\boldsymbol{p}^n, \quad \boldsymbol{p}^{n+1} = -\frac{1}{2}\mathbf{H}_{(1)}\mathbf{L}^{-1}\boldsymbol{F}(\boldsymbol{x}).$$

LH W4 is more efficient than UL W4.

Convergence region (Basin)

$$f_1(x,y) = x^2 + y^2 - 4 = 0,\ f_2(x,y) = x^2 y - 1 = 0$$

NR UL W4

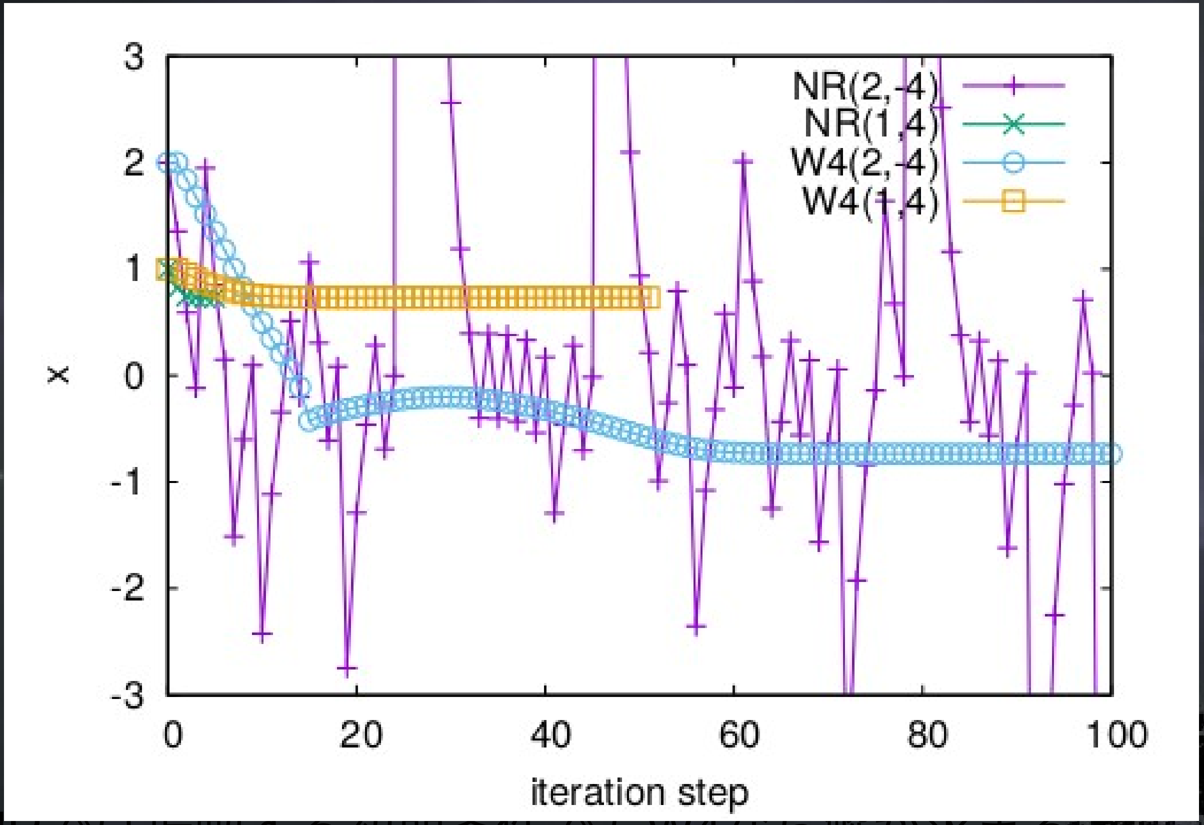


Global convergent method!

Numerical oscillation

$$f_1(x,y) = x^2 + y^2 - 4 = 0,\ f_2(x,y) = x^2 y - 1 = 0$$

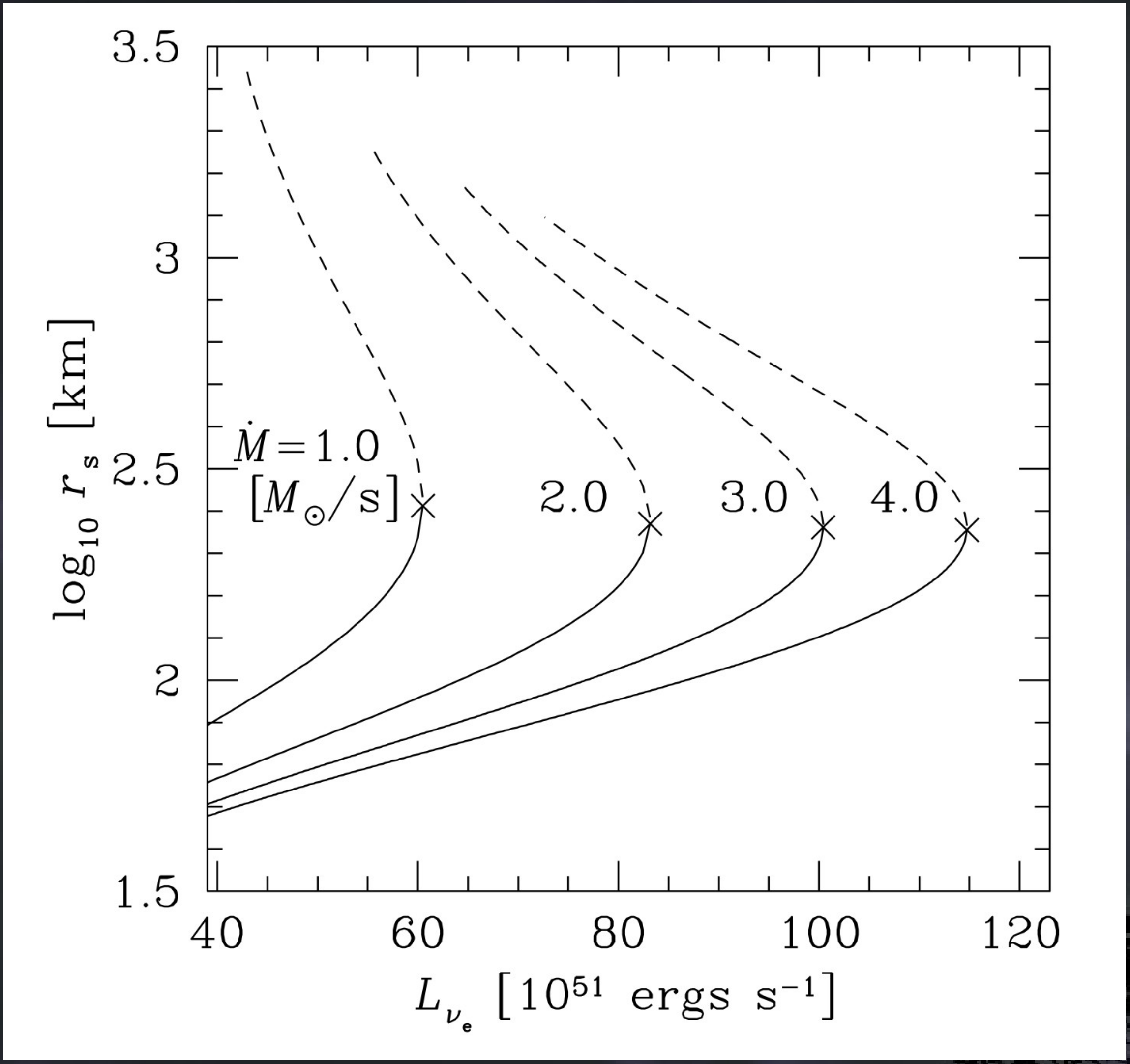
W4 avoids numerical oscillation of NR.



NRでは振動する初期条件でもW4なら解が求まる

Critical Luminosity

Yamasaki & Yamada (2005)



A numerical scheme for non-linear equations.

$$F_i(x_1, x_2, \cdots, x_N) = 0 \quad i = 1, 2, \cdots, N.$$

Newton-Raphson

$$\boldsymbol{x}^{n+1} = \boldsymbol{x}^n + \delta \boldsymbol{x},$$

$$\delta x_i = -\sum_{j=1}^{N} J_{ij}^{-1} F_j,$$

W4

$$\frac{d^2 \boldsymbol{x}}{d\tau^2} + 2\frac{d\boldsymbol{x}}{d\tau} + M\boldsymbol{F} = 0,$$

$$\boldsymbol{x}^{n+1} = \boldsymbol{x} + \frac{1}{2} L^{-1} \boldsymbol{p}^n, \quad \boldsymbol{p}^{n+1} = \frac{1}{2} U^{-1} \boldsymbol{F}(\boldsymbol{x}$$

$$J \equiv UL$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \ell_{21} & 1 & 0 & 0 \\ \ell_{31} & \ell_{42} & 1 & 0 \\ \ell_{41} & \ell_{42} & \ell_{43} & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix}$$